The Toric Code and the Quantum Double

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The toric code is a topological quantum error correcting code, which can be understood as a Z_2 gauge theory on a lattice [2]. Consider a $k \times k$ lattice on a plane with spins residing at each link. There are then altogether $N_s = k^2$ sites s and $N_p = k^2$ plaquettes p. The Hamiltonien of the model is given by

$$H = -\sum_s X_s - \sum_p Z_p, \tag{1}$$

where the site and plaquette operators,

$$X_s = \prod_{i \in s} \sigma_i^x, \qquad Z_p = \prod_{i \in p} \sigma_i^z, \tag{2}$$

are defined as the products of Pauli operators acting on all the spins meeting at each site or plaquette. In the gauge theory language the spins residing at the links are considered as Z_2 valued gauge potentials, the operators X_s are gauge transformations and the operators Z_p measure the magnetic field on plaquette p. Because $[X_s, Z_p] = 0$, the Hamiltonian is gauge-invariant and the ground states of the model are given by all the states $|\xi\rangle \in \mathcal{H}$ satisfying both of the stabilizer conditions

$$X_s|\xi\rangle = |\xi\rangle, \qquad \forall s, \tag{3}$$

$$Z_p|\xi\rangle = |\xi\rangle, \qquad \forall p. \tag{4}$$

The energy corresponding to the ground states is given by

$$H|\xi\rangle = -(N_p + N_s)|\xi\rangle. \tag{5}$$

Since the operators X_s are understood as the gauge transformations, the stabilizer condition (3) can be interpreted as the condition for the gauge invariance for all the physical states $|\psi\rangle \in \mathcal{H}$. If $|\psi\rangle$ satisfies also (4), the gauge field corresponds to a flat connection [2].

Excitations are created by violating the stabilizer conditions. Applying σ^x (σ^z) on a spin creates an excited state, which can be interpreted as flux - anti-flux (charge - anti-charge) pair living on the two plaquettes (sites) sharing the spin link. Since $(\sigma^x)^2 = (\sigma^z)^2 = 1$, both fluxes and charges are their own anti-particles. The corresponding quantum states can be expressed as

$$|X\rangle \equiv \sigma^x|\xi\rangle, \qquad H|X\rangle = -(N_s + (N_p - 2) - 2)|X\rangle = -(N_s + N_p - 4)|X\rangle, \tag{6}$$

$$|Z\rangle \equiv \sigma^z|\xi\rangle, \qquad H|Z\rangle = -((N_s - 2) - 2 + N_p)|Z\rangle = -(N_s + N_p - 4)|Z\rangle. \tag{7}$$

The energies corresponding to the excited states show that the system has an energy gap with a creation of a single excitation costing the energy $E = 2$. It is possible to have also a third kind of excitation Y, which can be thought of as a bound state of the X and Z particles. The quantum state corresponding to it can be defined by

$$|Y\rangle \equiv \sigma^y|\xi\rangle \equiv i\sigma^x\sigma^z|\xi\rangle, \tag{8}$$

$$H|Y\rangle = -((N_s - 2) - 2 + (N_p - 2) - 2)|Y\rangle = -(N_s + N_p - 8)|Y\rangle. \tag{9}$$

Since the particles are created or annihilated by applying the Pauli operators on the ground state, the fusion rules are obtained up to overall phases as the multiplication of the elements of the group $Z_2 \times Z_2$ generated by σ^x and σ^z. The respective fusion rules are

$$X \times X = 1, \quad Z \times Z = 1, \quad Y \times Y = 1, \quad X \times Z = Y, \quad X \times Y = Z, \quad Y \times Z = X, \tag{10}$$

where 1 denotes the vacuum. The X and Z particles are bosons, whereas Y particles are by fermions [5]. However, the mutual statistics between all the three distinct particle types are anyonic.

The Construction of $D(Z_2)$

The excitations appearing in the toric code model can be captured in the unified picture, which is provided by quasitringular Hopf algebras, i.e. quantum groups [1, 4]. In general, the excitations appearing in any two-dimensional system with a discrete symmetry H can be classified by the irreducible representations of the so called quantum double of H, $D(H)$. Furthermore, this algebraic structure allows also an elegant derivation of the quantum statistics.

Consider the discrete group $Z_2 = \{e, a\}, a^2 = e$, which describes the gauge symmetry in the toric code model. The elements of $D(Z_2) = F[Z_2] \times C[Z_2]$ are given by the set

$$D(Z_2) = \{P_e e, P_e a, P_a e, P_a a\}, \tag{11}$$

whose elements are defined to obey a multiplication rule

$$P_h g P_{h'} g' = \delta_{h, gh'g^{-1}} P_h g g', \qquad h, h', g, g' \in Z_2. \tag{12}$$

The irreducible representations Π_C^Γ of $D(Z_2)$ are carried by vector spaces $V_C^\Gamma = V_C \otimes V^\Gamma$ labeled by the conjugacy classes C of Z_2 and the irreducible representations Γ of the corresponding normalizer subgroups N [1]. The former describe the flux degrees of freedom whereas the latter account for the charge degrees of freedom. The orthonormal basis in the space V_C^Γ is given by the states

$$|k,i\rangle, \quad \langle k',i'|k,i\rangle = \delta_{k',k}\delta_{i',i}, \quad k \in C, \quad i = 1,\dots,\dim(\Gamma). \tag{13}$$

Since Z_2 is an abelian group, every element forms its own conjugacy class and the corresponding normalizers are given by the whole group,

$$C(e) = \{e\}, \qquad N(e) = \{e,a\} = Z_2, \tag{14}$$

$$C(a) = \{a\}, \qquad N(a) = \{e,a\} = Z_2. \tag{15}$$

One needs only the irreducible representations of Z_2, which are given by the trivial and sign representations

Z_2	e	a
Γ_1	1	1
Γ_{-1}	1	−1

(16)

All the conjugacy classes contain only a single element, and all the normalizer representations are one-dimensional, $\dim(V_C^\Gamma) = 1$. This means that the physical particles do not carry any internal degrees of freedom. This is a general property of all abelian models.

The irreducible representations Π_C^Γ of $D(Z_2)$ are given by matrices which act in the spaces V_C^Γ as

$$\Pi_C^\Gamma(P_h g)|k,i\rangle = \delta_{h,gkg^{-1}}|gkg^{-1},\Gamma(g)i\rangle, \quad \forall P_h g \in D(Z_2), \tag{17}$$

where $\Gamma(g)$ is the matrix assigned to the element g in the representation Γ (16). This action can be thought of as corresponding to first implementing a global $g \in Z_2$ gauge transformation and subsequently projecting onto the flux eigenstate labeled by $h \in Z_2$. Since $\dim(V_C^\Gamma) = 1$, the matrices Π_C^Γ are just numbers, which are summarized in the following table

$\Pi_A \equiv \Pi_C^\Gamma$	$\Pi_A(P_h g)$	e	a
$\Pi_1 \equiv \Pi_e^1$	P_e	1	1
	P_a	0	0
$\Pi_X \equiv \Pi_e^{-1}$	P_e	1	−1
	P_a	0	0
$\Pi_Z \equiv \Pi_a^1$	P_e	0	0
	P_a	1	1
$\Pi_Y \equiv \Pi_a^{-1}$	P_e	0	0
	P_a	1	−1

(18)

The conjugacy class $C(e)$ of the trivial element corresponds to having no flux degrees of freedom. Likewise, the trivial representation Γ_1 corresponds to having no charge degrees of freedom. Based on this observation, the four irreducible representations of $D(Z_2)$ have been identified with the four different excitations appearing in the toric code model.

The fusion rules for particles A and B can by evaluated by calculating the Clebsch-Gordan series

$$\Pi_A \otimes \Pi_B = \bigoplus_{C\in\{1,X,Z,Y\}} N_{AB}^C \Pi_C, \tag{19}$$

where the fusion multiplicities N_{AB}^C can be calculated by using the orthogonality of the characters

$$N_{AB}^C = \frac{1}{2} \sum_{g,h,h'\in Z_2} \mathrm{tr}\left(\Pi_A(P_{h'}g)\right) \mathrm{tr}\left(\Pi_B(P_{h'^{-1}h}g)\right) \mathrm{tr}\left(\Pi_C(P_h g)\right). \tag{20}$$

Now $\mathrm{tr}(\Pi_A) = \Pi_A$, and the fusion multiplicities can be calculated by using the appropriate entries from the table (18). This gives

$$\begin{array}{rcl} N_{1A}^A = 1, \quad N_{AA}^B & = & \delta_{B,1}, \quad \forall A, \\ N_{XZ}^Y = N_{XY}^Z & = & N_{YZ}^X = 1, \end{array} \tag{21}$$

which describe the same fusion rules as derived on the spin lattice level (10).

Braiding and Statistics

The fusion rules can also be derived using an alternative method, which provides also the mutual statistics between the particles. Quasitriangular Hopf algebras contain a special element called the universal R-matrix $\mathcal{R} \in D(Z_2) \otimes D(Z_2)$, which satisfies the Yang-Baxter equation [1]. This means that it can be used to construct representations of the braid group, which governs the statistics in two spatial dimensions. For $D(Z_2)$, it is defined by

$$\mathcal{R} = \sum_{h,g\in Z_2} P_g e \otimes P_h g. \tag{22}$$

A representation of $\mathcal{R}$ acting in the space $V_A \otimes V_B$ can be defined by

$$R_{AB} = \sigma \circ (\Pi_A \otimes \Pi_B)(\mathcal{R}), \tag{23}$$

where $\sigma : a \otimes b \mapsto b \otimes a$ is a transposition map. The operators R_{AB} can be taken to implement the counter-clockwise interchange of the particles A and B on the plane. The statistical phases associated with single interchanges can be expressed in a compact form in terms of the matrix

$$R = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}. \tag{24}$$

Similarly, one can define the monodromy operators $(R^2)_{AB} = R_{BA}R_{AB}$, which describe particle A propagating along a closed loop around particle B. In the matrix form it reads

$$R^2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}. \tag{25}$$

The mutual statistics of the particles can be inferred from the matrices (23) and (25). X and Z particles are bosons, because $R_{XX} = R_{ZZ} = (R^2)_{XX} = (R^2)_{ZZ} = 1$. On the other hand, the Y particles are fermions, because $R_Y Y = -1$, but $(R^2)_{YY} = 1$. In general, the elements of R^2 for which $(R^2)_{AB} = 1$ describe either bosonic or fermionic statistics between A and B. The elements $(R^2)_{AB} \neq 1$ describe mutual anyonic statistics. Based on these observations, one expects X and Z be spin 1 particles, whereas Y is expected to be spin $\frac{1}{2}$ particle. These can be verified by considering the representations of the central element

$$C_A = \Pi_A(\mathcal{C}), \qquad \mathcal{C} = \sum_{h \in Z_2} P_h h, \tag{26}$$

which implements the transformations corresponding to the flux sector on the charge sector. In the picture of anyons as fluxes attached to charges, this corresponds to rotating the particle by 2π. The object containing the spin factors is known as the modular T-matrix, whose elements are defined by

$$T_{AB} = \frac{\delta_{A,B}}{\dim(C_A)} \mathrm{tr}(C_A) = e^{2\pi i s_A}, \tag{27}$$

where s_A is the spin of particle A. Evaluating the matrix elements gives $T_{11} = T_{XX} = T_{ZZ} = 1$, which corresponds to integer spin, whereas $T_{YY} = -1$ corresponds to half-integer spin.

Finally, the fusion rules can be obtained by considering the so called modular S-matrix, which contains all the information concerning the fusion rules. Its elements are defined by

$$S_{AB} = \frac{1}{|H|} \mathrm{tr}\left((R^2)_{AB}\right). \tag{28}$$

Since the elements of R^2 are numbers, the modular S-matrix is given by $S = \frac{1}{2}R^2$. The fusion rules are obtained by using the Verlinde formula [1]

$$N_{AB}^C = \sum_{D \in \{1,X,Z,Y\}} \frac{S_{AD} S_{BD} S_{CD}^*}{S_{1D}}, \tag{29}$$

which gives the same fusion multiplicities as (21).

Discussion

The quantum double construction $D(H)$ can be used to derive the particle spectrum as well as the braid statistics for any two-dimensional model with a discrete and finite symmetry group H. This was verified by considering the toric code model, which can be understood as a Z_2 gauge theory. This case is trivial in the sense that same properties could be inferred directly by considering the group operations of $Z_2 \times Z_2$ describing the fusion and propagation of the excitations. The real advantage of the quantum double formalism lies in applying it to non-abelian models, i.e. to cases when H is a non-abelian group. The generalization is straightforward, but more involved. The standard reference is [1]. It is hoped that the presented elementary example gives a flavor of how quantum groups can be used provide a unified language for describing any anyon model. This language is closely related to topological quantum field theories, which are discussed in the context of anyons models in [3].

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