NUI MAYNOOTH

MATHEMATICAL PHYSICS

science foundation ireland

Topological quantum computation

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Outline

Introduction

Topological quantum computation and algorithms

- Links and knots
- Jones polynomial
- Braid group
- Approximating Jones polynomial

Topological phases and natural fault-tolerance

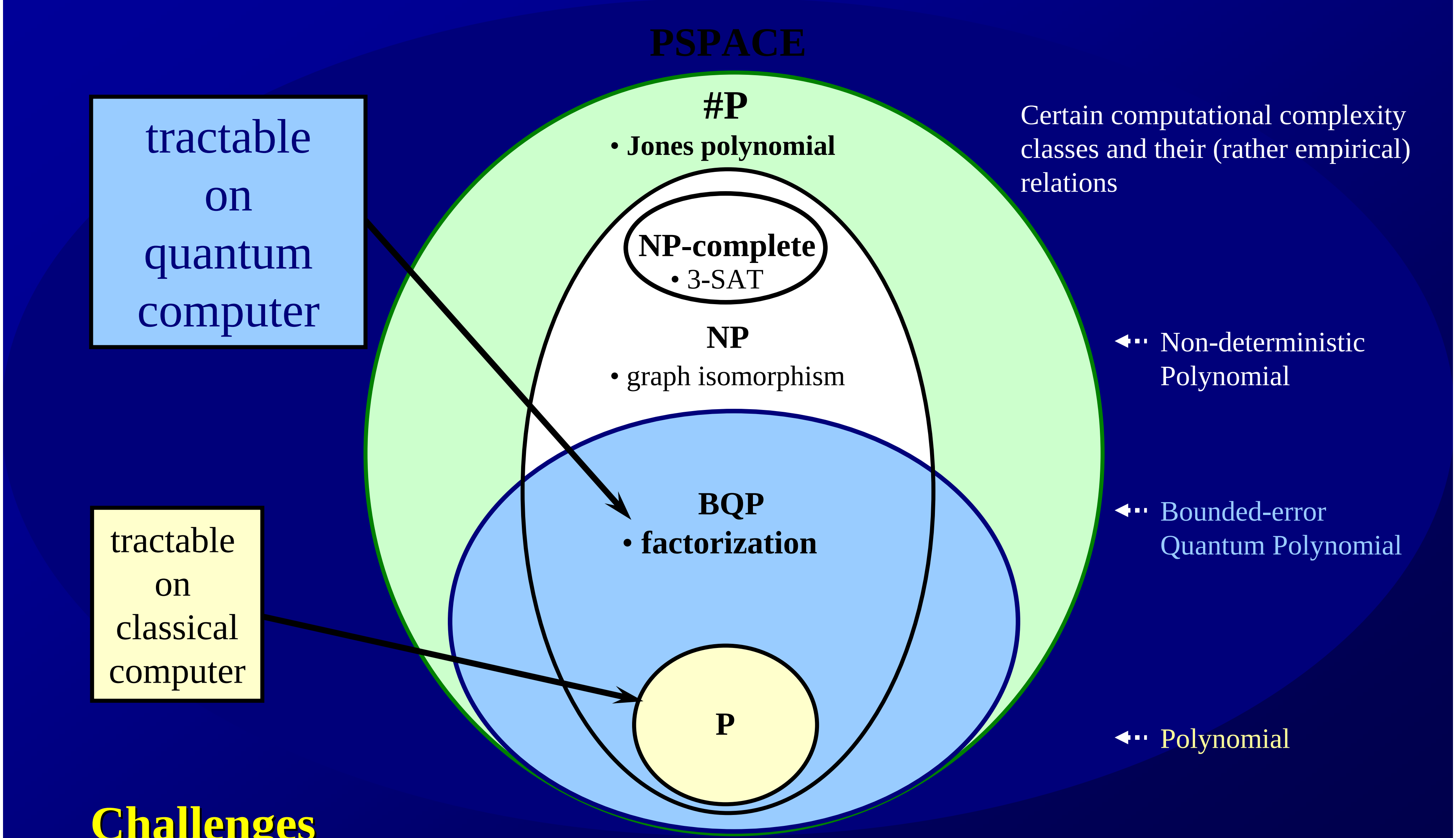
- Topological quantum field theory
- Spectral properties of topological phases

Anyons

Physical realization

- Fractional quantum Hall systems
- Lattice models of topological phases

Quantum computation



PSPACE

#P

• Jones polynomial

NP-complete

• 3-SAT

NP

• graph isomorphism

BQP

• factorization

P

tractable on quantum computer

tractable on classical computer

Certain computational complexity classes and their (rather empirical) relations

Non-deterministic Polynomial

Bounded-error Quantum Polynomial

Polynomial

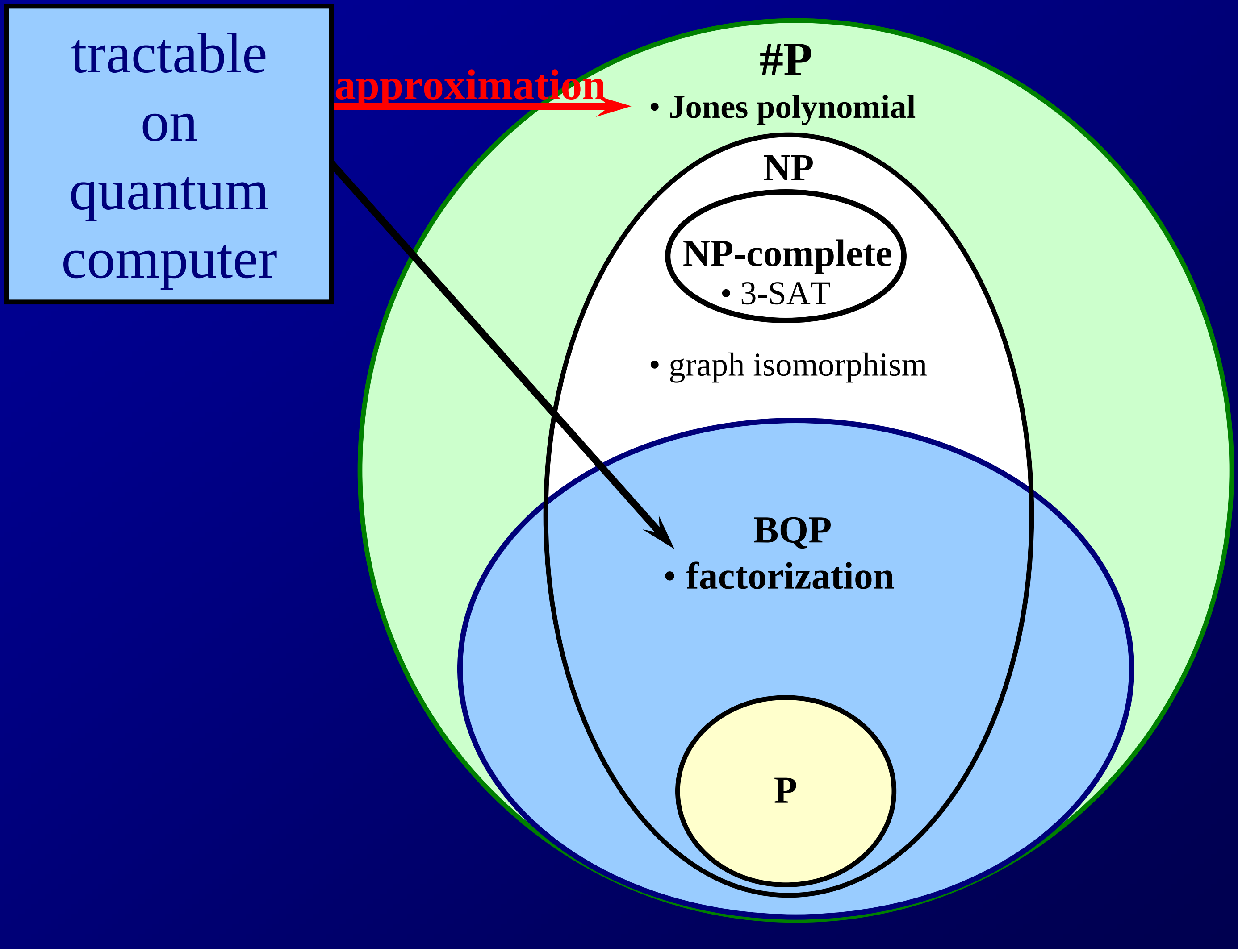
Challenges

Theory: algorithms, computational complexity, …

Implementations: fault-tolerance, …

Topological quantum computation

- is a unique QC model (though equivalent to standard QCM) => new algorithms
- natural fault tolerance



tractable on quantum computer

approximation

#P

- **Jones polynomial**

NP

NP-complete

- 3-SAT

- graph isomorphism

BQP

- **factorization**

P

Freedman et al., Commun. Math. Phys. 227, 587 and 605 (2002)

Aharonov, Jones, Landau, STOC’06, quant-ph/0511069

Links and knots

Link

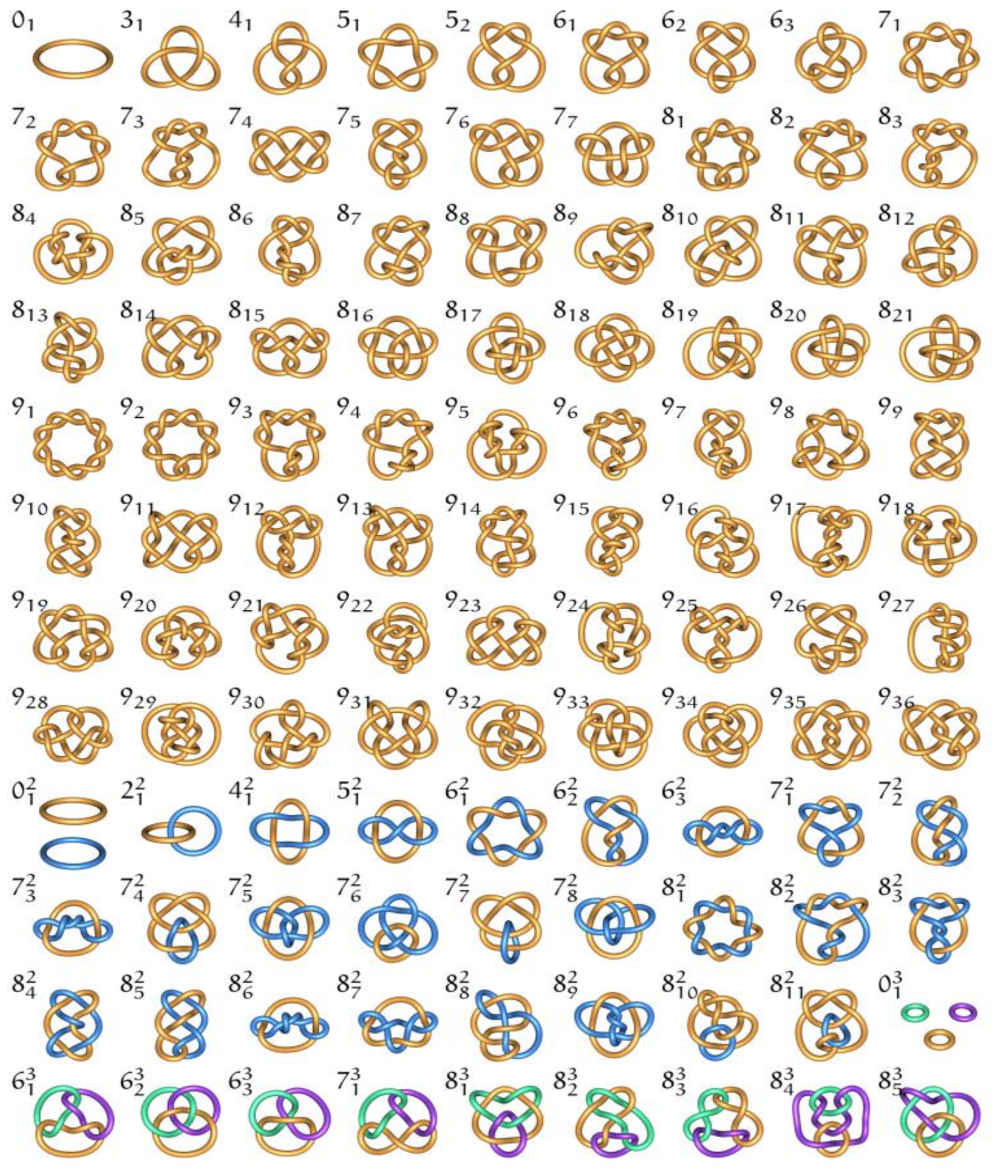
- a finite family of disjoint, smooth, closed curves in $\mathbb{R}^3$ or equivalently in S^3

Knot

- a link with one component

Knot complexity grows very fast:

# of crossings	# of knots
0	1
3	1
4	1
5	2
6	3
7	7
8	21
9	36



http://www.pims.math.ca/knotplot/zoo/

Jones polynomial

- Laurent polynomial in $t^{1/2}$,$V_L(t)$
- invariant of links and knots under isotopy
- #P-hard combinatorial problem

Jones, Bull. AMS 12, 103 (1985)

- classical algorithm is exponential in the number of crossings

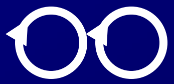
skein relation

$t^{-1} V$  $- t V$  $= (t^{1/2} - t^{-1/2}) V$ 

trivial knot

V  $= 1$

Examples:

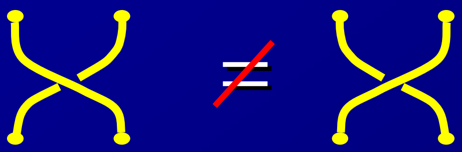
V  $= - (t^{-1/2} + t^{1/2})$ “unlink”

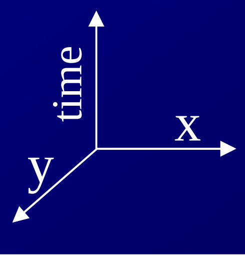
V  $= t + t^3 - t^4$ right hand trefoil knot

Braid group B_n

Artin, Ann. Math. 48, 101 (1947)

Exchanging particles on a plane is not permutation but braiding:

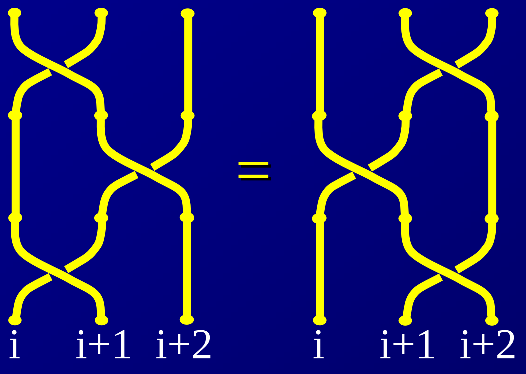

clockwise $\neq$ counterclockwise



A braid group for n strands (particles) has n generators $\{1, \sigma_1, \ldots, \sigma_{n-1}\}$ which satisfy:

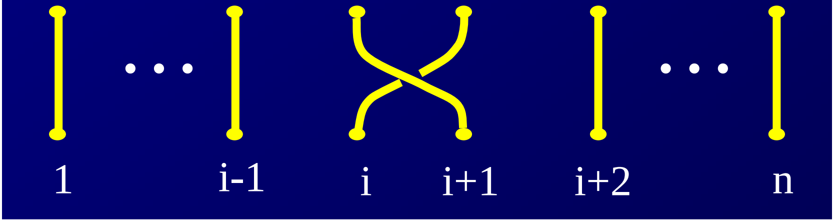
$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{for } |j - i| > 1$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$$



Yang-Baxter equation

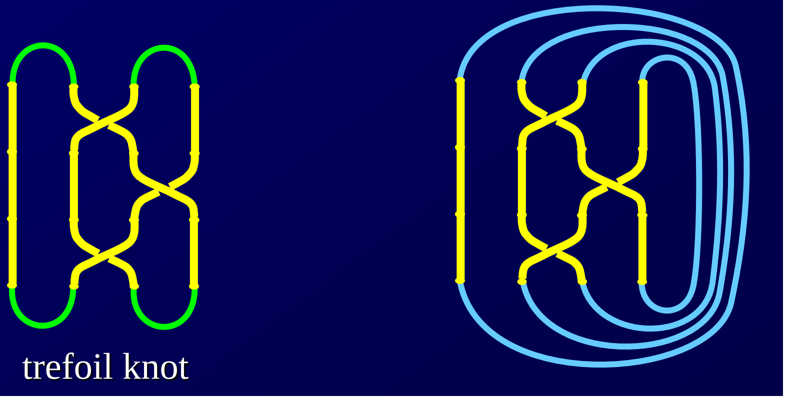
Example: σ_i



Closures of braids are links and knots

plat closure

trace closure



trefoil knot

A braid group can be usefully represented in Temperley-Lieb algebra $TL_n(d)$ which permits a unitary representation (qubits!!!)

Approximating Jones polynomial

Freedman et al., Commun. Math. Phys. 227, 587 and 605 (2002)
Aharonov, Jones, Landau, STOC'06, quant-ph/0511069

There is an efficient, explicit and simple quantum algorithm to approximate Jones polynomial for all $t = e^{2\pi i/k}$:

Theorem 1

the trace closure case

For a given braid B with n strands and m crossings, and a given integer k, there is a quantum algorithm which is polynomial in n, m, k which with all but exponentially small probability, outputs a complex number r with $|r - V_{Btr}(e^{2\pi i/k})| < \varepsilon d^{n-1}$ where $d = -A^2 - A^{-2}$, and ε is inverse polynomial in n, m, k.

Theorem 2

the plat closure case

For a given braid B with n strands and m crossings, and a given integer k, there is a quantum algorithm which is polynomial in n, m, k which with all but exponentially small probability, outputs a complex number r with $|r - V_{Bpl}(e^{2\pi i/k})| < \varepsilon d^{3n/2}/N$ where $d = -A^2 - A^{-2}$, and ε is inverse polynomial in n, m, k (N is an exponentially large factor).

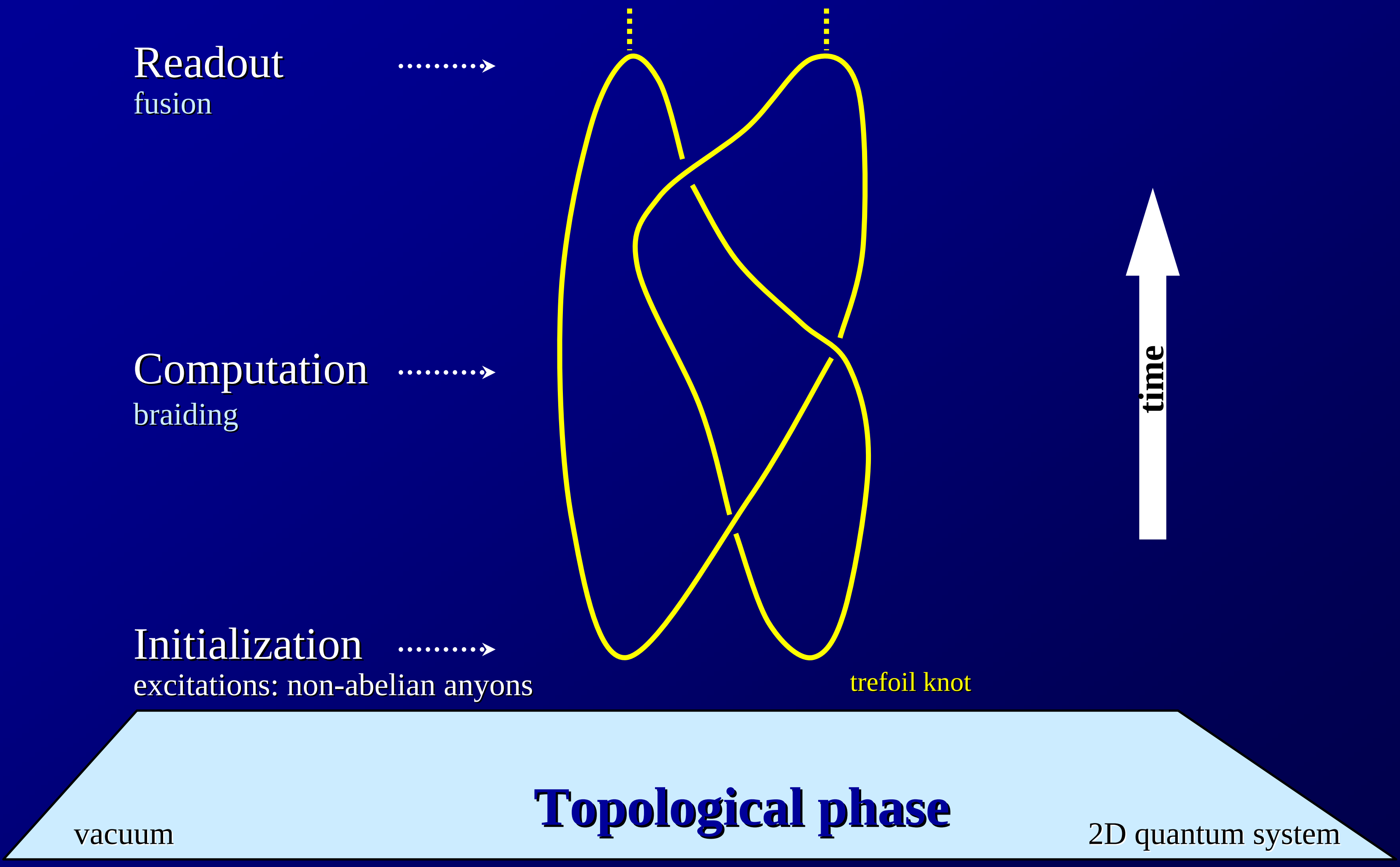
Theorem 3

Approximating the Jones polynomial of the plat closure (Th.2) is BQP-complete.

Aharonov, Arad, quant-ph/0605181
Wocjan, Yard, quant-ph/0603069

Topological quantum computation

- is a unique QC model (though equivalent to standard QCM) => new algorithms
- natural fault tolerance



Readout
fusion

Computation
braiding

Initialization
excitations: non-abelian anyons

trefoil knot

Topological phase

vacuum

2D quantum system

Topological phases: effective theory

- topological phases are phases of two-dimensional many-body quantum systems whose properties depend only on topology of the manifold on whose surface a given phase is realized

- their effective description is given by topological quantum field theory (3 dimensional) defined e.g. by the Chern-Simons action:

Witten, Commun. Math. Phys. 121, 351 (1989)

$$S = k/4\pi \int_{\Gamma} dt\, d^2x\, \varepsilon^{\mu\nu\rho}\, a_\mu \partial_\nu a_\rho$$

level of theory (integer)

$\Gamma \longleftarrow$ (2+1)D manifold

gauge field

no metric!!!

Example: doubled SU(2)$_k$ Chern-Simons theory (PT invariant theory):

Freedman,et al., CMP 227, 605 (2002)

$k = 1$	- abelian topological phase	- quantum memory
$k \geq 2$	- non-abelian	
$k = 3, 5 \ldots$	- non-abelian and universal	- universal QC

- topological phases are invariant with local geometry and hence quantum information stored in them is invariant with local error processes

no metric, no error!!!

Topological phases: Hamiltonian spectrum

- are ground states of certain strongly correlated many-body quantum systems

e.g. in Coulomb gauge, $a_0 = 0$: $\mathcal{L} = a_2 \partial_0 a_1 - a_1 \partial_0 a_2$

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial(\partial_0 a_1)} \partial_0 a_1 + \frac{\partial \mathcal{L}}{\partial(\partial_0 a_2)} \partial_0 a_2 - \mathcal{L} = \mathbf{0}$$

no metric, no energy!!!

- energy spectrum of matter in a topological phase is characterized by

finite topology-dependent ground state degeneracy,

e.g. for the doubled $SU(2)_k$ Chern-Simons theory: $(k+1)^{2g}$

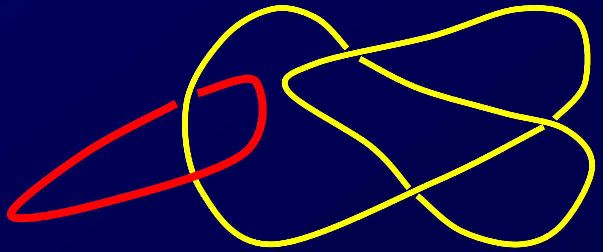
genus

Freedman et al. Ann. Phys. 310, 428 (2004)

$\Downarrow$

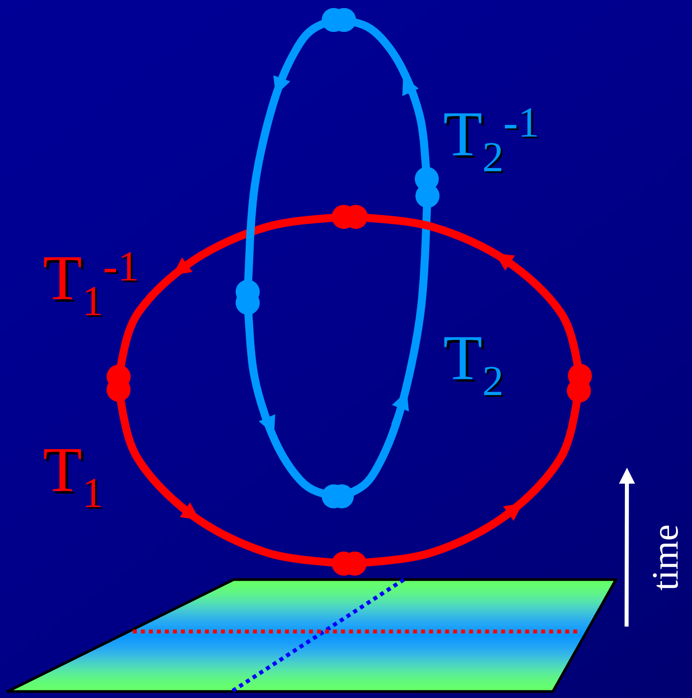
spectral gap

- excitations of stray anyons, which may cause errors via non-local processes, are exponentially suppressed due to the spectral gap !!!

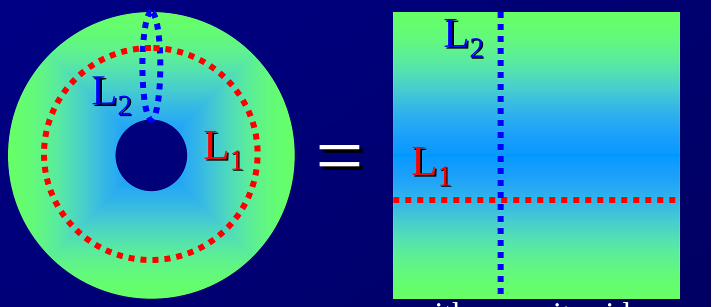


Topological phases: ground state degeneracy

Example: abelian anyons on torus



torus (genus g=1 surface)



with opposite sides identified

one anyon winds clockwise around the other:

$$T_2^{-1}T_1^{-1}T_2T_1 = e^{-i2\theta}\,\mathbb{1}$$

$[T_1, H] = 0$ and $[T_2 , H] = 0$, so $T_1 |\alpha> = e^{i\alpha} |\alpha>$: $T_1(T_2)|\alpha> = e^{i2\theta}T_2T_1|\alpha> = e^{i2\theta}e^{i\alpha}(T_2)|\alpha>$ suppose that θ is a rational multiple of π: $\theta = \pi\, p/q$, then T_1 has q distinct eigenvalues (orbits: $\alpha + (2\pi p/q)k(\text{mod } 2\pi)$, where k=0,1,…,q-1) and the ground state degeneracy is q.

For genus g surface: **the ground state degeneracy is q^g**

Preskill, Lecture notes

Anyons

are excitations, quasiparticles, of a topological phase

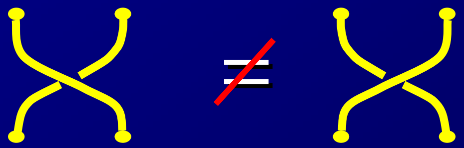
Configuration space of **n** indistinguishable particles in **d** dimensional space excluding diagonal points D:

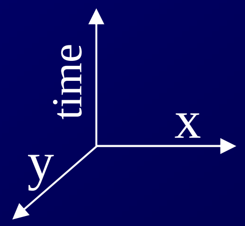
$$\boxed{\mathbf{M_n = (R^{nd} - D)/S_n}}$$

Leinaas and Myrheim'77
Wilczek'82

- in two spatial dimensions the configuration space is multiply connected

Exchanging particles on a plane is an element of braid group B_n:





One-dimensional irreps of B_n correspond to abelian fractional statistics:

$$\chi_\theta\ (\sigma) = e^{i\theta} \qquad \in\ U(1)$$

Higher dimensional irreps correspond to nonabelian fractional statistics:

$$\chi_\theta\ (\sigma) = e^{i\theta\Lambda} \qquad \text{e.g.} \in\ SU(2)$$

Example: Fibonacci anyons

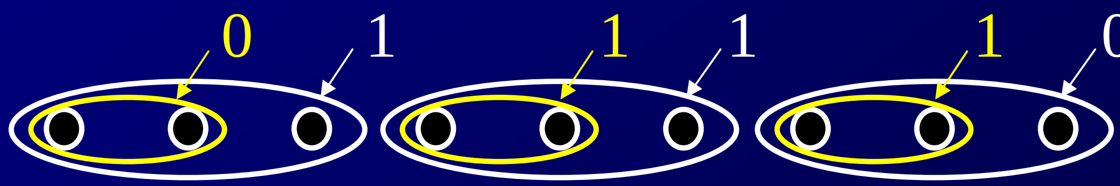
J. Preskill, Lecture notes

- are characterized by two possible values of “q-deformed” spin quantum number
  0 (trivial) and 1

- composition of q-spins is dictated by **fusion** rules (CFT):
  1 x 1 = 0 + 1
  0 x 0 = 0
  0 x 1 = 1

- Hilbert space dimension for the trivial sector grows with the number of anyons as the Fibonacci series: 0, 1, 1, 2, 3, 5, 8 …

- one logical qubit can be constructed with q-spin=1 anyons as follows (reminiscence of encoded universality)

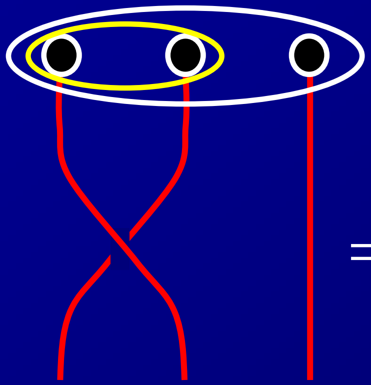


$|0>_L$ $|1>_L$ $|NC>$ = non-computational

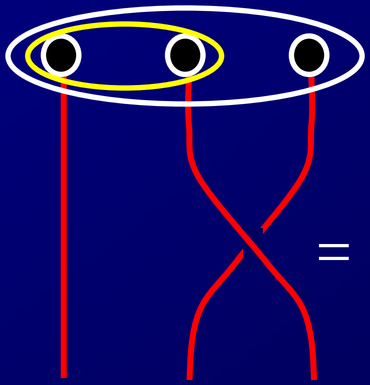
QC operations with Fibonacci anyons

are derived from fusion rules and consistency relations between braiding and fusion operations know as pentagon and hexagon equations (quantum groups), the result is:

single-qubit operations:



$$= \begin{pmatrix} e^{-i\pi/5} & 0 \\ 0 & -e^{-i2\pi/5} \end{pmatrix}$$



$$= \begin{pmatrix} -\tau e^{-i\pi/5} & -i\tau^{1/2}e^{-i\pi/10} \\ -i\tau^{1/2}e^{-i\pi/10} & -\tau \end{pmatrix}$$

two-qubit operations

- braiding between anyons of different logical qubits - requires optimization
- analogous to the concept of encoded universality

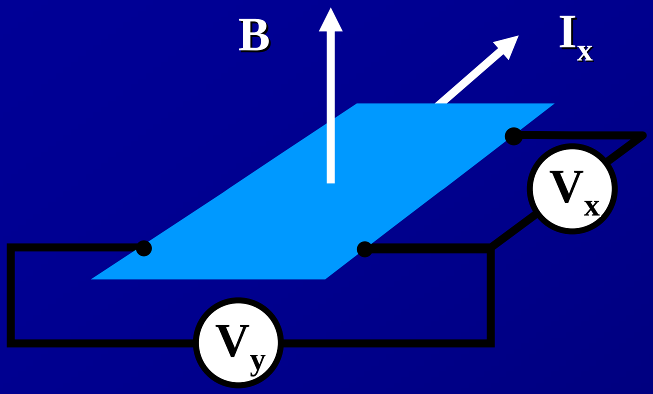
Bonesteel, et al.'05

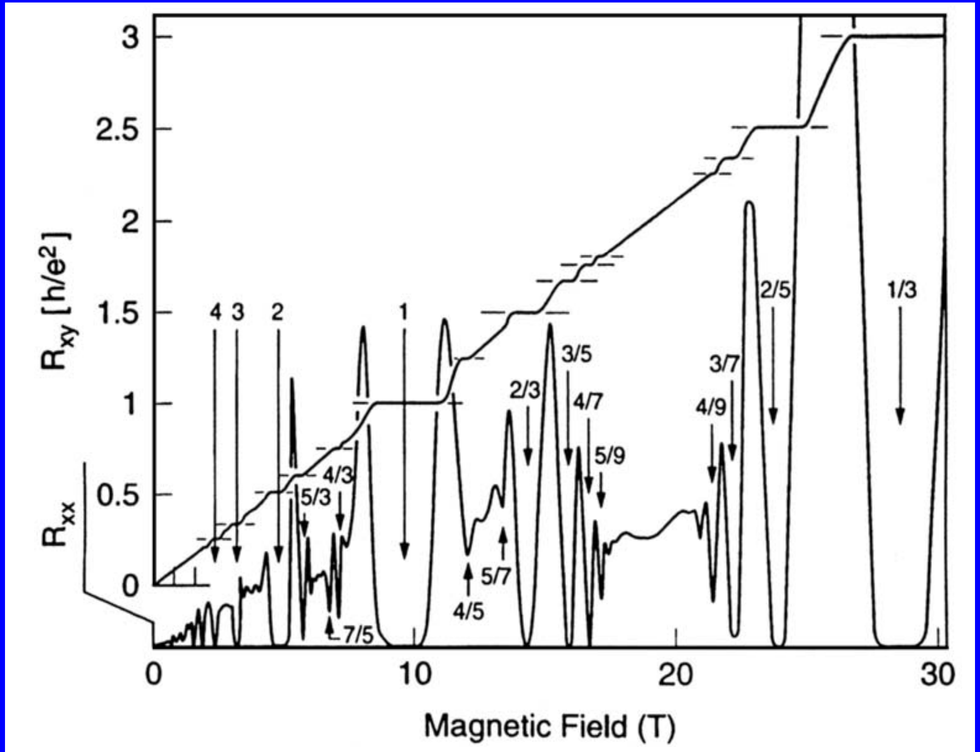
Topological phases in physical systems

- fractional quantum Hall systems — Das Sarma, et al., Phys. Rev. Lett. 94, 166802 (2005)
  - particularly promising !!!
- p_x+ip_y superconductors
  - Sr_2RuO_4 — Das Sarma, et al., Phys. Rev. B 73, 220502 (2006)
  - Helium-3 — Salomaa, Volovik, Rev. Mod. Phys. 59, 533 (1989)
- quantum lattice systems
  - atoms in optical lattices — Duan, et al., Phys. Rev. Lett. 91, 040902 (2003)
  - polar molecules — Micheli et al., Nature Phys. 2, 341 (2006)
  - Josephson-junction arrays — Ioffe et al., Nature 415, 503 (2002)
- rotating Bose-Einstein condensates
- nuclear matter

Fractional quantum Hall effect

Stormer, Tsui, Gossard, Phys. Rev. Lett. 48, 1559 (1982)
Rev. Mod. Phys. 71, S298 (1999)





Eisenstein, Stormer, Science 248, 1461 (1990)

Longitudinal resistance

$R_{xx} = V_x / I_x$

Transverse (Hall) resistance

$R_{xy} = V_y / I_x = h / \nu e^2$

Theory

nonabelian quantum Hall phases at ν=5/2 and 12/5

Read, Rezayi, Phys. Rev.B 59, 8084 (1999)

Experiment

detecting these phases in high mobility samples

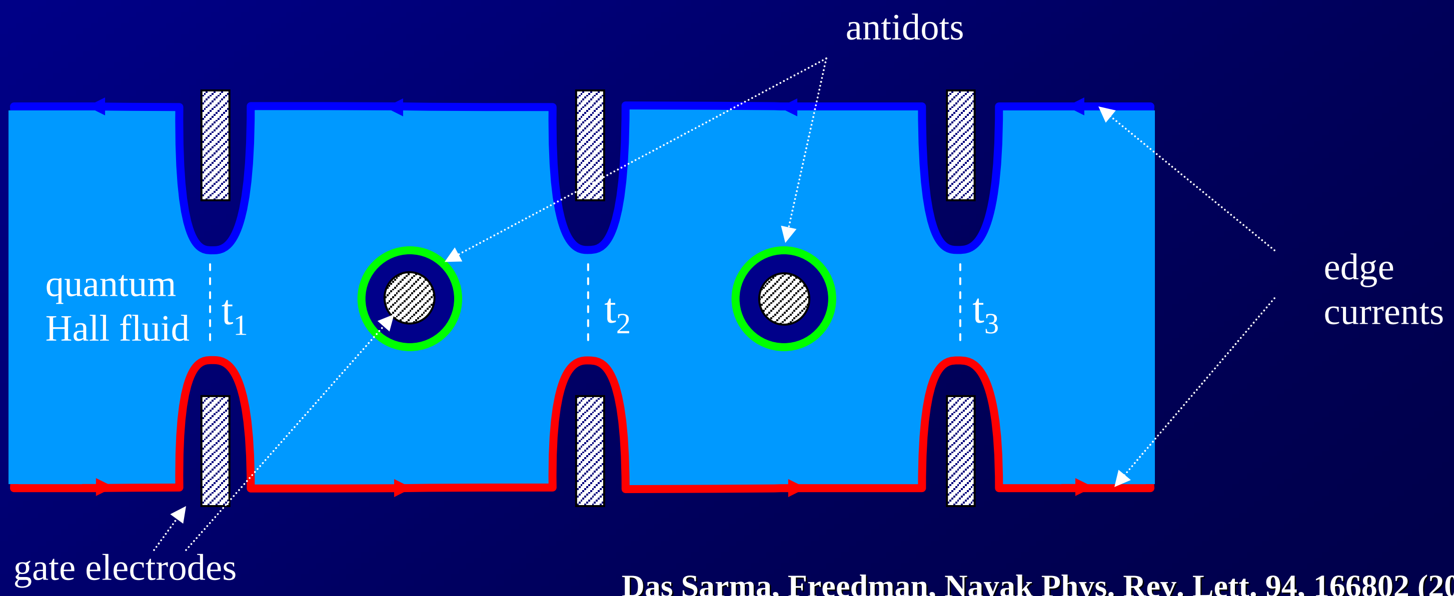
Xia et al., Phys. Rev. Lett. 93, 176809 (2004)

Fractional quantum Hall systems

- non-abelian topological phases predicted in FQH systems at the filling ν=5/2 and 12/5
- experimental tests of fractional statistics using Laughlin interferometer

Camino, Zhou, Goldman, Phys. Rev. B 72, 075342 (2005)

- relation between boundary (CFT) and bulk (TQFT) – holographic principle
- topologically protected qubit



Das Sarma, Freedman, Nayak Phys. Rev. Lett. 94, 166802 (2005)

Topological phases in quantum lattice systems

- Toric code (Kitaev)
  - abelian topological phase
  - quantum memory

Kitaev,
quant-ph/9707021
Ann. Phys. 303, 2 (2003)

- Kitaev honeycomb lattice model
  - abelian topological field in zero magnetic field
  - non-abelian topological phase in the presence of magnetic field
  - realizations proposed using atoms in optical lattices and polar molecules
  - graphene

Kitaev,
cond-mat/0506438

- Quantum loop gas model
  - abelian topological phases (k=1)
  - non-abelian topological phases (k=2)
  - concrete physical representation – extended Hubbard model

Freedman et al.,
cond-mat/0309120;
Phys. Rev. Lett.
94, 066401 (2005).

- Trivalent graph (spin-1) model
  - hierarchy of topological phases

Fendley, Fradkin,
Phys. Rev. B
72, 024412

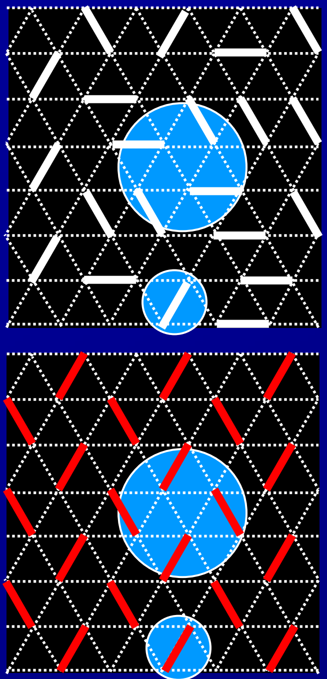
- String nets condensation
  - general hierarchy of topological phases

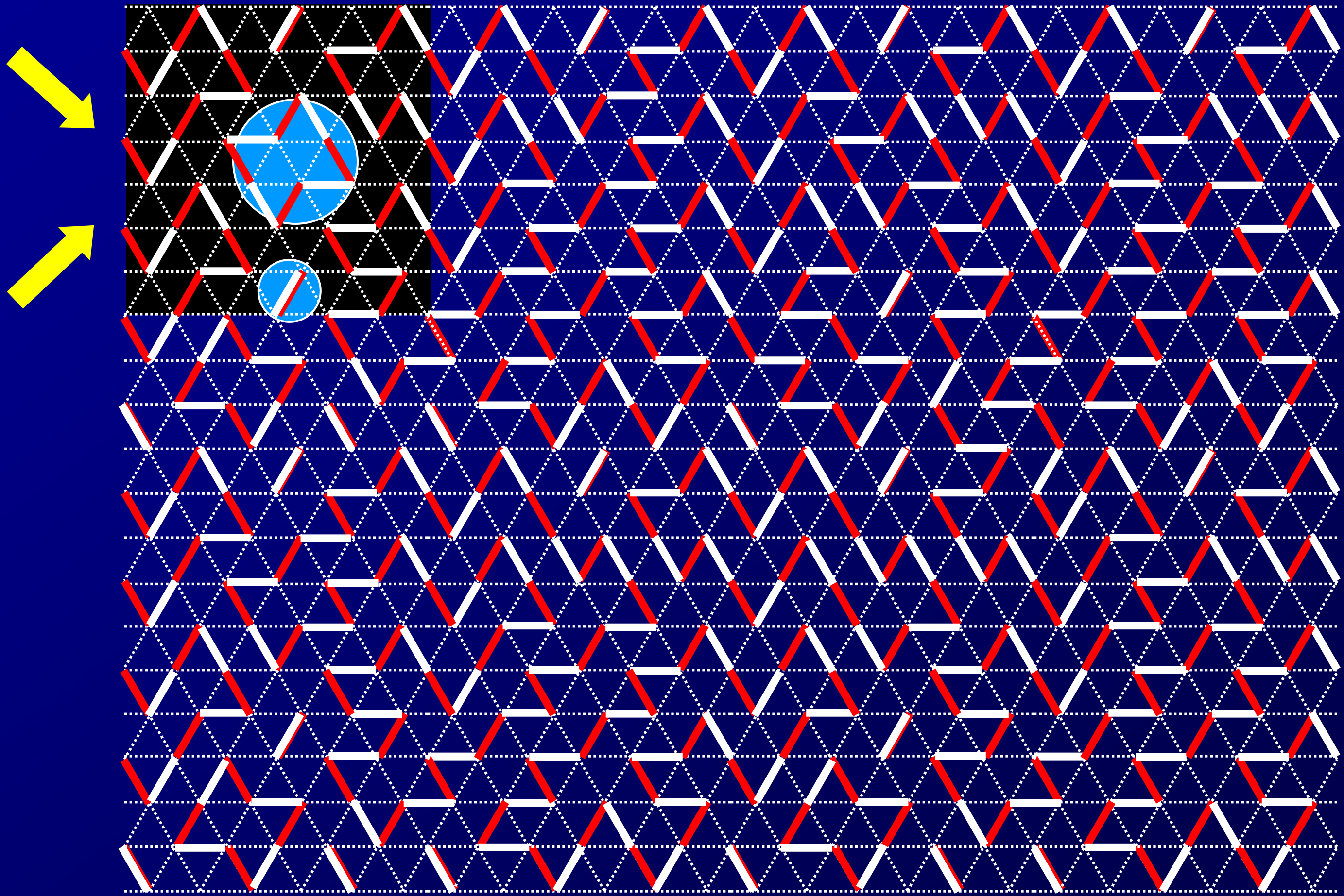
Levin, Wen,
Phys. Rev. B
71, 045110 (2005)

Quantum loop gas

Two-dimensional sea of fluctuating loops

formed for example by **dimers** on a quantum lattice with a fixed **background dimer covering**





Extended Hubbard model

- 1/6 filled Kagome lattice

$$H = U_0 \Sigma_i (n_i-1)n_i +$$

$$\Sigma_i \mu_i n_i +$$

$$U \underset{⬡}{\Sigma_{i,j}} n_i n_j +$$

$$\underset{\times}{\Sigma_{i,j}} V_{ij} n_i n_j +$$

$$\Sigma_{i,j} t_{ij} (c_i^+ c_j + c_j^+ c_i)$$

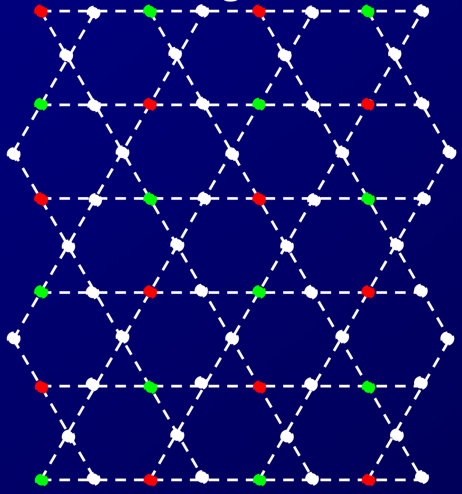
- U_0 is infinite, excluding doubly-occupied sites and hence preventing collisions

- potential terms are diagonal in the occupation basis, V and μ terms are color dependent

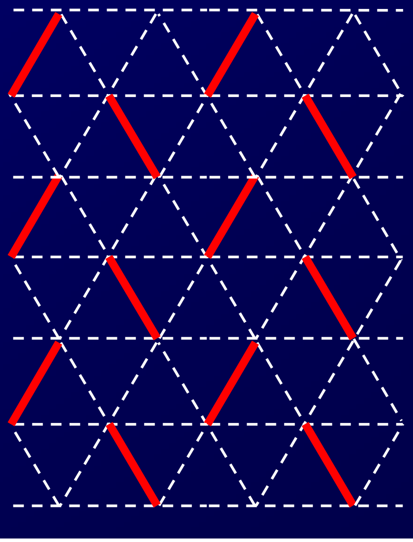
- tunneling between nearest neighbors, color dependent

- $U >> \mu_i, V_{ij}, t_{ij}$

Kagome lattice



Dual lattice

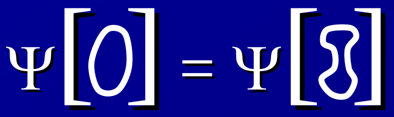


M. H. Freedman, C. Nayak, K. Shtengel, cond-mat/0309120; PRL 94, 066401 (2005).

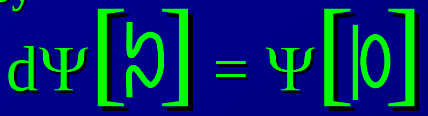
Topological phase in EHM

Topological conditions:

1. isotopy

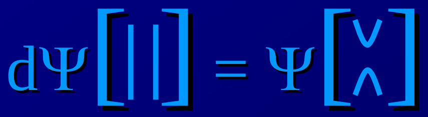


2. d-isotopy



➡ extensively degenerate ground state

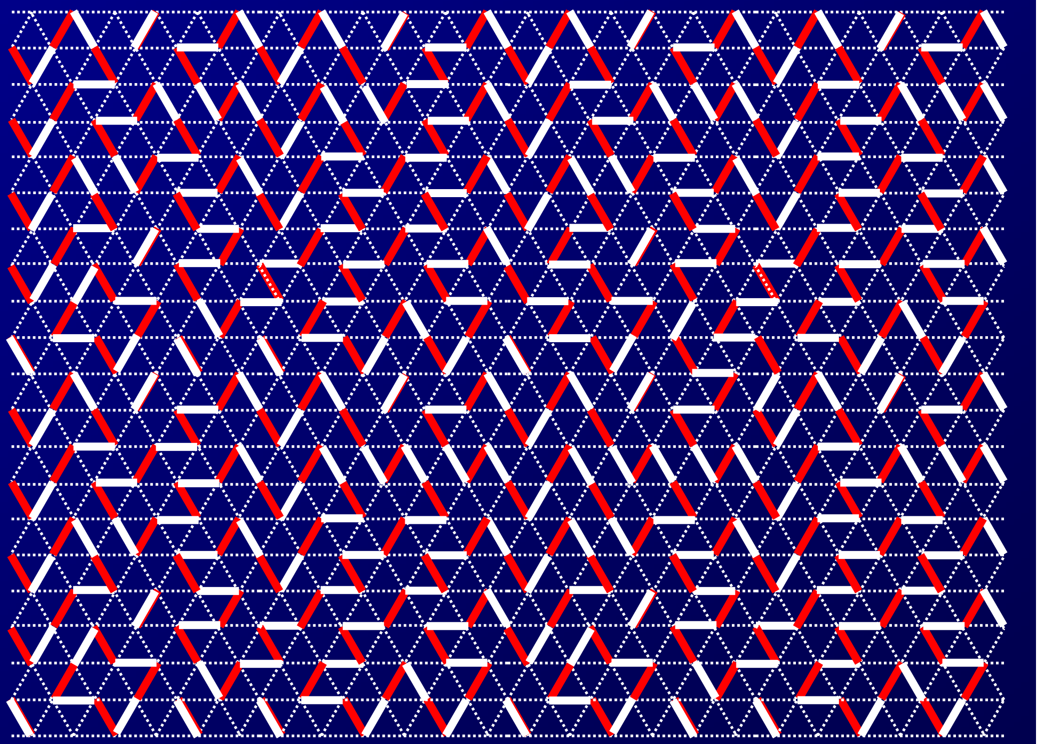
3. consistent surgery conditions – Jones-Wenzl idempotents, e.g



where $d=2\cos(\pi/(k+2))$

➡ **Topological phase**

quantum loop gas



M. H. Freedman, C. Nayak, K. Shtengel, cond-mat/0309120; PRL 94, 066401 (2005).

Challenges

Topological phases

- microscopic models which are
  - universal for quantum computation
  - based on local interactions
  - experimentally conceivable
- physical realization
  - quantum Hall systems, etc.
- classification

Topological quantum computing operations

- braiding
- measurement (Aharonov-Bohm-like experiment)

Algorithms

- approximation of certain statistical mechanical problems (e.g. Potts model)
- approximation of NP-complete problems
- graph theoretical problems

Complexity

- BQP-complete problems