NUI MAYNOOTH

MATHEMATICAL PHYSICS

science foundation ireland

Topological quantum computation

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Tutorial Presentation, Symposium on Quantum Technologies, University of Cambridge, August 29, 2006

Outline

Introduction

Topological quantum computation and algorithms

- Links and knots
- Jones polynomial
- Braid group
- Approximating Jones polynomial

Topological phases and natural fault-tolerance

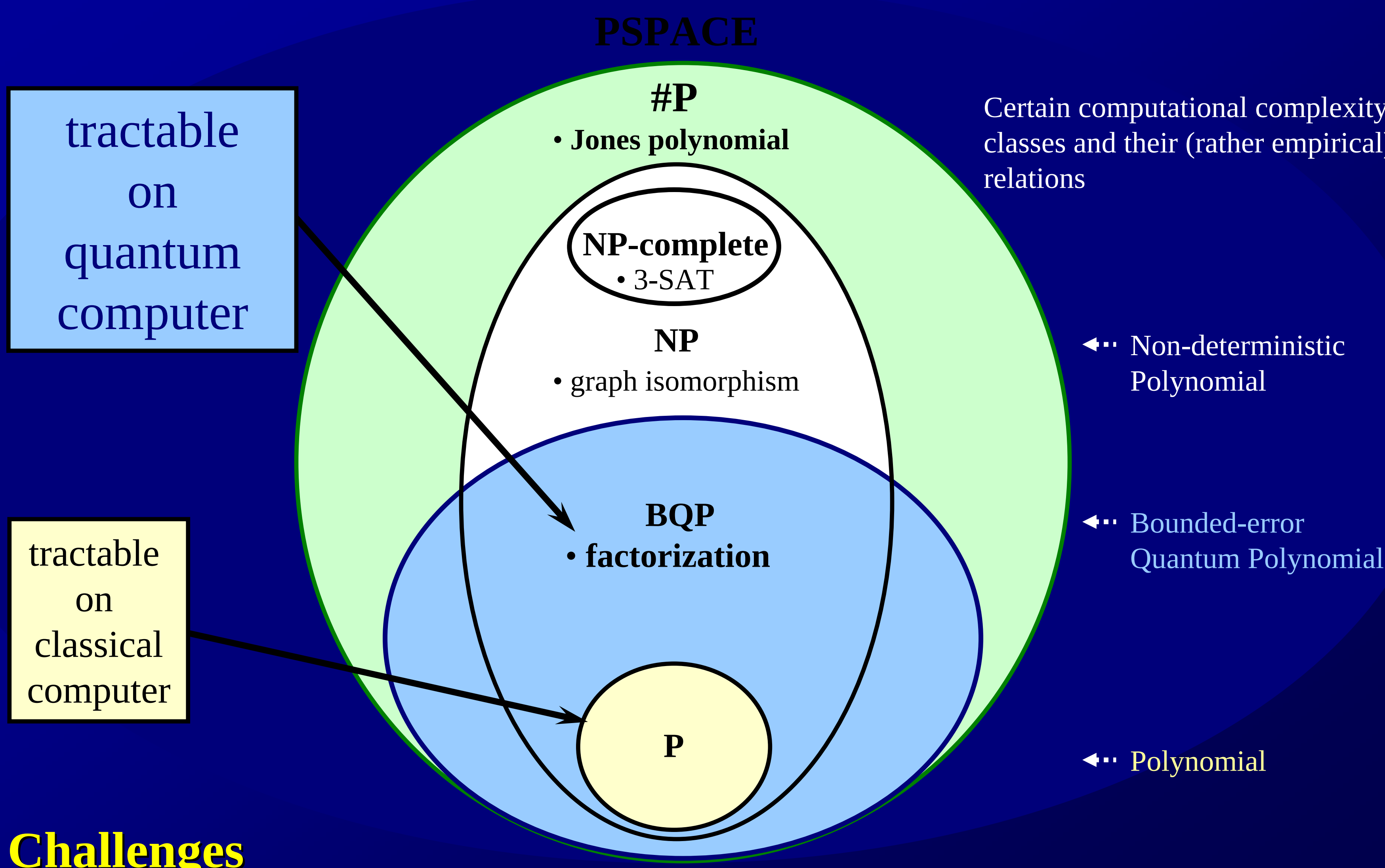
- Topological quantum field theory
- Spectral properties of topological phases

Anyons

Physical realization

- Fractional quantum Hall systems
- Lattice models of topological phases

Quantum computation



Certain computational complexity classes and their (rather empirical) relations

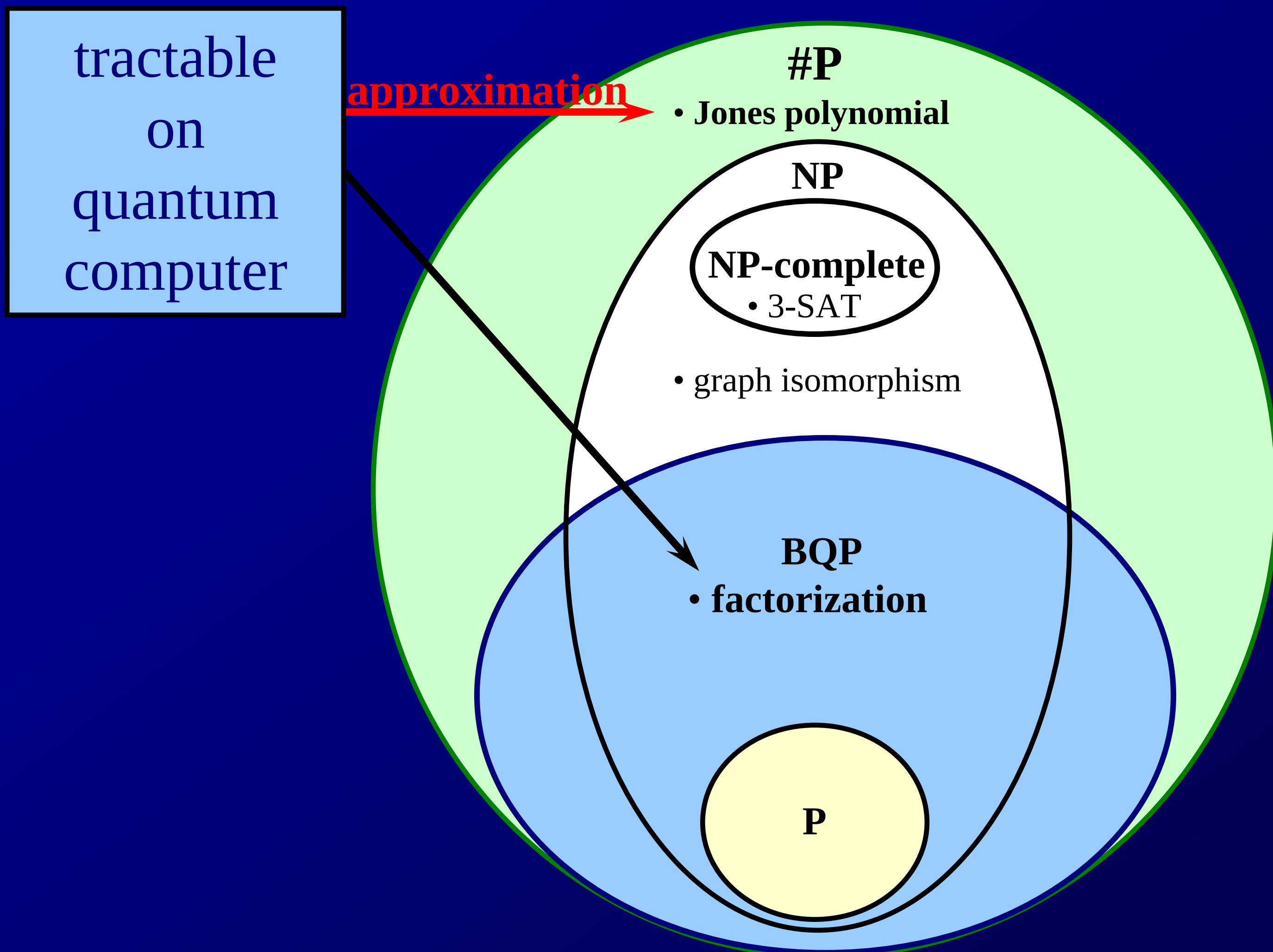
Challenges

Theory: algorithms, computational complexity, …

Implementations: fault-tolerance, …

Topological quantum computation

• is a unique QC model (though equivalent to standard QCM) => new algorithms
• natural fault tolerance



Freedman et al., Commun. Math. Phys. 227, 587 and 605 (2002)

Aharonov, Jones, Landau, STOC'06, quant-ph/0511069

Links and knots

Link

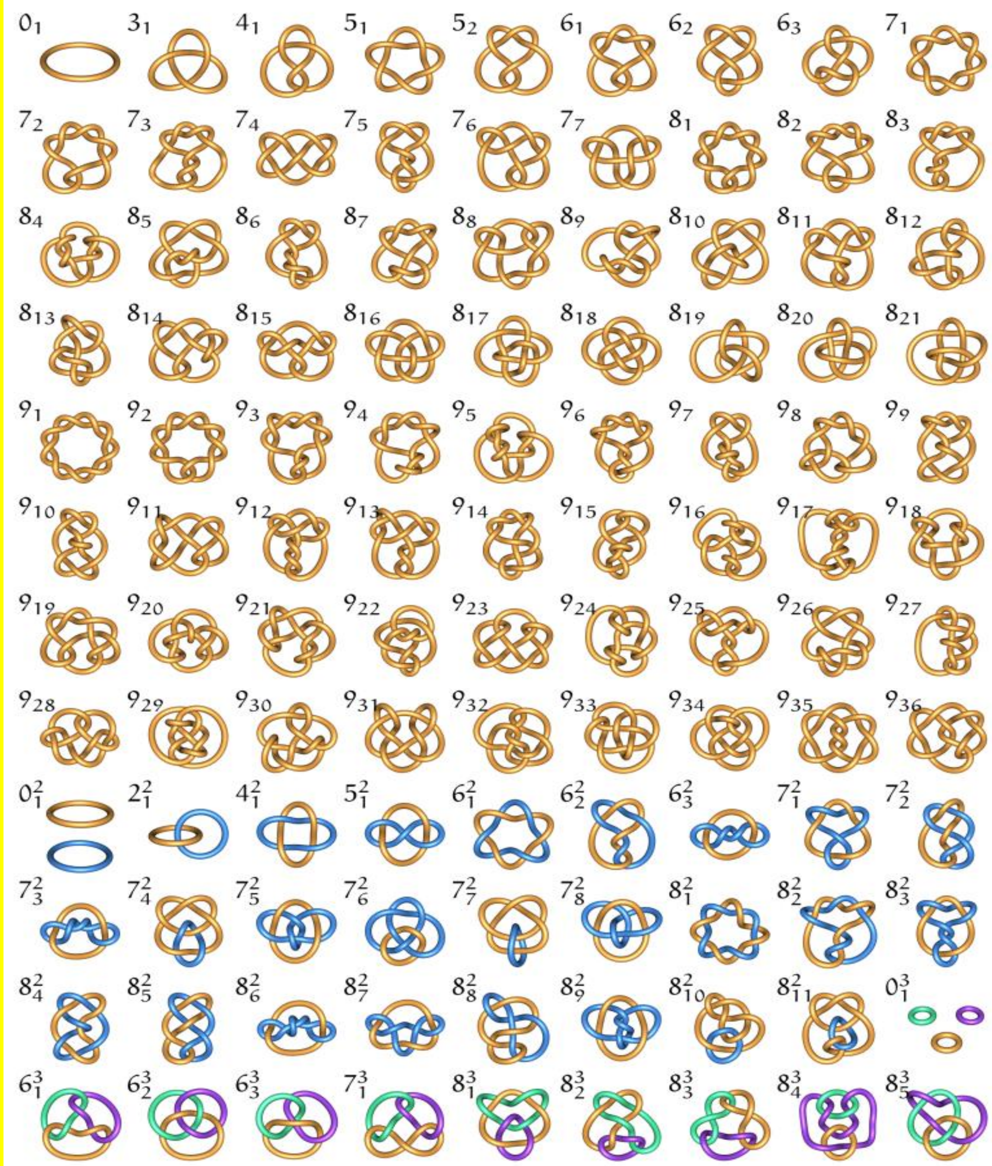
- a finite family of disjoint, smooth, closed curves in $\mathbb{R}^3$ or equivalently in S^3

Knot

- a link with one component

Knot complexity grows very fast:

# of crossings	# of knots
0	1
3	1
4	1
5	2
6	3
7	7
8	21
9	36



http://www.pims.math.ca/knotplot/zoo/

Jones polynomial

- Laurent polynomial in $t^{1/2}$, $V_L(t)$

Jones, Bull. AMS 12, 103 (1985)

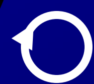
- invariant of links and knots under isotopy
- #P-hard combinatorial problem

  - classical algorithm is exponential in the number of crossings

skein relation

$t^{-1}\,V$ $-\ t\,V$ $= (t^{1/2} - t^{-1/2})\,V$

trivial knot

V $= 1$

Examples:

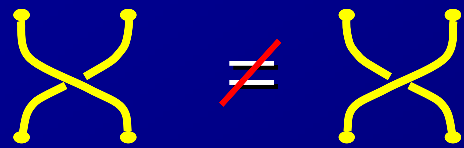
V $= -\,(t^{-1/2} + t^{1/2})$ "unlink"

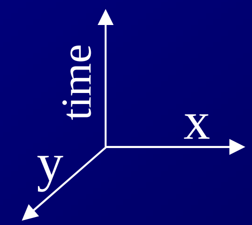
V $= t + t^{3} - t^{4}$ right hand trefoil knot

Braid group B_n

Artin, Ann. Math. 48, 101 (1947)

Exchanging particles on a plane is not permutation but braiding:

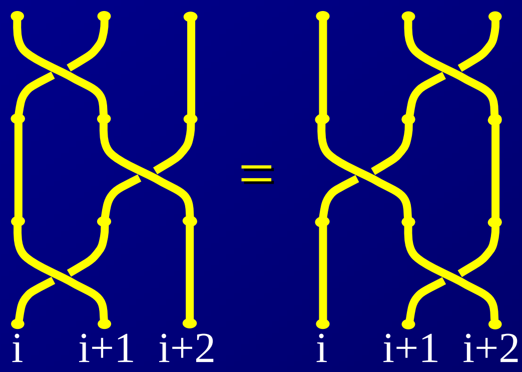
clockwise  counterclockwise



A braid group for n strands (particles) has n generators $\{1, \sigma_1, \ldots, \sigma_{n-1}\}$ which satisfy:

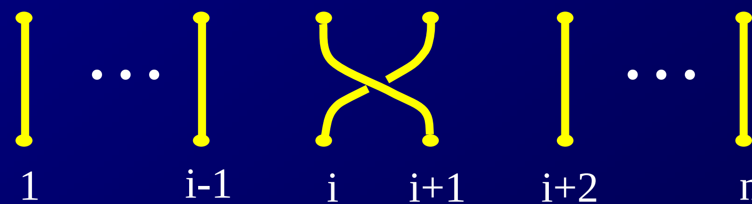
$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad \text{for } |j - i| > 1$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$



Yang-Baxter equation

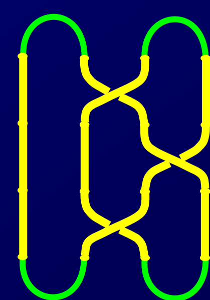
Example: σ_i



A braid group can be usefully represented in Temperley-Lieb algebra $TL_n(d)$ which permits a unitary representation (qubits!!!)

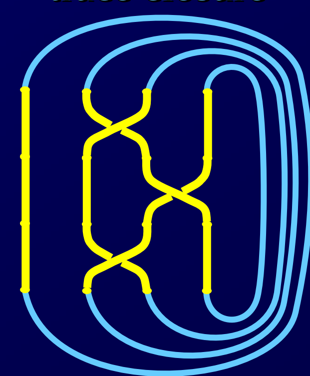
Closures of braids are links and knots

plat closure



trefoil knot

trace closure



Approximating Jones polynomial

Freedman et al., Commun. Math. Phys. 227, 587 and 605 (2002)
Aharonov, Jones, Landau, STOC'06, quant-ph/0511069

There is an efficient, explicit and simple quantum algorithm to approximate Jones polynomial for all $t = e^{2\pi i/k}$:

Theorem 1 — **the trace closure case**

For a given braid B with n strands and m crossings, and a given integer k, there is a quantum algorithm which is polynomial in n, m, k which with all but exponentially small probability, outputs a complex number r with $|r - V_{Btr}(e^{2\pi i/k})| < \varepsilon d^{n-1}$ where $d = -A^2 - A^{-2}$, and ε is inverse polynomial in n, m, k.

Theorem 2 — **the plat closure case**

For a given braid B with n strands and m crossings, and a given integer k, there is a quantum algorithm which is polynomial in n, m, k which with all but exponentially small probability, outputs a complex number r with $|r - V_{Bpl}(e^{2\pi i/k})| < \varepsilon d^{3n/2}/N$ where $d = -A^2 - A^{-2}$, and ε is inverse polynomial in n, m, k (N is an exponentially large factor).

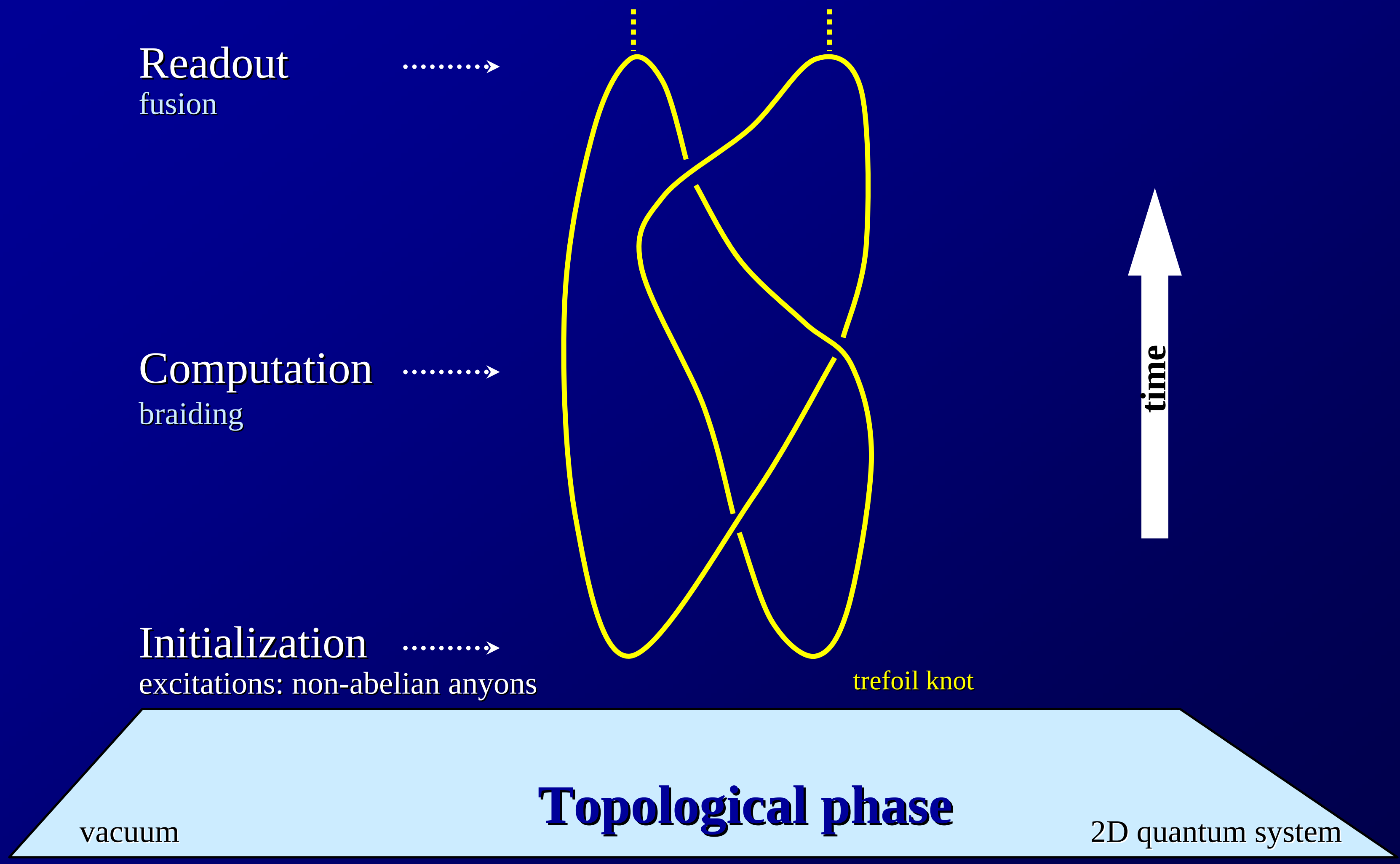
Theorem 3

Approximating the Jones polynomial of the plat closure (Th.2) is BQP-complete.

Aharonov, Arad, quant-ph/0605181
Wocjan, Yard, quant-ph/0603069

Topological quantum computation

- is a unique QC model (though equivalent to standard QCM) => new algorithms
- natural fault tolerance



Topological phases: effective theory

- topological phases are phases of two-dimensional many-body quantum systems whose properties depend only on topology of the manifold on whose surface a given phase is realized

- their effective description is given by topological quantum field theory (3 dimensional) defined e.g. by the Chern-Simons action:

Witten, Commun. Math. Phys. 121, 351 (1989)

$$S = k/4\pi \int_{\Gamma} dt\, d^2x\, \varepsilon^{\mu\nu\rho}\, a_\mu \partial_\nu a_\rho$$

level of theory (integer)   (2+1)D manifold   gauge field

no metric!!!

Example: doubled $SU(2)_k$ Chern-Simons theory (PT invariant theory):

Freedman,et al., CMP 227, 605 (2002)

$k = 1$	- abelian topological phase	- quantum memory
$k \geq 2$	- non-abelian	
$k = 3, 5 \ldots$	- non-abelian and universal	- universal QC

- topological phases are invariant with local geometry and hence quantum information stored in them is invariant with local error processes

no metric, no error!!!

Topological phases: Hamiltonian spectrum

- are ground states of certain strongly correlated many-body quantum systems

  e.g. in Coulomb gauge, $a_0 = 0$: $\mathcal{L} = a_2 \partial_0 a_1 - a_1 \partial_0 a_2$

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial(\partial_0 a_1)} \partial_0 a_1 + \frac{\partial \mathcal{L}}{\partial(\partial_0 a_2)} \partial_0 a_2 - \mathcal{L} = \mathbf{0}$$

no metric, no energy!!!

- energy spectrum of matter in a topological phase is characterized by

finite topology-dependent ground state degeneracy,

e.g. for the doubled $SU(2)_k$ Chern-Simons theory: $(k+1)^{2g}$

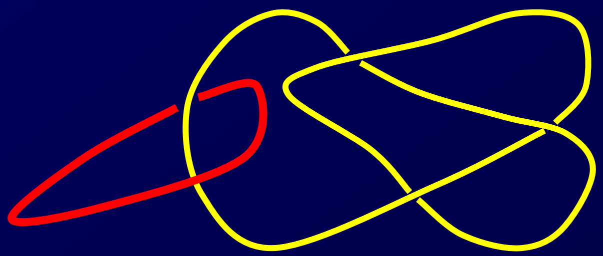
genus

Freedman et al. Ann. Phys. 310, 428 (2004)

$\Downarrow$

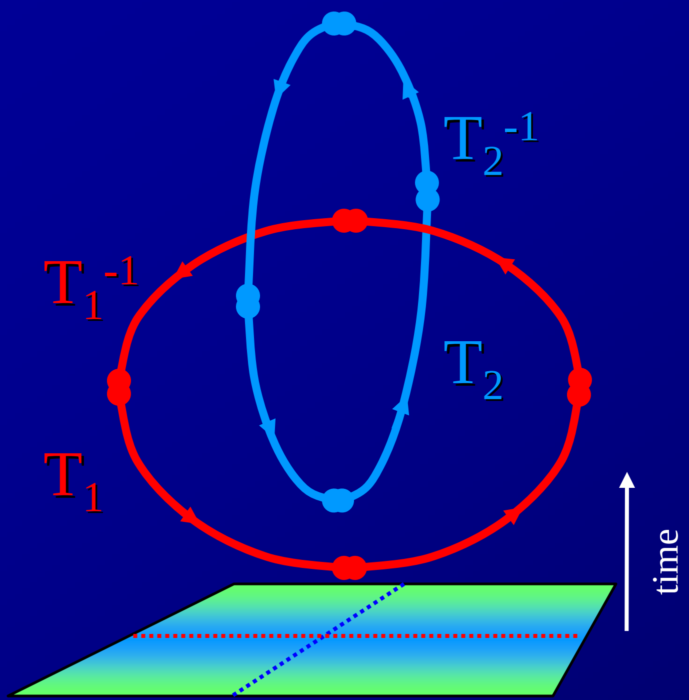
spectral gap

- excitations of stray anyons, which may cause errors via non-local processes, are exponentially suppressed due to the spectral gap !!!

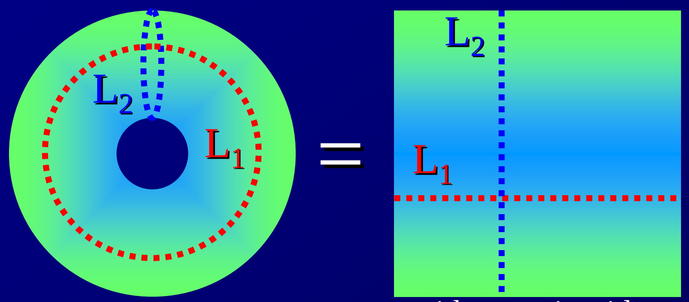


Topological phases: ground state degeneracy

Example: abelian anyons on torus



torus (genus g=1 surface)



with opposite sides identified

one anyon winds clockwise around the other:

$$T_2^{-1}T_1^{-1}T_2T_1 = e^{-i2\theta}\,\mathbb{1}$$

$[T_1, H] = 0$ and $[T_2, H] = 0$, so $T_1\,|\alpha\rangle = e^{i\alpha}\,|\alpha\rangle$: $T_1(T_2)|\alpha\rangle = e^{i2\theta}T_2T_1|\alpha\rangle = e^{i2\theta}e^{i\alpha}(T_2)|\alpha\rangle$
suppose that θ is a rational multiple of π: $\theta = \pi\, p/q$, then T_1 has q distinct eigenvalues (orbits: $\alpha + (2\pi p/q)k (\mathrm{mod}\ 2\pi)$, where k=0,1,…,q-1) and the ground state degeneracy is q.

For genus g surface: **the ground state degeneracy is q^g**

Preskill, Lecture notes

Anyons

are excitations, quasiparticles, of a topological phase

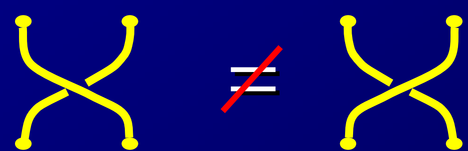
Configuration space of **n** indistinguishable particles in **d** dimensional space excluding diagonal points D:

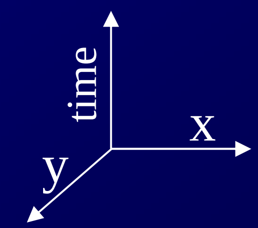
$$\boxed{M_n = (R^{nd} - D)/S_n}$$

Leinaas and Myrheim'77
Wilczek'82

- in two spatial dimensions the configuration space is multiply connected

Exchanging particles on a plane is an element of braid group B_n:





One-dimensional irreps of B_n correspond to abelian fractional statistics:

$$\chi_\theta\ (\sigma) = e^{i\theta} \qquad \in U(1)$$

Higher dimensional irreps correspond to nonabelian fractional statistics:

$$\chi_\theta\ (\sigma) = e^{i\theta\Lambda} \qquad \text{e.g.} \in SU(2)$$

Example: Fibonacci anyons

J. Preskill, Lecture notes

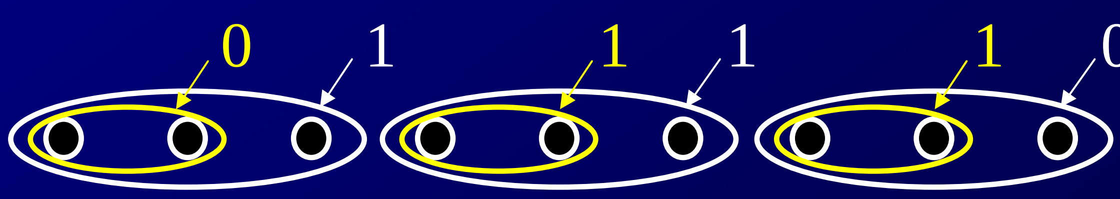
• are characterized by two possible values of "q-deformed" spin quantum number
0 (trivial) and 1

• composition of q-spins is dictated by **fusion** rules (CFT):

$$1 \times 1 = 0 + 1$$
$$0 \times 0 = 0$$
$$0 \times 1 = 1$$

• Hilbert space dimension for the trivial sector grows with the number of anyons as the Fibonacci series: 0, 1, 1, 2, 3, 5, 8 …

• one logical qubit can be constructed with q-spin=1 anyons as follows (reminiscence of encoded universality)

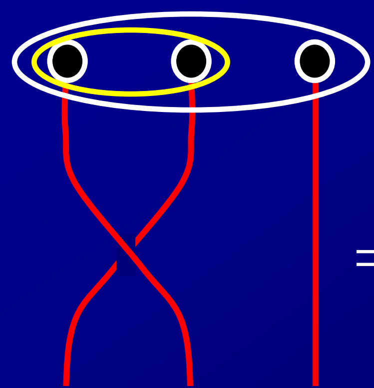


$|0>_L$ $|1>_L$ $|NC>$ = non-computational

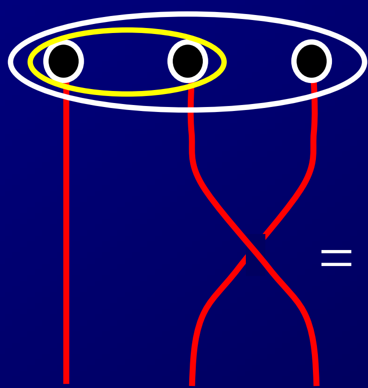
QC operations with Fibonacci anyons

are derived from fusion rules and consistency relations between braiding and fusion operations know as pentagon and hexagon equations (quantum groups), the result is:

single-qubit operations:



$$= \begin{pmatrix} e^{-i\pi/5} & 0 \\ 0 & -e^{-i2\pi/5} \end{pmatrix}$$



$$= \begin{pmatrix} -\tau e^{-i\pi/5} & -i\tau^{1/2} e^{-i\pi/10} \\ -i\tau^{1/2} e^{-i\pi/10} & -\tau \end{pmatrix}$$

two-qubit operations

- braiding between anyons of different logical qubits - requires optimization
- analogous to the concept of encoded universality

Bonesteel, et al.'05

Topological phases in physical systems

- fractional quantum Hall systems
  particularly promising !!!

Das Sarma, et al., Phys. Rev. Lett. 94, 166802 (2005)

- p_x+ip_y superconductors
  Sr_2RuO_4
  Helium-3

Das Sarma, et al., Phys. Rev. B 73, 220502 (2006)

Salomaa, Volovik, Rev. Mod. Phys. 59, 533 (1989)

- quantum lattice systems
  atoms in optical lattices
  polar molecules
  Josephson-junction arrays

Duan, et al., Phys. Rev. Lett. 91, 040902 (2003)

Micheli et al., Nature Phys. 2, 341 (2006)

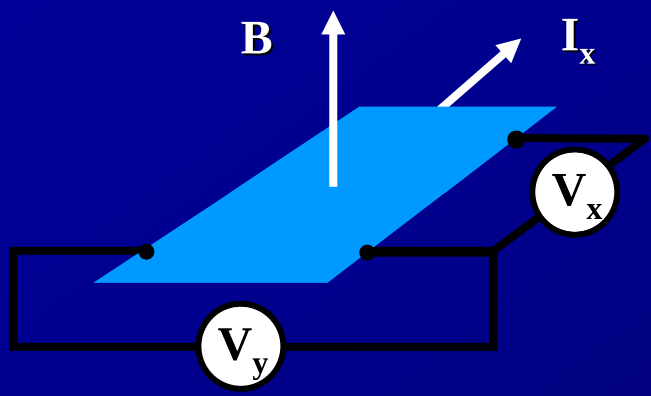
Ioffe et al., Nature 415, 503 (2002)

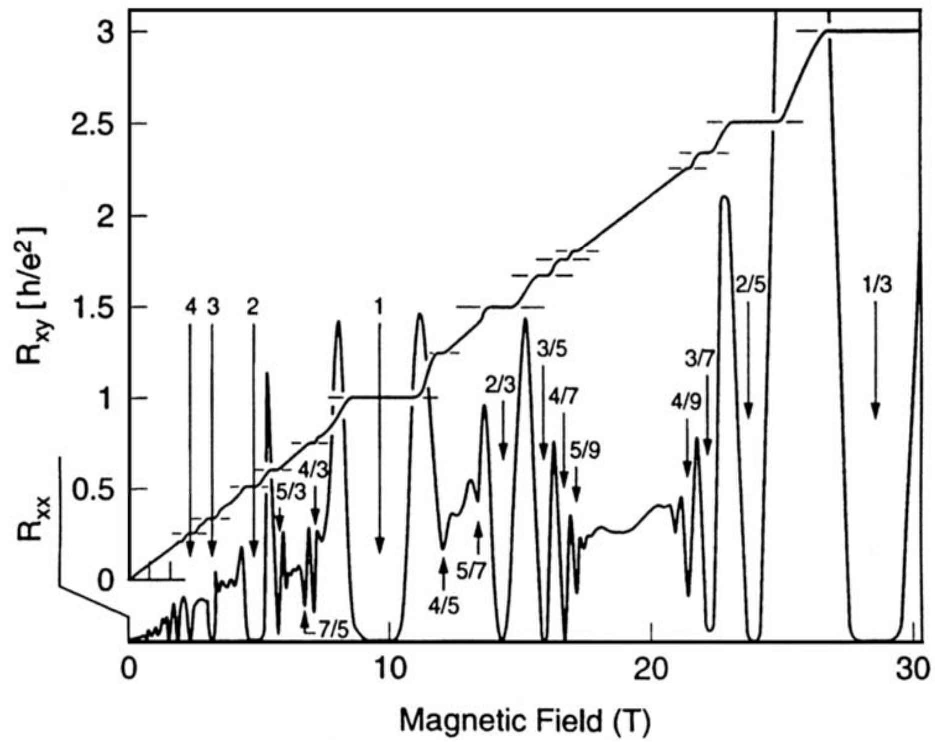
- rotating Bose-Einstein condensates

- nuclear matter

Fractional quantum Hall effect

Stormer, Tsui, Gossard, Phys. Rev. Lett. 48, 1559 (1982)
Rev. Mod. Phys. 71, S298 (1999)





Eisenstein, Stormer, Science 248, 1461 (1990)

Longitudinal resistance
$R_{xx} = V_x / I_x$

Transverse (Hall) resistance
$R_{xy} = V_y / I_x = h / \nu e^2$

Theory

nonabelian quantum Hall phases at ν=5/2 and 12/5

Read, Rezayi, Phys. Rev.B 59, 8084 (1999)

Experiment

detecting these phases in high mobility samples

Xia et al., Phys. Rev. Lett. 93, 176809 (2004)

Fractional quantum Hall systems

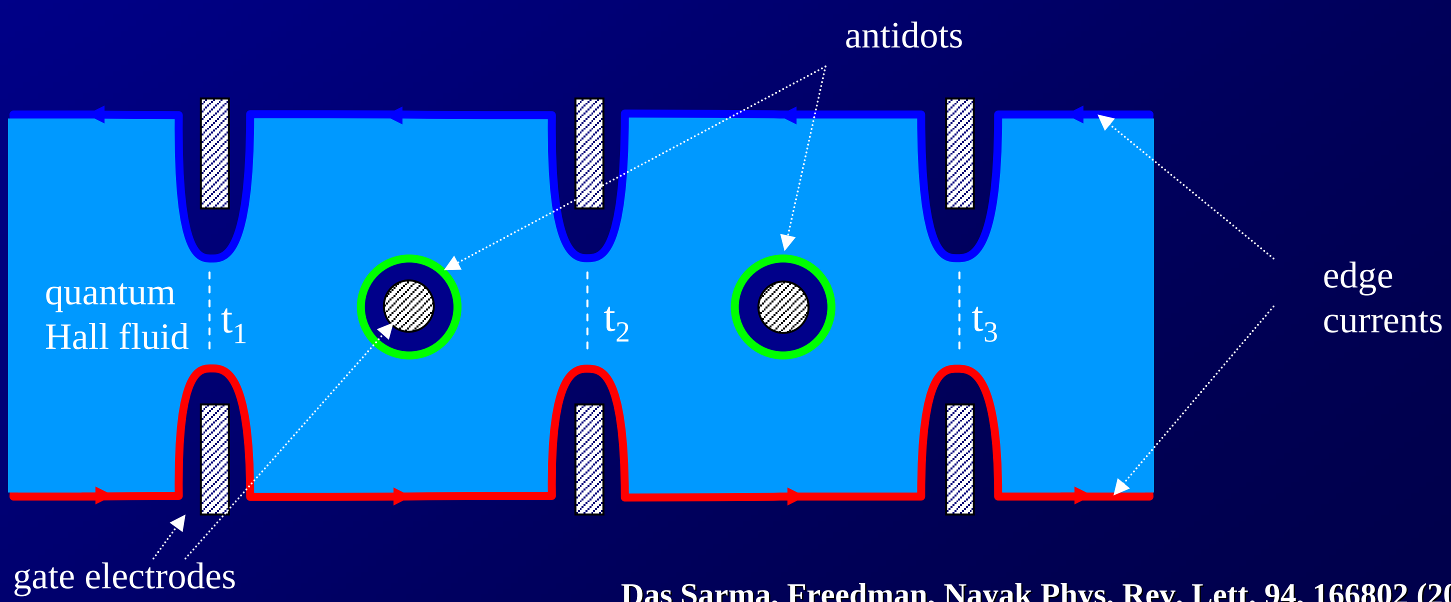
- non-abelian topological phases predicted in FQH systems at the filling ν=5/2 and 12/5

- experimental tests of fractional statistics using Laughlin interferometer

Camino, Zhou, Goldman, Phys. Rev. B 72, 075342 (2005)

- relation between boundary (CFT) and bulk (TQFT) – holographic principle

- topologically protected qubit



Das Sarma, Freedman, Nayak Phys. Rev. Lett. 94, 166802 (2005)

Topological phases in quantum lattice systems

- Toric code (Kitaev)
  abelian topological phase
  quantum memory

Kitaev, quant-ph/9707021 Ann. Phys. 303, 2 (2003)

- Kitaev honeycomb lattice model
  abelian topological field in zero magnetic field
  non-abelian topological phase in the presence of magnetic field
  realizations proposed using atoms in optical lattices and polar molecules
  graphene

Kitaev, cond-mat/0506438

- Quantum loop gas model
  abelian topological phases (k=1)
  non-abelian topological phases (k=2)
  concrete physical representation – extended Hubbard model

Freedman et al., cond-mat/0309120; Phys. Rev. Lett. 94, 066401 (2005).

- Trivalent graph (spin-1) model
  hierarchy of topological phases

Fendley, Fradkin, Phys. Rev. B 72, 024412

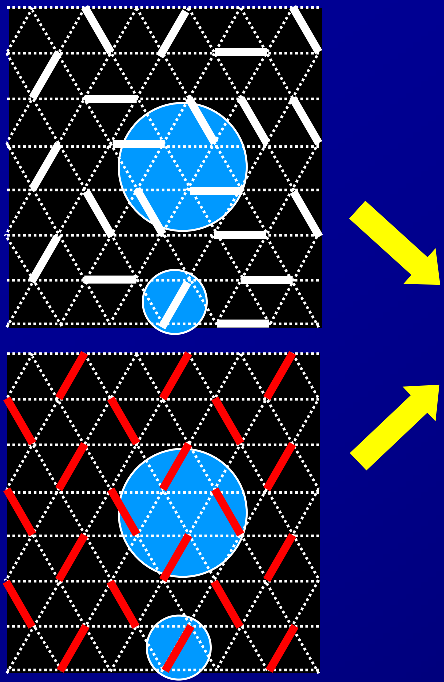
- String nets condensation
  general hierarchy of topological phases

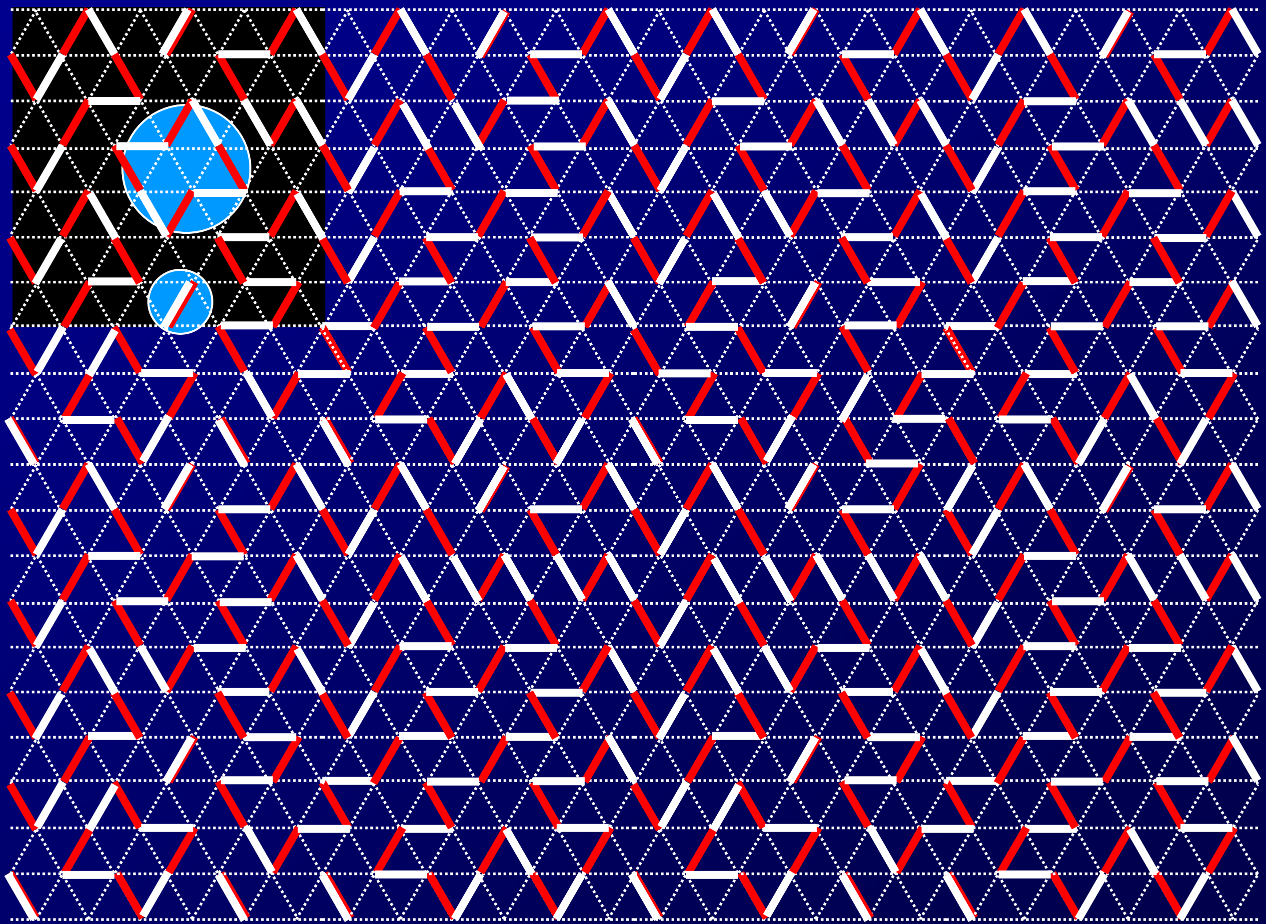
Levin, Wen, Phys. Rev. B 71, 045110 (2005)

Quantum loop gas

Two-dimensional sea of fluctuating loops

formed for example by **dimers** on a quantum lattice with a fixed **background dimer covering**





Extended Hubbard model

- 1/6 filled Kagome lattice

$$H = U_0 \Sigma_i (n_i - 1) n_i +$$

$$\Sigma_i \mu_i n_i +$$

$$U \Sigma_{\substack{i,j \\ \hexagon}} n_i n_j +$$

$$\Sigma_{\substack{i,j \\ \times}} V_{ij} n_i n_j +$$

$$\Sigma_{i,j} t_{ij} (c_i^+ c_j + c_j^+ c_i)$$

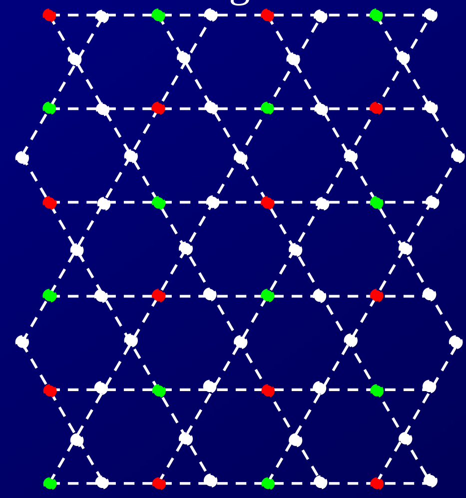
- U_0 is infinite, excluding doubly-occupied sites and hence preventing collisions

- potential terms are diagonal in the occupation basis, V and μ terms are color dependent

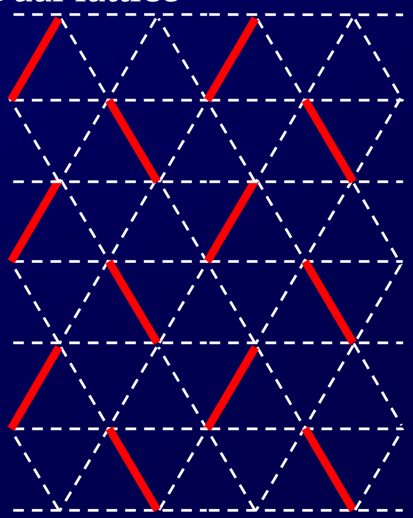
- tunneling between nearest neighbors, color dependent

- $U >> \mu_i, V_{ij}, t_{ij}$

Kagome lattice



Dual lattice

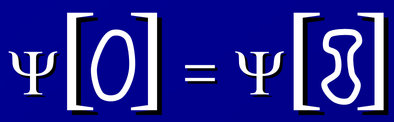


M. H. Freedman, C. Nayak, K. Shtengel, cond-mat/0309120; PRL 94, 066401 (2005).

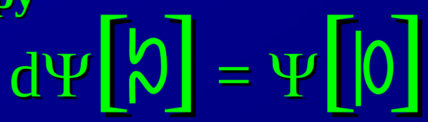
Topological phase in EHM

Topological conditions:

1. isotopy

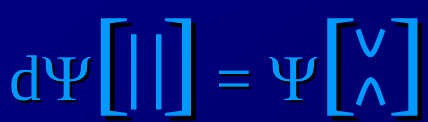


2. d-isotopy



➡ extensively degenerate ground state

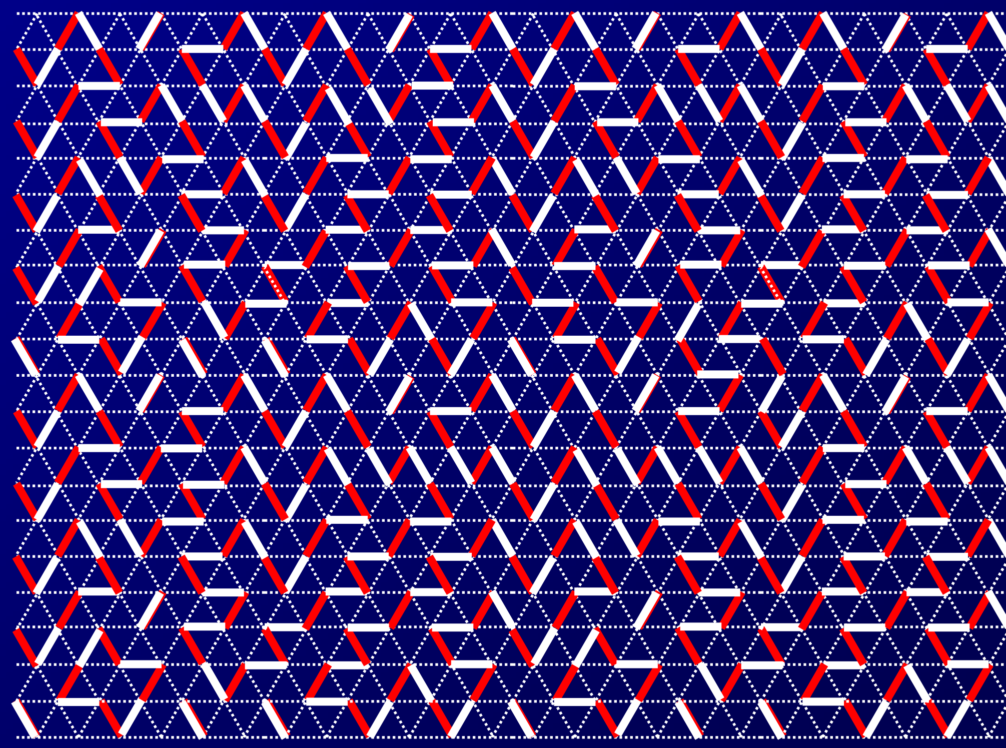
3. consistent surgery conditions –
Jones-Wenzl idempotents, e.g



where $d=2\cos(\pi/(k+2))$

➡ **Topological phase**

quantum loop gas



M. H. Freedman, C. Nayak, K. Shtengel,
cond-mat/0309120; PRL 94, 066401 (2005).

Challenges

Topological phases
- microscopic models which are
  - universal for quantum computation
  - based on local interactions
  - experimentally conceivable
- physical realization
  - quantum Hall systems, etc.
- classification

Topological quantum computing operations
- braiding
- measurement (Aharonov-Bohm-like experiment)

Algorithms
- approximation of certain statistical mechanical problems (e.g. Potts model)
- approximation of NP-complete problems
- graph theoretical problems

Complexity
- BQP-complete problems