Topology in quantum states: Many-body ansätze

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Contents

1. Topology and quantum computation

- Use topology to protect QI from local errors.

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- Proposals use topological phases of matter:

— (Non-Abelian) Anyonic excitations.

— Ground level degeneracy depends on topology of space.

— This degeneracy unrelated to spontaneous local group symmetry breakdown.

— Governed by TQFT, quantum group symmetries

(Levin + Wen, PR **B71** (2005) 045110.)

— Topological entanglement entropy S^{top} (see later.)

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  Gates performed by braiding of anyons.

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- FQHE $\nu = 5/2$, $\nu = 12/5$: non-Abelian anyonic excitations?

- $|GS(\nu = 5/2)\rangle \sim$ Moore-Read Pfaffian Ψ:

  LL excitations $\sim \mathrm{SU}(2)_{k=2}$ CS effective theory

  (but see Tőke + Jain, PRL **96** (2006) 246805.)

  Cannot achieve universal QC by pure braiding

  (but see Freedman et al., PR **B73** (2006) 245307.)

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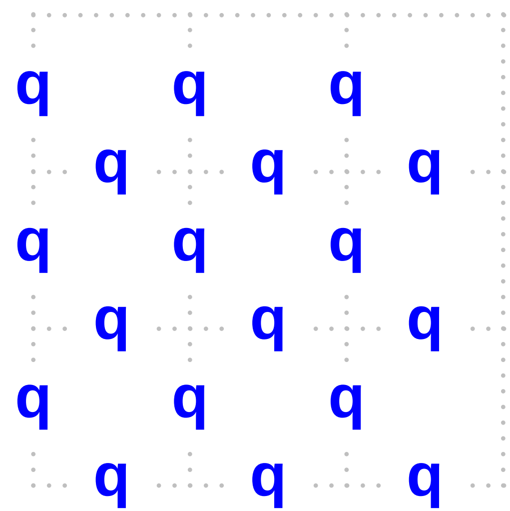

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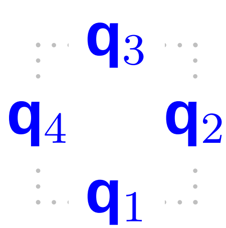
  Plaquette constraints (even parity).



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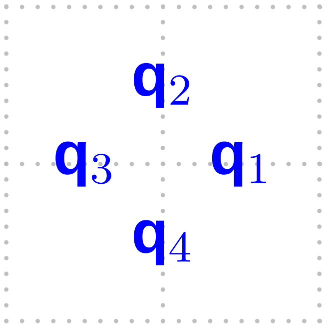

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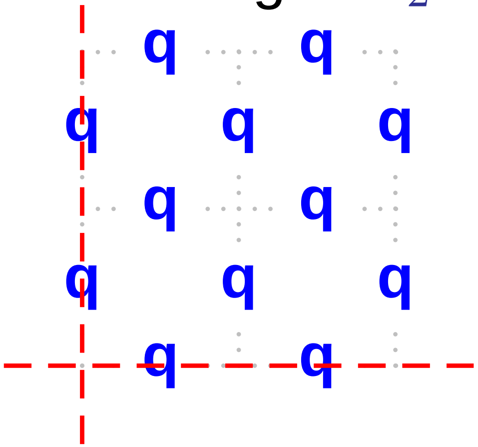

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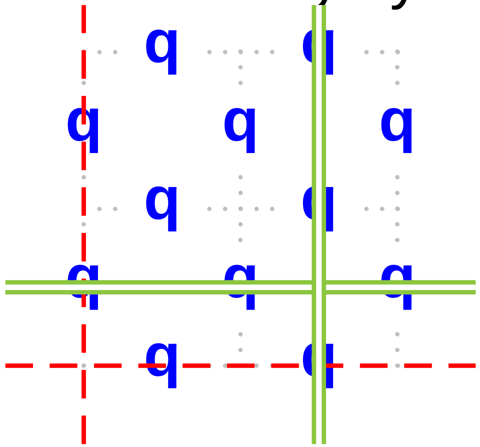

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- Reinterpretation of $\mathbb{Z}_2$ gauge theory (Wegner, JMP **12** (1971) 2259.)

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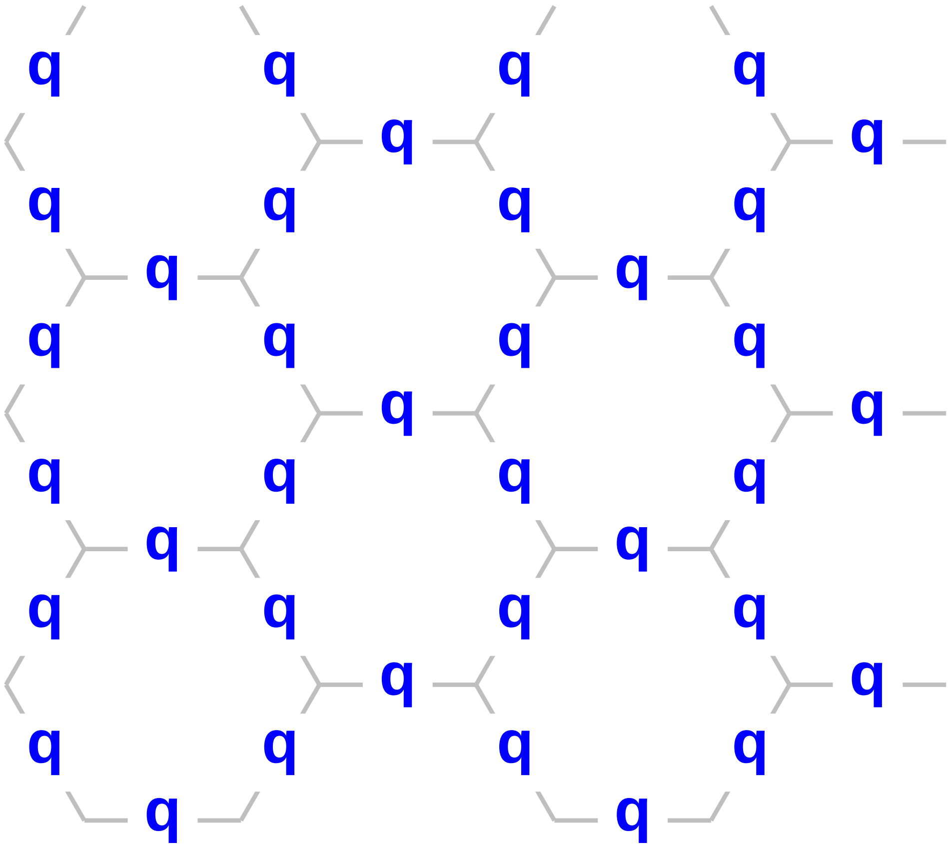

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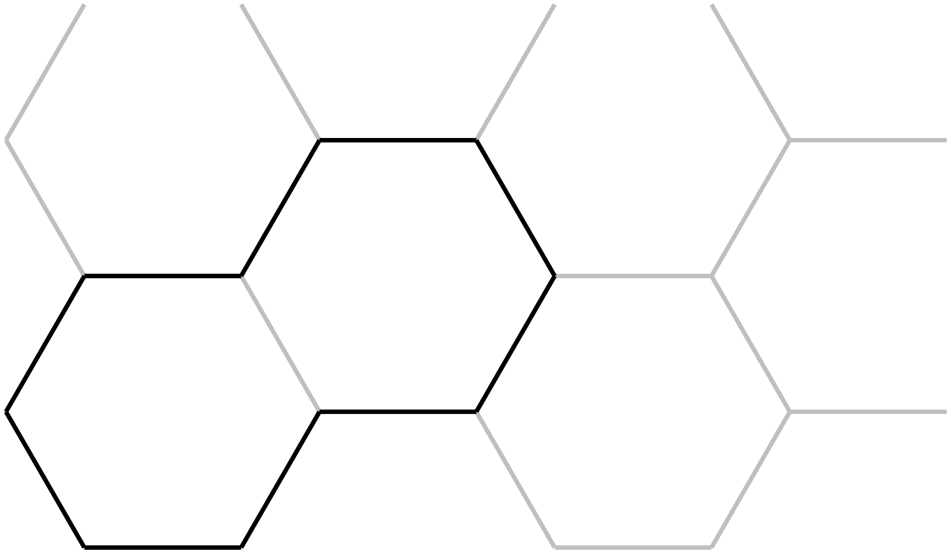

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- Place qubits at edges of honeycomb lattice.
- Dual $\mathbb{Z}_2$ gauge theory $\equiv$ loop gas model (mark $|1\rangle$'s.)
- Can compute topological entropy easily

  (Kitaev + Preskill, PRL **96** (2006) 110404.)

$$S_L^{2d}(\rho) = -\mathrm{Tr}_{N-L}(\rho \log_2 \rho) \sim \alpha L - \gamma + \cdots$$

- Nonzero $S^{\mathrm{top}} \stackrel{\mathrm{def}}{=} -\gamma$: characterisation of topological order.

  Relation to holography (Fendley et al., `cond-mat/0609072`.)

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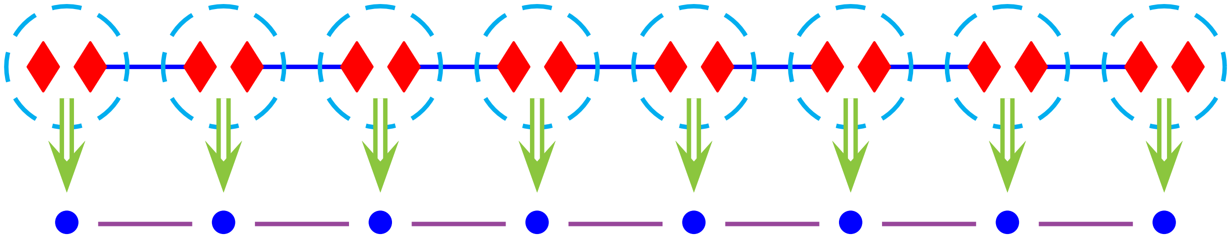

$$|\Psi\rangle = \sum_{i_1,\ldots,i_N} \mathrm{Tr}\left(A^{i_1}\cdots A^{i_N}\right)|i_1\cdots i_N\rangle$$

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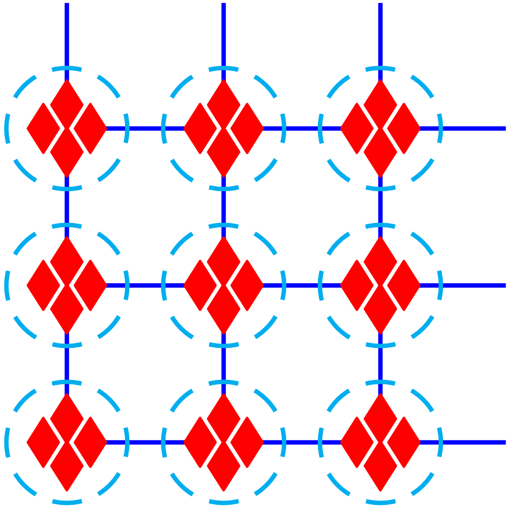
  Project onto d-dim physical site spaces with projectors $A^{i}_{\alpha\beta}$
- Related to Density Matrix Renormalisation Group (White '92)
  Algorithm to compute GS energies and correlations.
  White's truncation: keep as much entanglement as possible!
  $\sim$ Variational procedure over the set of MPS.
  Successful for 1d noncritical systems.

2. MPS, PEPS, and topology

- Projected entangled-pair states (PEPS): Ansatz for $d \geq 2$.

  (Verstraete + Cirac, Phys. Rev. **A70** (2004) 060302(R).)

  Generalise MPS construction to higher dimensions.



$$|\Psi\rangle = \sum_{i_\mathcal{R}} \mathcal{C}_\mathcal{R}(\{A^{i_\mathcal{R}}\})|i_\mathcal{R}\rangle$$

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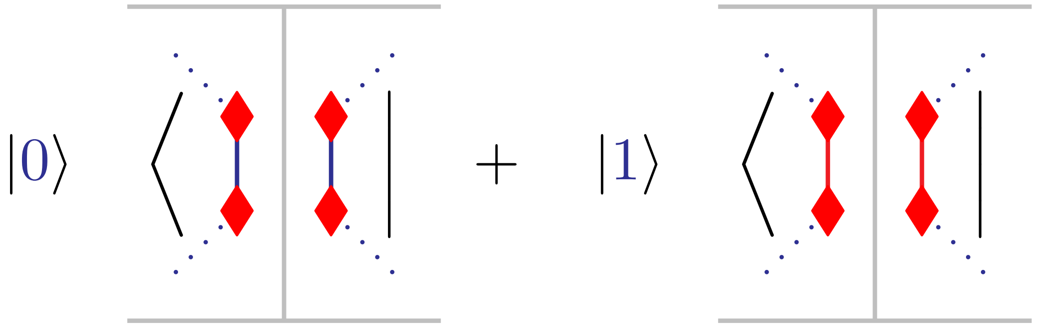
- The toric code as a PEPS: Verstraete et al., PRL **96** (2006) 220601.

  Projectors ($= |00\rangle + |11\rangle$, $= |00\rangle - |11\rangle$):



(rotated for horizontal bonds.)

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- Essentially the same construction applies to Levin + Wen.

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- MERA: Multi-scale entanglement renormalisation ansatz.

  Alternative ansatz based on renormalisation group.

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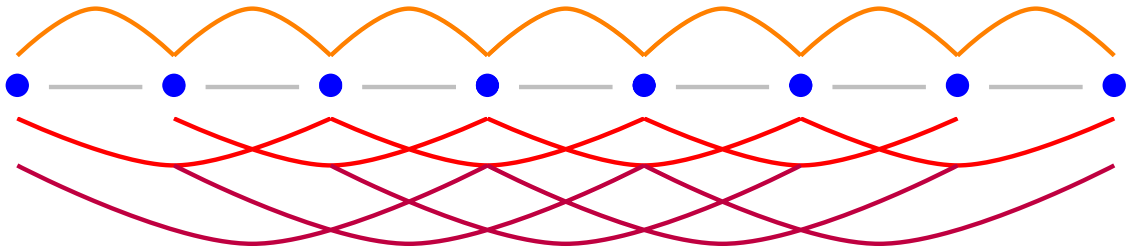

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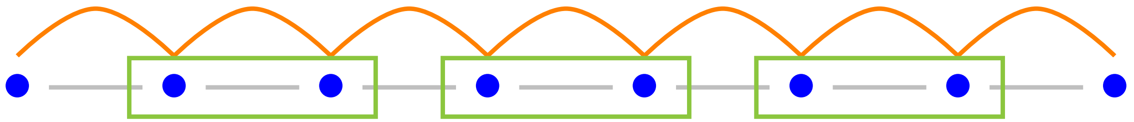

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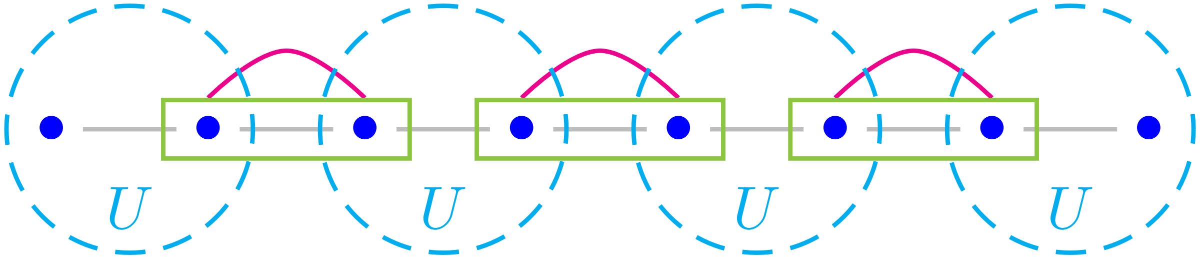

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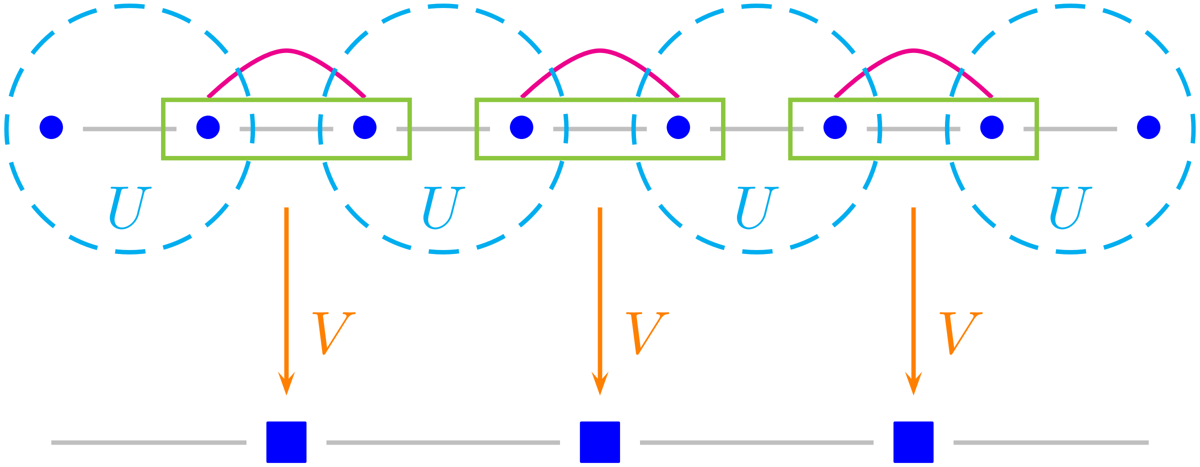

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- Idea: real-space RG, White's optimal truncation, and
  get rid of short-range entanglement before coarse-graining.
- Information about original state:
  encoded in RG steps $\{(\bigotimes V) \circ (\bigotimes U)\}_i$
  Efficient representation of quantum states.
  Well suited to describe quantum criticality.

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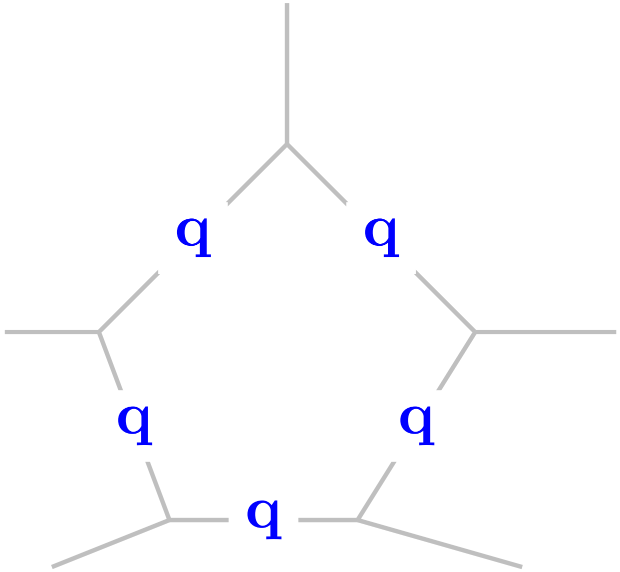
- General $\mathbb{Z}_2$ planar code: can add faces



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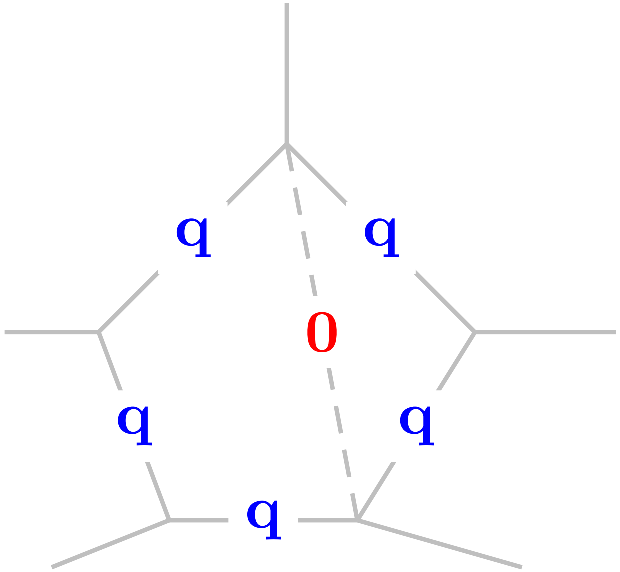

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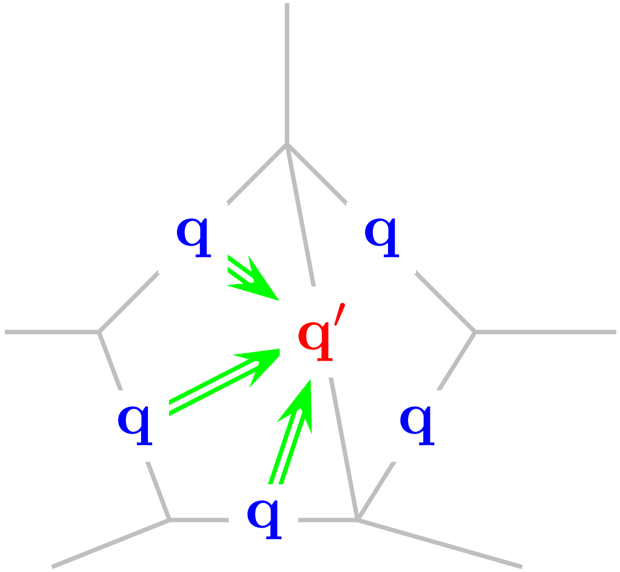

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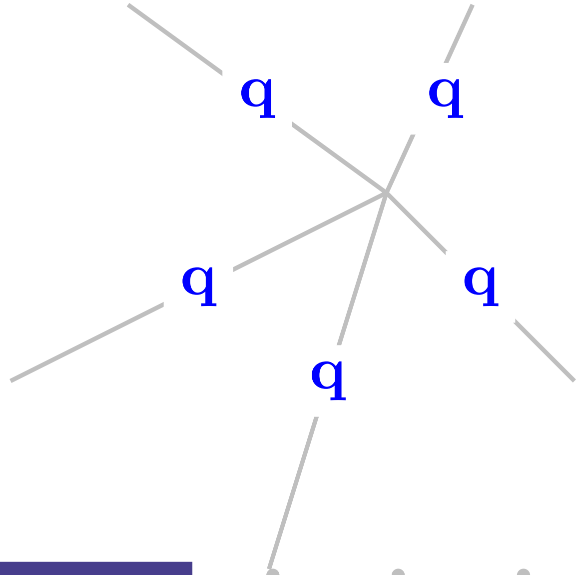

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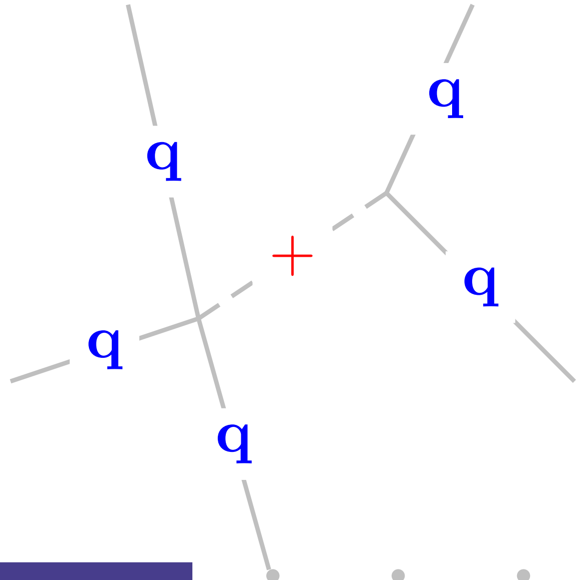

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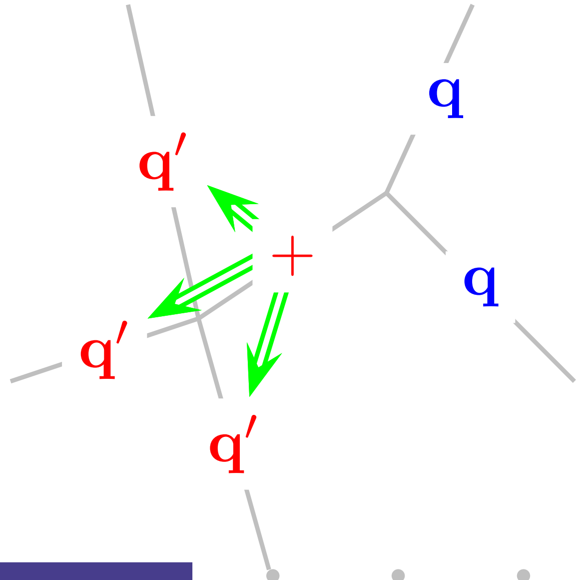

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- Intermediate step involves dual of Levin-Wen loop model.

Conclusions

- PEPS shown to describe known topological states. Useful to analyse S_{top}.
- MERA ansatz allows to describe and grow toric codes sequentially.
- Outlook: Characterisation of topological properties in terms of different ansätze. Computation of entanglement entropies for interesting models.