

# MP 472 Quantum Information and Computation

<http://www.thphys.may.ie/staff/jvala/MP472.htm>

## Outline

Open quantum systems

The density operator

- ensemble of quantum states
- general properties
- the reduced density operator

Quantum noise (decoherence)

Quantum error correction

Fault-tolerant quantum computation

## Density operator (review)

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

Characterization of density operators:

An operator  $\rho$  is the density operator associated to some ensemble  $\{p_i, |\psi_i\rangle\}$  iff it satisfies the conditions:

- (1) (Trace condition)  $\text{tr}(\rho) = 1$
- (2) (Positivity)  $\rho$  is positive operator

Proof:

(1)  $\text{tr}(\rho) = \sum_i p_i \text{tr}(|\psi_i\rangle\langle\psi_i|) = \sum_i p_i = 1$

(2) Suppose  $|\phi\rangle$  is an arbitrary vector in the Hilbert space

$$\langle\phi|\rho|\phi\rangle = \sum_i p_i \langle\phi|\psi_i\rangle\langle\psi_i|\phi\rangle = \sum_i p_i |\langle\phi|\psi_i\rangle|^2 \geq 0$$

Conversely, suppose  $\rho$  is any operator satisfying (1) and (2). Since  $\rho$  is positive it must have a spectral decomposition

$$\rho = \sum_j \lambda_j |j\rangle\langle j|$$

where the vectors  $|j\rangle$  are orthogonal and  $\lambda_j \in \mathbb{R}$  are nonnegative eigenvalues of  $\rho$ . From the trace condition  $\sum_j \lambda_j = 1$ . Therefore a system in state  $|j\rangle$  with probability  $\lambda_j$  will have the density operator  $\rho$ .

# Schmidt decomposition

Theorem (Schmidt's decomposition):

Suppose  $|\psi\rangle$  is a pure state of a bipartite composite system, AB. Then there exist orthonormal states  $|i_A\rangle$  for system A, and  $|i_B\rangle$  for system B s.t.

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle$$

where  $\lambda_i$  are non-negative real numbers satisfying  $\sum_i \lambda_i^2 = 1$  known as Schmidt coefficients.

Proof:

Let  $|j\rangle$  and  $|k\rangle$  be any fixed orthonormal bases for A and B (the relevant state spaces are here of the same dimension). Then  $|\psi\rangle$  can be written

$$|\psi\rangle = \sum_{jk} a_{jk} |j\rangle |k\rangle$$

for some matrix  $\mathbf{a}$  of complex numbers  $a_{jk}$ .

By the singular value decomposition  $\mathbf{a} = \mathbf{u} \mathbf{d} \mathbf{v}$ , where  $\mathbf{d}$  is a nonnegative diagonal matrix, and  $\mathbf{u}$  and  $\mathbf{v}$  are unitary matrices,

$$|\psi\rangle = \sum_{ijk} u_{ji} d_{ii} v_{ik} |j\rangle |k\rangle$$

Defining  $|i_A\rangle = \sum_j u_{ji} |j\rangle$ , and  $|i_B\rangle = \sum_k v_{ik} |k\rangle$ , and  $\lambda_i = d_{ii}$ , this gives

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle$$

$|i_A\rangle$  form orthonormal set due to unitarity of  $\mathbf{u}$  and orthonormality of  $|j\rangle$ , and similarly  $|i_B\rangle$  form an orthonormal set.

## Schmidt decomposition: examples

The number of non-zero values of the Schmidt coefficients  $\lambda_i$  is called the Schmidt number. If it equals to 1 then the state is the product state.

A)  $|\psi\rangle = (2^{-1/2})(|00\rangle + |01\rangle)$

1) construct matrices  $a$  and  $a^+a$ :  $a = (2^{-1/2}) \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$        $a^+a = (2^{-1}) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

2) diagonalize  $a^+a$ :      characteristic equation  $\det(a^+a - \alpha I) = 0$   
 $\alpha_1 = \lambda_1^2 = 1$  and  $\alpha_2 = \lambda_2^2 = 0$

There is only one nonzero Schmidt coefficient and thus  $|\psi\rangle$  is a product state.

B)  $|\psi\rangle = (2^{-1/2})(|00\rangle + |11\rangle)$

1) construct  $a$  and  $a^+a$ :  $a = (2^{-1/2}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$        $a^+a = (2^{-1}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

2) diagonalize  $a^+a$ :      characteristic equation  $\det(a^+a - \alpha I) = 0$   
 $\alpha_1 = \lambda_1^2 = 1/2$  and  $\alpha_2 = \lambda_2^2 = 1/2$

There are two nonzero Schmidt coefficients and thus  $|\psi\rangle$  is an entangled state.

Homework:

What is the Schmidt number for the state  $(3^{-1/2})(|00\rangle + |01\rangle + |11\rangle)$ ?

# Reduced density operator

Suppose we have physical systems A and B, whose state is described by a density matrix  $\rho^{AB}$ . The reduced density operator for system A is

$$\rho^A = \text{tr}_B(\rho^{AB})$$

where  $\text{tr}_B$  is an operator map known as partial trace over system B. It is defined as

$$\rho^A = \text{tr}_B(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) = |a_1\rangle\langle a_2| \text{tr}(|b_1\rangle\langle b_2|)$$

where  $|a_1\rangle$  and  $|a_2\rangle$  are any two vectors in A and  $|b_1\rangle$  and  $|b_2\rangle$  are any two vectors in B.  $\text{tr}(|b_1\rangle\langle b_2|)$  is the usual trace, so  $\text{tr}(|b_1\rangle\langle b_2|) = \langle b_2|b_1\rangle$  (via completeness relation!)

## Physical interpretation:

The reduced density matrix  $\rho^A$  above provides correct measurement statistics for measurements on system A.

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### Completeness relation:

Let  $\{|i\rangle\}$  be orthonormal basis for the vector space  $V$ , so an arbitrary vector can be written  $|v\rangle = \sum_i v_i |i\rangle$  for some set of  $v_i \in \mathbb{C}$ . Note that  $v_i = \langle i|v\rangle$ , thus:  $(\sum_i |i\rangle\langle i|)|v\rangle = \sum_i |i\rangle\langle i|v\rangle = \sum_i v_i |i\rangle = |v\rangle$ . Since this is true for all  $|v\rangle$ , it follows that  $\sum_i |i\rangle\langle i| = I$ .

$$\text{tr}(|b_1\rangle\langle b_2|) = \sum_i \langle i|b_1\rangle\langle b_2|i\rangle = \sum_i \langle b_2|i\rangle\langle i|b_1\rangle = \langle b_2|(\sum_i |i\rangle\langle i|)|b_1\rangle = \langle b_2|b_1\rangle$$

## Reduced density operator: independent subsystems

A state of the composite system AB consisting of independent subsystems is a product state which is described by the density operator

$$\rho^{AB} = \rho \otimes \sigma$$

Example:

$$|\Psi\rangle = (2^{-1/2})(|00\rangle + |10\rangle) = [(2^{-1/2})(|0\rangle + |1\rangle)] |0\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$$

$$\rho^{AB} = |\Psi\rangle\langle\Psi| = |\psi_A\rangle\langle\psi_A| \otimes |\psi_B\rangle\langle\psi_B| = \rho \otimes \sigma$$

(Remark: in general  $\rho$  and  $\sigma$  can describe mixed states!)

The reduced density operators for the composite system in a product state is calculated as follows:

$$\rho^A = \text{tr}_B(\rho^{AB}) = \text{tr}_B(\rho \otimes \sigma) = \rho \text{tr}(\sigma) = \rho$$

$$\rho^B = \text{tr}_A(\rho^{AB}) = \text{tr}_A(\rho \otimes \sigma) = \text{tr}(\rho) \sigma = \sigma$$

## Reduced density operator: entangled subsystems

Example:

Two-qubit system in the Bell state  $|\beta_{00}\rangle = (2^{-1/2})(|00\rangle + |11\rangle)$ :

$$\rho = (2^{-1/2})^2 (|00\rangle + |11\rangle)(\langle 00| + \langle 11|) = (1/2)(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|)$$

Partial trace over the second qubit is

$$\begin{aligned}\rho_1 &= \text{tr}_2[(1/2)(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|)] = \\ &= (1/2)(|0\rangle\langle 0|\langle 0|0\rangle + |0\rangle\langle 1|\langle 1|0\rangle + |1\rangle\langle 0|\langle 0|1\rangle + |1\rangle\langle 1|\langle 1|1\rangle) = \\ &= (1/2)(|0\rangle\langle 0| + |1\rangle\langle 1|) = I/2\end{aligned}$$

Is this single qubit mixed state?

$$\text{tr}((I/2)^2) = \frac{1}{2} < 1$$

The state of the joint system of two qubits is a pure state, i.e. it is known exactly, however, the first qubit is in mixed state, i.e. a state about which we do not have maximal knowledge. **characteristics of quantum entanglement**

Homework:

$\rho_1 = (1/2)I$  is a state of one qubit - what is a Bloch vector of this state?

# Reduced density operators and the Schmidt decomposition

Suppose  $|\psi\rangle$  is a pure state of a bipartite composite system, AB. The Schmidt decomposition is given as

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle$$

where the Schmidt coefficients  $\lambda_i$  are real and satisfy  $\sum_i \lambda_i^2 = 1$ . The density matrix of the system is

$$\rho = |\psi\rangle\langle\psi| = \sum_i \lambda_i^2 |i_A\rangle\langle i_A| \otimes |i_B\rangle\langle i_B|$$

and the reduced density matrices are

$$\rho^A = \sum_i \lambda_i^2 |i_A\rangle\langle i_A|$$

$$\rho^B = \sum_i \lambda_i^2 |i_B\rangle\langle i_B|$$

Note that the eigenvalues of  $\rho^A$  and  $\rho^B$  are identical and are equal to  $\lambda_i^2$ . (recall that  $|i_A\rangle$  and  $|i_B\rangle$  form orthonormal sets).