# MP 472 Quantum Information and Computation

http://www.thphys.may.ie/staff/jvala/MP472.htm

## Outline

Open quantum systems

The density operator
- ensemble of quantum states
- general properties
- the reduced density operator

Quantum noise (decoherence)

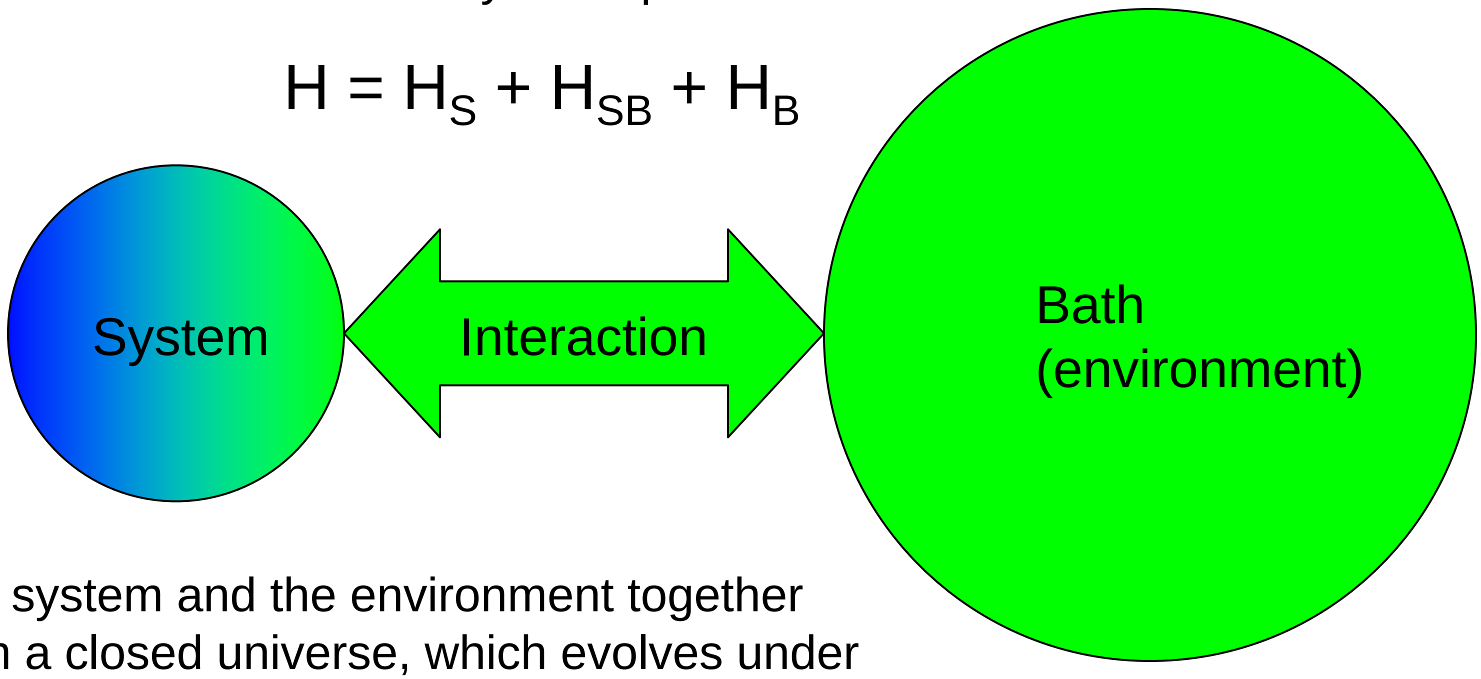
Quantum error correction

Fault-tolerant quantum computation

# Open quantum systems

No physical systems are closed (isolated). They are open as they interact with environment formed by other particles and fields:

$$H = H_S + H_{SB} + H_B$$




The system and the environment together form a closed universe, which evolves under unitary dynamics generated by the total Hamiltonian H

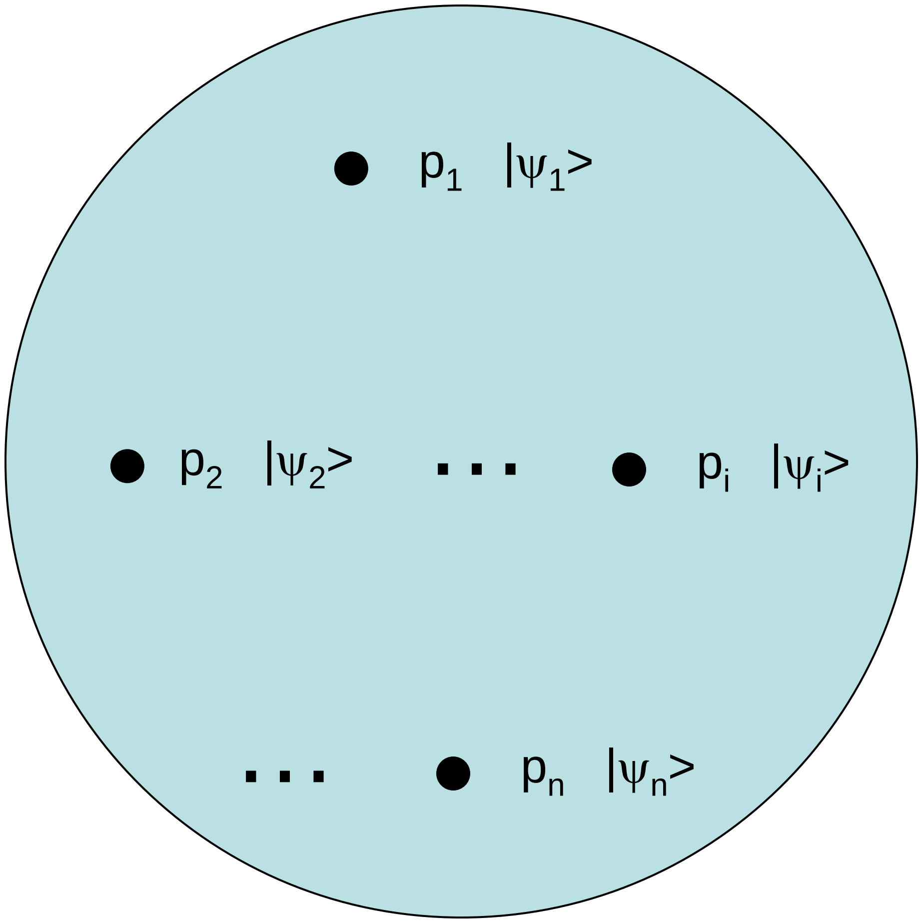
$$i\hbar \, d|\psi(t)\rangle/dt = H(t)|\psi(t)\rangle$$

The system interacting with environment however evolves as open quantum system under reduced dynamics which is NOT unitary. The effect of environment appears as noise onto the system intrinsic dynamics (generated by $H_s$). Quantum states of the system and of the environment interact and become entangled, they are loosing their purity and become MIXED.

**Novel description of a quantum state needed!!!**

# Ensemble of quantum states

Suppose a quantum system is in one of a number of pure states $|\psi_i\rangle$ with a probability $p_i$. We call the set $\{p_i, |\psi_i\rangle\}$ an ensemble of pure states:




The state of the ensemble is described by the density operator

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

# Density operator $\rho = \Sigma_i p_i |\psi_i\rangle\langle\psi_i|$

Postulates of quantum mechanics can be reformulated using density operator

Examples:

Quantum evolution $|\psi\rangle \longrightarrow U_t(H)|\psi\rangle = U|\psi\rangle$

$$\rho = \Sigma_i p_i |\psi_i\rangle\langle\psi_i| \longrightarrow \Sigma_i p_i U|\psi_i\rangle\langle\psi_i|U^+ = U \rho U^+$$

Measurement

- when the measurement described by the operator $M_m$ is performed on the state $|\psi_i\rangle$, the result m is obtained with probability

$$p(m|i) = \langle\psi_i| M_m^+ M_m |\psi_i\rangle = \Sigma_k \langle\psi_i|k\rangle\langle k| M_m^+ M_m |\psi_i\rangle = \Sigma_k \langle k| M_m^+ M_m |\psi_i\rangle\langle\psi_i|k\rangle =$$

$$= tr(M_m^+ M_m |\psi_i\rangle\langle\psi_i|)$$

- and the probability of the result m to be measured on the ensemble described by $\rho = \Sigma_i p_i |\psi_i\rangle\langle\psi_i|$ is then

$$p(m) = \Sigma_i p_i\, tr(M_m^+ M_m |\psi_i\rangle\langle\psi_i|) = tr(M_m^+ M_m \rho)$$

- and the state after the measurement is

$$\rho_m = M_m \rho M_m^+ / tr(M_m^+ M_m \rho)$$

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Completeness relation:

Let $\{|i\rangle\}$ be orthonormal basis for the vector space V, so an arbitrary vector can be written $|v\rangle = \Sigma_i v_i |i\rangle$ for some set of $v_i \in \mathbb{C}$. Note that $v_i = \langle i|v\rangle$, thus: $(\Sigma_i |i\rangle\langle i|)|v\rangle = \Sigma_i |i\rangle\langle i|v\rangle = \Sigma_i v_i |i\rangle = |v\rangle$. Since this is true for all $|v\rangle$, it follows that $\Sigma_i |i\rangle\langle i| = I$.

# Density operator

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

Characterization of density operators:

An operator $\rho$ is the density operator associated to some ensemble $\{p_i, |\psi_i\rangle\}$ iff it satisfies the conditions:

(1) (Trace condition) $\quad tr(\rho) = 1$

(2) (Positivity) $\quad \rho$ is positive operator

Proof:

(1) $tr(\rho) = \Sigma_i p_i \, tr(|\psi_i\rangle\langle\psi_i|) = \Sigma_i p_i = 1$

(2) Suppose $|\phi\rangle$ is an arbitrary vector is state space

$$\langle\phi|\rho|\phi\rangle = \Sigma_i p_i \langle\phi|\psi_i\rangle\langle\psi_i|\phi\rangle = \Sigma_i p_i |\langle\phi|\psi_i\rangle|^2 \geqslant 0$$

Conversely, suppose $\rho$ is any operator satisfying (1) and (2). Since $\rho$ is positive it must have a spectral decomposition

$$\rho = \Sigma_j \lambda_i |j\rangle\langle j|$$

where the vectors $|j\rangle$ are orthogonal and $\lambda_i \in \mathbb{R}$ are nonnegative eigenvalues of $\rho$. From the trace condition $\Sigma_j \lambda_i = 1$. Therefore a system in state $|j\rangle$ with probability $\lambda_i$ will have the density operator $\rho$.

# Is a quantum state mixed?

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

Purity $tr(\rho^2)$

pure states: $tr(\rho^2) = 1$

$tr(\rho^2) = tr(|\psi\rangle\langle\psi|\psi\rangle\langle\psi|) = tr(|\psi\rangle\langle\psi|) = tr(\rho) = 1$

mixed states: $tr(\rho^2) < 1$

Homework: Can unitary operation change purity?

Von Neumann entropy $S = - tr(\rho \log \rho)$

pure states: $S = 0$

mixed states: $S > 0$

# Bloch sphere for mixed states

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

An arbitrary single qubit density matrix can be written as

$$\rho = (1/2)\,(I + r.\sigma)$$

where r is a three dimensional vector s.t. $||r|| \leqslant 1$, known as the Bloch vector.
$\sigma = (X,Y,Z)$ is the vector of Pauli matrices.

Bloch representation for a pure state of quantum bit

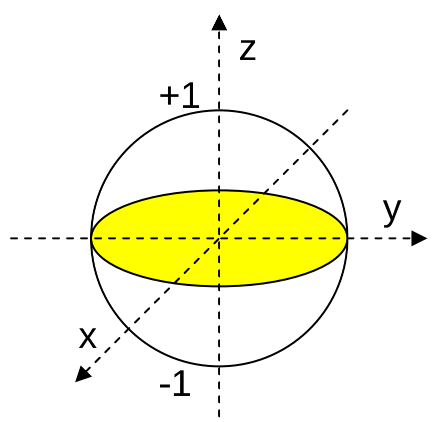
$$\rho = |\phi\rangle\langle\phi| = |c_0|^2|0\rangle\langle 0| + c_0^*c_1|0\rangle\langle 1| + c_1^*c_0|1\rangle\langle 0| + |c_1|^2|1\rangle\langle 1| =$$

$$= \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} \begin{pmatrix} c_0^* & c_1^* \end{pmatrix} = \begin{pmatrix} |c_0|^2 & c_0c_1^* \\ c_0^*c_1 & |c_1|^2 \end{pmatrix} = (1/2)(\, I + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z \,)$$

$r_x = 2\mathrm{Re}(c_0^*c_1)$

$r_y = 2\mathrm{Im}(c_0^*c_1)$

$r_z = |c_0|^2 - |c_1|^2$



# Pure state and the Bloch sphere (review)

$r_x = 2\mathrm{Re}(c_0{}^*c_1)$

$r_y = 2\mathrm{Im}(c_0{}^*c_1)$

$r_z = |c_0|^2 - |c_1|^2$

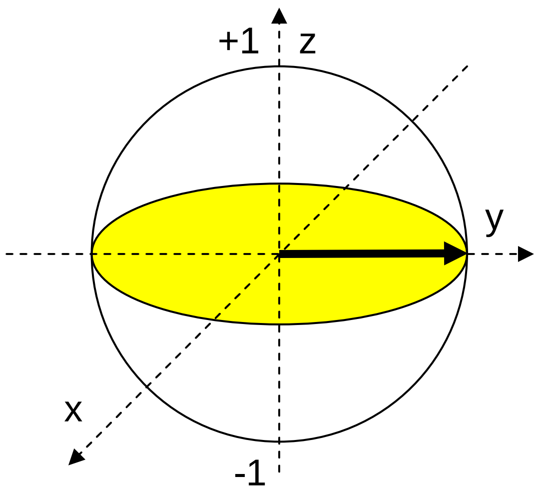
$$\begin{pmatrix} |c_0|^2 & c_0 c_1{}^* \\ c_0{}^* c_1 & |c_1|^2 \end{pmatrix}$$

Example:

$|\phi\rangle = 2^{-1/2}(|0\rangle + i|1\rangle)$

$$|\phi\rangle = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = 2^{-1/2} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\rho = 1/2 \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$



# Mixed state and the Bloch sphere

$r_x = 2\mathrm{Re}(c_0{}^*c_1)$
$r_y = 2\mathrm{Im}(c_0{}^*c_1)$
$r_z = |c_0|^2 - |c_1|^2$

$$\begin{pmatrix} |c_0|^2 & c_0c_1{}^* \\ c_0{}^*c_1 & |c_1|^2 \end{pmatrix}$$

Example: $|\phi_1\rangle = 2^{-1/2}(|0\rangle + i|1\rangle)$ $\quad |\phi_2\rangle = 2^{-1/2}(|0\rangle - i|1\rangle)$

$$|\phi_1\rangle\langle\phi_1| = 1/2 \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \qquad |\phi_2\rangle\langle\phi_2| = 1/2 \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

$$\rho = 3/4\ |\phi_1\rangle\langle\phi_1| + 1/4\ |\phi_2\rangle\langle\phi_2|$$

$$\rho = 1/2 \begin{pmatrix} 1 & -i/2 \\ i/2 & 1 \end{pmatrix}$$

$r_x = 0$
$r_y = 1/2$
$r_z = 0$