MP 472 Quantum Information Processing

http://www.thphys.may.ie/staff/jvala/MP472.htm

Outline

Quantum bits

Quantum operations

Quantum measurement

- **projective measurement**
- **POVM measurement**
- **measurement and quantum circuit**

Postulates of quantum mechanics

Quantum state

• At a fixed time t, the state of a physical system is defined by specifying a ket $|\psi(t)>$ belonging to the state space $\mathscr{H}$.

Quantum observable

• Every measurable physical quantity $\mathscr{A}$ is described by an operator **A** acting on $\mathscr{H}$; this operator is an observable.

Quantum Dynamics

• The time evolution of the state vector $|\psi(t)>$ is governed by the Schroedinger equation $i\hbar\, d|\psi(t)>/dt = \mathbf{H}(t)|\psi(t)>$ where **H**(t) is the observable associated with the total energy of the system.

Measurement

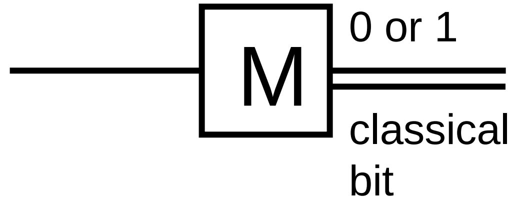
• The only possible result of the measurement of a physical quantity $\mathscr{A}$ is one of the eigenvalues of the corresponding observable **A**.

• When the physical quantity $\mathscr{A}$ is measured on a system in the normalized state $|\psi>$, the probability $p(a_n)$ of obtaining the (non-degenerate) eigenvalue a_n of the corresponding observable **A** is: $p(a_n) = |<u_n|\psi>|^2$, where $|u_n>$ is the normalized eigenvector of **A** with the eigenvalue a_n.

• If the measurement of the physical quantity $\mathscr{A}$ on the system in the state $|\psi>$ gives the result a_n, the state of the system immediately after the measurement is the normalized projection, $P_n|\psi>/(<\psi|\psi>)^{1/2}$, of $|\psi>$ onto the eigenspace associated with a_n.

(Projective) measurement of a qubit in computational basis

$|\phi\rangle = c_0|0\rangle + c_1|1\rangle$



$|\psi\rangle = ?$

Measurement of one qubit (in the computational basis $\{|0\rangle,|1\rangle\}$) gives a classical bit of information:

- we define the measurement operator as:

$$M_0 = |0\rangle\langle 0| \qquad \text{projector}$$

in matrix representation in the computational basis

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Note: measurement is not unitary !

- the measurement gives the result M = 0 with a probability:

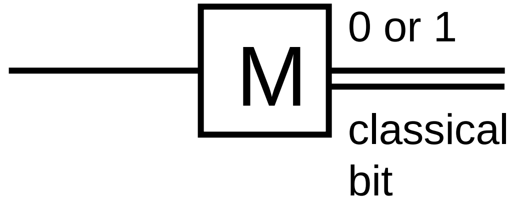
$$p_0 = \langle\phi|M_0^+ M_0|\phi\rangle = |c_0|^2$$

and the quantum state transforms as:

$$|\psi\rangle = |0\rangle = M_0|\phi\rangle / ||M_0|\phi\rangle||^{1/2}$$

(Projective) measurement of a qubit in computational basis

$|\phi\rangle = c_0|0\rangle + c_1|1\rangle$



$|\psi\rangle = ?$

• we define the measurement operator as:

$M_1 = |1\rangle\langle 1|$ projector

in matrix representation in the computational basis

$$M_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Note: measurement is not unitary !

• the result is M = 1 with the probability:

$$p_1 = \langle\phi|M_1^+ M_1|\phi\rangle = |c_1|^2$$

$M_1 = |1\rangle\langle 1|$

and the quantum state transforms as

$$|\psi\rangle = |1\rangle = M_1|\phi\rangle / ||M_1|\phi\rangle||^{1/2}$$

General measurement

Measurement is defined by the set of measurement operators $\{M_m\}$ where m refers to the measurement outcomes.

If the state of the system before the measurement is $|\phi>$, then the probability that result m occurs is

$$p_m = <\phi|M_m^+ M_m|\phi>$$

and the state after the measurement is

$$|\psi> = M_m|\phi>/||M_m|\phi>||^{1/2}$$

The measurement operators satisfy the completeness equation

$$\Sigma_m M_m^+ M_m = 1$$

which expresses the fact that the probabilities sum to one

$$1 = \Sigma_m p_m = \Sigma_m <\phi|M_m^+ M_m|\phi>$$

Distinguishing quantum states

Two-parties game:

Alice chooses a state $|\psi_i\rangle$, $1 \leqslant i \leqslant n$ from some fixed set of states known to both parties, and sends it to Bob whose task is to identify the index i of the state

$|\psi_i\rangle$ ⟶

Alice Bob

If the states $\{|\psi_i\rangle\}$ are **orthonormal** than Bob can peform a quantum measurement to distinguish the states:

Bob has to define the measurement operators

$M_i = |\psi_i\rangle\langle\psi_i|$

M_0 – a positive square root of $I - \sum_{i\neq 0} |\psi_i\rangle\langle\psi_i|$

which satisfy the completenes relation (Homework: show it!) and thus can be used to distingush the orthonormal state $|\psi_i\rangle$.

If the states $\{|\psi_i\rangle\}$ are **non-orthonormal** than there is no quantum measurement to reliably distinguish the states.

Projective measurement

A projective measurement is described by an observable, M, a Hermitian operator on a state space of the system being observed. The observable has the spectral decomposition

$$M = \Sigma_m mP_m$$

where P_m is the projector onto the eigenspace of M with eigenvalue m.

The possible outcomes of the measurement correspond to the eigenvalues m of the observable.

If the state of the system before the measurement is $|\phi\rangle$, then the probability that the result m occurs is

$$p_m = \langle\phi|P_m|\phi\rangle$$

and the state after the measurement is

$$|\psi\rangle = P_m|\phi\rangle/p_m^{1/2}$$

Projective measurement has useful properties:
it allows to calculate easily average values

$$E(M) = \Sigma_m mp_m = \Sigma_m m\langle\phi|P_m|\phi\rangle = \langle\phi|\Sigma_m mP_m|\phi\rangle = \langle\phi| M |\phi\rangle$$

e.g. standard deviation $\Delta(M) = \langle(M - \langle M\rangle)^2\rangle = \langle M^2\rangle - \langle M\rangle^2$

Heisenberg uncertainty relation

Let A and B be Hermitian operators, and $|\phi\rangle$ is a quantum state.
Suppose $\langle\phi|AB|\phi\rangle = x + iy$, where $x,y \in \mathbb{R}$, and note that

$$\langle\phi|[A,B]|\phi\rangle = 2iy \text{ and}$$

$$\langle\phi|\{A,B\}|\phi\rangle = 2x$$

This implies $$|\langle\phi|[A,B]|\phi\rangle|^2 + |\langle\phi|\{A,B\}|\phi\rangle|^2 = 4\,|\langle\phi|AB|\phi\rangle|^2$$

By the Cauchy-Schwarz inequality

$$|\langle\phi|AB|\phi\rangle|^2 \leqslant \langle\phi|A^2|\phi\rangle\langle\phi|B^2|\phi\rangle$$

and (using the previous relation and dropping negative terms)

$$|\langle\phi|[A,B]|\phi\rangle|^2 \leqslant 4\langle\phi|A^2|\phi\rangle\langle\phi|B^2|\phi\rangle$$

Suppose C and D are two observables. Substituting $A=C-\langle C\rangle$ and $B=D-\langle D\rangle$ into the last equation, we obtain the Heisenberg uncertainty principle:

$$\Delta(C)\,\Delta(D) \geqslant |\langle\phi|[C,D]|\phi\rangle|/2$$

Example: Consider the observables X and Y (the Pauli matrices) when measured for the quantum state $|0\rangle$. We know (or we can easily calculate) that $[X,Y]=2iZ$, so the uncertainty relation is

$$\Delta(X)\,\Delta(Y) \geqslant \langle 0|Z|0\rangle = 1$$

POVM measurement

Suppose a measurement described by the set of measurement operators $\{M_m\}$ is performed upon a quantum system in the state $|\phi\rangle$.

Then the probability that result m occurs is $p_m = \langle\phi|M_m^+ M_m|\phi\rangle$.

Let us define

$$E_m = M_m^+ M_m$$

then E_m is a positive operator such that

$$\Sigma_m E_m = 1 \quad \text{and} \quad p_m = \langle\phi|E_m|\phi\rangle$$

Thus the set of operators E_m, which are known as the POVM elements associated with the measurement, are sufficient to determine the probabilities of the different measurement outcomes.

The set $\{E_m\}$ is known as a POVM (Positive Operator-Valued Measure).

Example (trivial one): projective measurement

$$E_m = P_m$$

POVM measurement: example

$|\psi_1\rangle = |0\rangle$

$|\psi_2\rangle = 2^{-1/2}(|0\rangle+|1\rangle)$

Alice

Bob

The states $|\psi_1\rangle$ and $|\psi_2\rangle$ are not orthonormal so Bob can not distinguish them reliably. However he can perform a measurement which distinguishes the states some of the time, and NEVER makes an error of mis-identification.

Consider a POVM:

$$E_1 = [2^{1/2}/(1+2^{1/2})]\ |1\rangle\langle 1|$$

$$E_2 = [2^{1/2}/2(1+2^{1/2})]\ (|0\rangle - |1\rangle)(\langle 0| - \langle 1|)$$

$$E_3 = I - E_1 - E_2$$

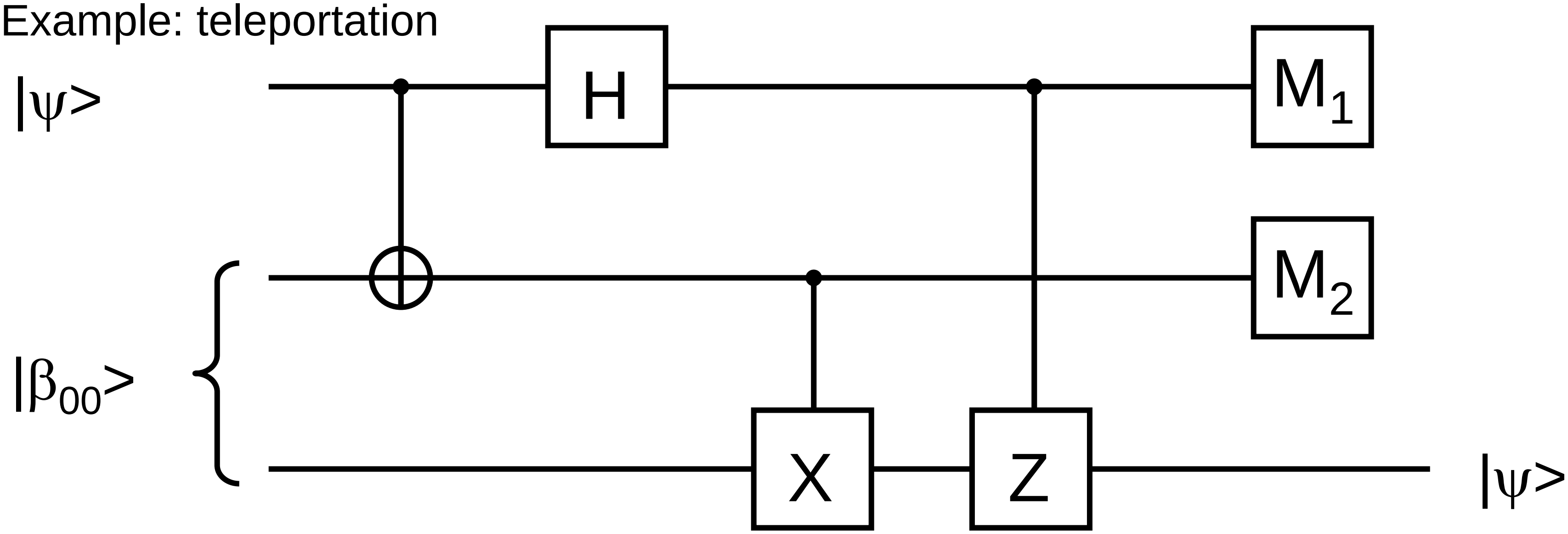
Homework: verify that these operators form a POVM

If the result is E_1 then the state was $|\psi_2\rangle$, and if the result E_2 occurs then the state was $|\psi_1\rangle$. Some of the time however Bob will obtain E_3 from which he can infer nothing about the state.

Measurement and quantum circuit

Principle of deferred measurement:
Measurement can always be moved from an intermediate stage of a quantum circuit to the end of the circuit; if the measurement results are used at any stage of the circuit than the classically controlled operations can be replaced by conditional quantum operations.

Example: teleportation



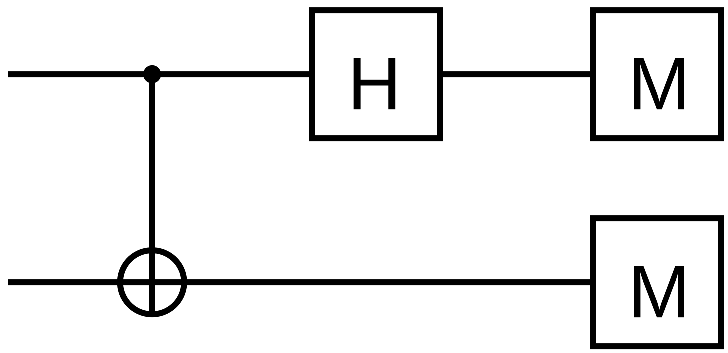
Homework: verify that this circuit is indeed a teleportation circuit.

Principle of implicite measurement:
Without loss of generality, any unterminated quantum wires (qubits which are not measured) at the end of a quantum circuit may be assumed to be measured.

Measurement in other than computational basis

Recipe:
first unitarily transform from the basis we wish to perform a measurement in to the computational basis, and then measure in the computational basis

Example: measurement in the Bell basis



Homework: Show that the circuit performs the measurement in the Bell basis.