MP 472 Quantum Information and Computation

http://www.thphys.may.ie/staff/jvala/MP472.htm

Outline

Quantum bits
- Bloch representation
- composition of quantum states – tensor product of Hilbert spaces
- separable and entangled states
- superdense coding

Quantum operations
- postulate of quantum mechanics regarding observables
- matrix representation of operators
- eigenvalues and eigenvectors
- adjoint and hermitian operators
- normal, unitary, positive operators

Quantum measurement

Postulates of quantum mechanics

Quantum state

- At a fixed time t, the state of a physical system is defined by specifying a ket $|\psi(t)\rangle$ belonging to the state space $\mathcal{H}$.

Quantum observable

- Every measurable physical quantity $\mathcal{A}$ is described by an operator **A** acting on $\mathcal{H}$; this operator is an observable.

Quantum Dynamics

- The time evolution of the state vector $|\psi(t)\rangle$ is governed by the Schroedinger equation $i\hbar\, d|\psi(t)\rangle/dt = \mathbf{H}(t)|\psi(t)\rangle$ where **H**(t) is the observable associated with the total energy of the system.

Measurement

- The only possible result of the measurement of a physical quantity $\mathcal{A}$ is one of the eigenvalues of the corresponding observable **A**.

- When the physical quantity $\mathcal{A}$ is measured on a system in the normalized state $|\psi\rangle$, the probability $p(a_n)$ of obtaining the (non-degenerate) eigenvalue a_n of the corresponding observable **A** is: $P(a_n) = |\langle u_n|\psi\rangle|^2$, where $|u_n\rangle$ is the normalized eigenvector of **A** with the eigenvalue a_n.

- If the measurement of the physical quantity $\mathcal{A}$ on the system in the state $|\psi\rangle$ gives the result a_n, the state of the system immediately after the measurement is the normalized projection, $P_n|\psi\rangle/(\langle\psi|\psi\rangle)^{1/2}$, of $|\psi\rangle$ onto the eigenspace associated with a_n.

Quantum bit

Classical bit is a set of two elements: $\mathbb{B} = \{0,1\}$

Quantum bit is a two-dimensional Hilbert space: $\mathcal{H}^2 \simeq \mathbb{C}^2$

- orthonormal basis: $\{|0\rangle, |1\rangle\}$ $\langle i|j\rangle = \delta_{ij}$, $i,j = 0,1$ (Kronecker delta)

- a state of a quantum bit:

$$|\phi\rangle = c_0|0\rangle + c_1|1\rangle \in \mathcal{H}^2 \simeq \mathbb{C}^2$$

$c_k \in \mathbb{C}$, for $k = 0, 1$, s.t. $|c_0|^2 + |c_1|^2 = 1$

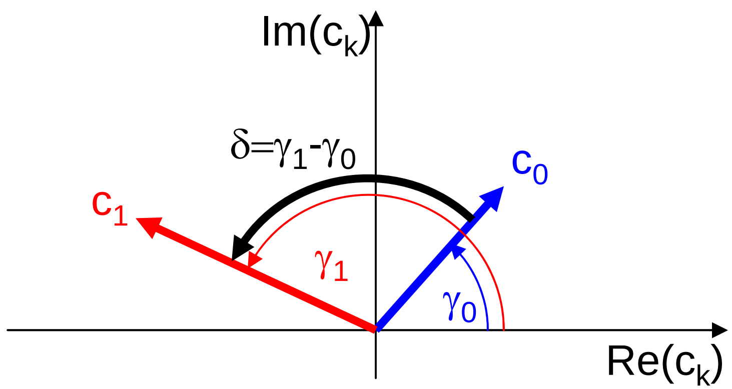
Towards Bloch representation

$|\phi\rangle = c_0|0\rangle + c_1|1\rangle$ $c_k \in \mathbb{C}$, for k =0, 1, s.t. $|c_0|^2 + |c_1|^2 = 1$

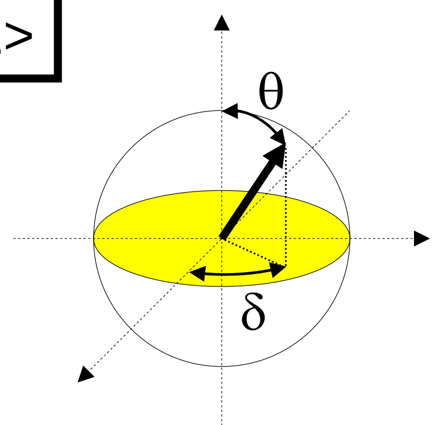
Is a quantum state really a vector?
Actually it is a ray, i.e. $e^{i\gamma}|\psi\rangle$, $\gamma \in \mathbb{R}$, because the global phase, $e^{i\gamma}$, is not observable in quantum mechanics and what matters is only a relative phase between individual basis states, e.g.:

$c_0 = |c_0| \exp(i\gamma_0)$

$c_1 = |c_1| \exp(i\gamma_1)$



$$|\phi\rangle = \cos\theta|0\rangle + e^{i\delta}\sin\theta|1\rangle$$



Density matrix for a qubit

- a state of a quantum bit:

$$|\phi\rangle = c_0|0\rangle + c_1|1\rangle \qquad c_k \in \mathbb{C}, \text{ for } k = 0, 1, \text{ s.t. } |c_0|^2 + |c_1|^2 = 1$$

$$|\phi\rangle = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}$$

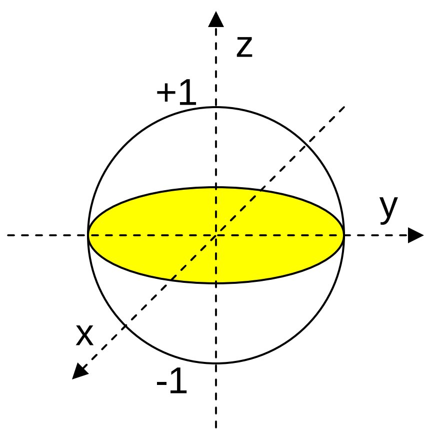
- a density matrix of a qubit (pure state case)

$$\rho = |\phi\rangle\langle\phi| = |c_0|^2|0\rangle\langle 0| + c_0{}^*c_1|0\rangle\langle 1| + c_1{}^*c_0|1\rangle\langle 0| + |c_1|^2|1\rangle\langle 1|$$

$$|\phi\rangle\langle\phi| = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} \begin{pmatrix} c_0{}^* & c_1{}^* \end{pmatrix} = \begin{pmatrix} |c_0|^2 & c_0c_1{}^* \\ c_0{}^*c_1 & |c_1|^2 \end{pmatrix}$$

- Bloch representation

$$x = 2\mathrm{Re}(c_0{}^*c_1)$$
$$y = 2\mathrm{Im}(c_0{}^*c_1)$$
$$z = |c_0|^2 - |c_1|^2$$



Bloch sphere

Bloch representation

$x = 2\mathrm{Re}(c_0 c_1^*)$

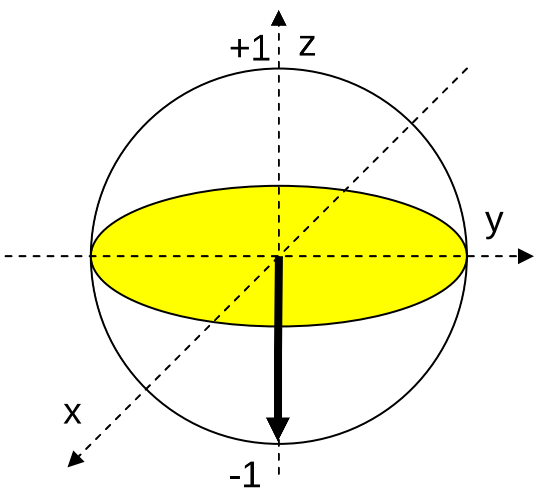
$y = 2\mathrm{Im}(c_0 c_1^*)$

$z = |c_0|^2 - |c_0|^2$

Examples:

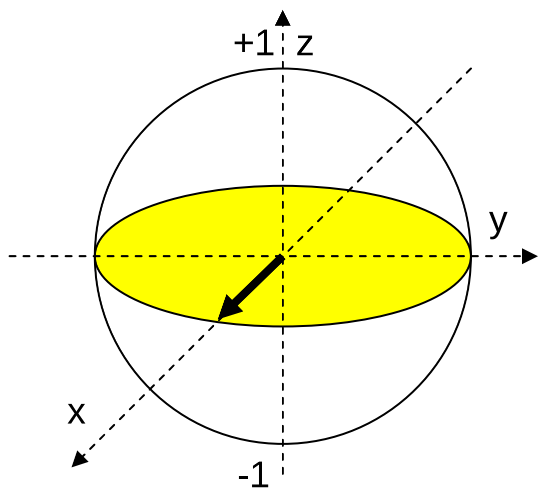
$|\phi\rangle = |1\rangle$

$$|\phi\rangle = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



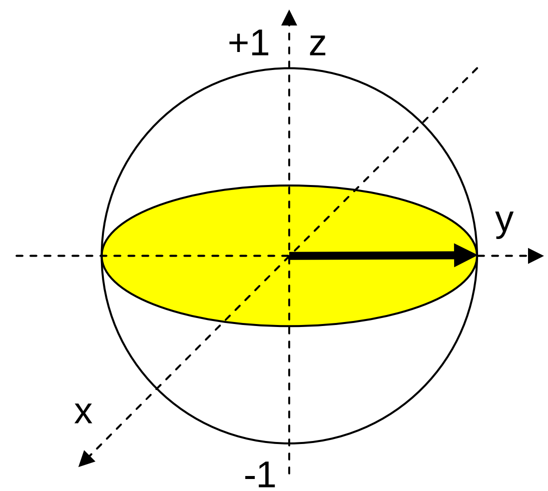
$|\phi\rangle = 2^{-1/2}(|0\rangle + |1\rangle)$

$$|\phi\rangle = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = 2^{-1/2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



$|\phi\rangle = 2^{-1/2}(|0\rangle + i|1\rangle)$

$$|\phi\rangle = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = 2^{-1/2} \begin{pmatrix} 1 \\ i \end{pmatrix}$$



Composition of Hilbert spaces

A tensor product of vector space V and U is a vector space W whose dimension is (dim V).(dim U). Thus a Hilbert space of n qubits is

$$\underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \ \dots \ \otimes \mathbb{C}^2}_{\text{n-times}} = \mathbb{C}^{2^n}$$

Let $\mathcal{B}_U = \{|u_1>,\dots,|u_n>\}$ be a basis of U and $\mathcal{B}_V=\{|v_1>,\dots,|v_n>\}$ be a basis of V, then a basis of W = V$\otimes$U is $\{|u_1v_1>,\dots,|u_nv_n>\}$ where $|u_kv_l> = |u_k>\otimes|v_l>$.

Example:

- $\mathcal{B}_U = \{|0>,|1>\}$ be a basis of U and $\mathcal{B}_V=\{|0>,|1>\}$ a basis of V, then a basis of W = V$\otimes$U is {|00>,|01>,|10>,|11>};
- tensor product of two vectors:

$$|\psi> = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \in U \quad |\phi> = \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} \in V \qquad |\phi> = \begin{pmatrix} a_0b_0 \\ a_0b_1 \\ a_1b_0 \\ a_1b_1 \end{pmatrix} \in W$$

- example: a basis vector |01>:

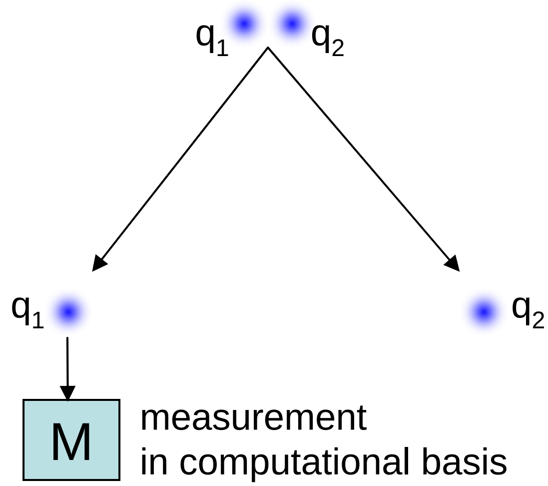
$$|0> = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in U \quad |1> = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in V \qquad |01> = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \in W$$

Separable and entangled states

What measurement on one part of a composite system tells us about the other parts? Let us try a bipartite system:

separable

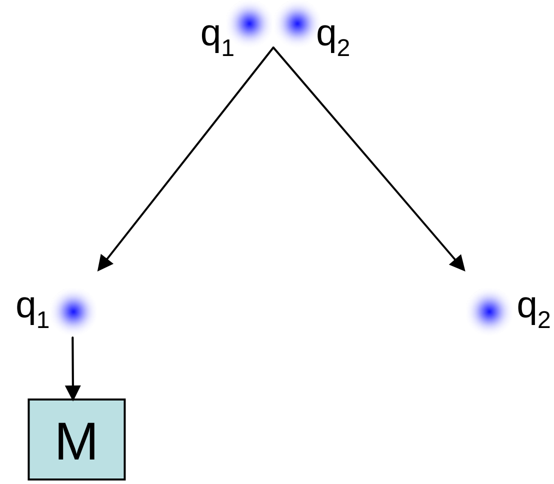
e.g. $2^{-1/2}|0\rangle \otimes (|0\rangle + |1\rangle)$



M=0, |0>　　?

entangled

e.g. $2^{-1/2}(|00\rangle + |11\rangle)$



if M=0, then q_1 is in |0> and q_2 in |0>

if M=1, then q_1 is in |1> and q_2 in |1>

If the state is entangled, measurement on one of its parts completely determines the state of the other part.

Separable states

Composite states which can be expressed as a tensor product of states of individual qubits is called a separable state.

Separable states of a composite system should be regarded as states of the uncorrelated (individual) constitients.

Example:

$$|\psi\rangle = a|011\rangle + b|111\rangle =$$
$$= (a|0\rangle + b|1\rangle) \otimes |1\rangle \otimes |1\rangle$$

$$|\psi\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ a \\ 0 \\ 0 \\ 0 \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

in the standard, i.e. computational, basis

Entanglement

Examples: Bell states

$$|\beta_{00}\rangle = 2^{-1/2}(|00\rangle + |11\rangle)$$
$$|\beta_{01}\rangle = 2^{-1/2}(|00\rangle - |11\rangle)$$
$$|\beta_{10}\rangle = 2^{-1/2}(|01\rangle + |10\rangle)$$
$$|\beta_{11}\rangle = 2^{-1/2}(|01\rangle - |10\rangle)$$

Show that the Bell states form a basis of two-qubit Hilbert space

An entangled state can not be expressed as a tensor product of states of individual constituents:

$$|\beta_{00}\rangle = 2^{-1/2}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

Try to express this state as a tensor product

Example: multi-qubit entangled state

$$2^{-1/2}(|0000\rangle + |1111\rangle)$$

Is the following state entangled? How?

$$2^{-1/2}(|001\rangle + |111\rangle)$$

Superdense coding

Task: Alice wants to send two classical bits of information, i.e. one of the bit strings {00,01,10,11} to Bob

Resources:
- Alice and Bob share two qubits in the Bell state $|\beta_{00}>$
- Alice can send her (one!) qubit to Bob

Alice

$$|\beta_{00}> = 2^{-1/2}(|00> + |11>)$$

Bob

Superdense coding

$|\beta_{00}\rangle = 2^{-1/2}(|00\rangle + |11\rangle)$

Alice Bob

1) Depending on what bit string, 00, 01,10,11, Alice wants to send to Bob, she applies the following transformations on her qubit:

00: I (identity)	$|\beta_{00}\rangle = 2^{-1/2}(|00\rangle + |11\rangle) \longrightarrow$	$|\beta_{00}\rangle = 2^{-1/2}(|00\rangle + |11\rangle)$
01: Z	$|\beta_{00}\rangle = 2^{-1/2}(|00\rangle + |11\rangle) \longrightarrow$	$|\beta_{01}\rangle = 2^{-1/2}(|00\rangle - |11\rangle)$
10: X	$|\beta_{00}\rangle = 2^{-1/2}(|00\rangle + |11\rangle) \longrightarrow$	$|\beta_{10}\rangle = 2^{-1/2}(|01\rangle + |10\rangle)$
11: ZX=iY	$|\beta_{00}\rangle = 2^{-1/2}(|00\rangle + |11\rangle) \longrightarrow$	$|\beta_{11}\rangle = 2^{-1/2}(|01\rangle - |10\rangle)$

2) The resulting Bell states are orthogonal and hence Bob can distinguish them by a measurement (in the Bell basis).

> The task could not be accomplished by sending one classical bit only but one quantum bit is sufficient to transmit two classical bits of information.

Entanglement in bipartite systems

Theorem (Schmidt's decomposition):

Suppose $|\psi>$ is a pure state of a bipartite composite system, AB. Then there exist orthonormal states $|i_A>$ for system A, and $|i_B>$ for system B s.t.

$$|\psi> = \sum_i \lambda_i |i_A>|i_B>$$

where λ_i are non-negative real numbers satisfying $\sum_i \lambda_i^2 = 1$ known as Schmidt coefficients.

The number of non-zero values of λ_i is called the Schmidt number.

If the Schmidt number is 1, then the quantum state is a product state, otherwise it is an entangled state.

$$|\beta_{00}> = 2^{-1/2}(|00> + |11>)$$
$$|\beta_{01}> = 2^{-1/2}(|00> - |11>)$$
$$|\beta_{10}> = 2^{-1/2}(|01> + |10>)$$
$$|\beta_{11}> = 2^{-1/2}(|01> - |10>)$$

What is the Schimdt number for these states?

Postulates of quantum mechanics

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Quantum observable

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Quantum Dynamics

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• When the physical quantity $\mathcal{A}$ is measured on a system in the normalized state $|\psi>$, the probability $p(a_n)$ of obtaining the (non-degenerate) eigenvalue a_n of the corresponding observable **A** is: $P(a_n) = |<u_n|\psi>|^2$, where $|u_n>$ is the normalized eigenvector of **A** with the eigenvalue a_n.

• If the measurement of the physical quantity $\mathcal{A}$ on the system in the state $|\psi>$ gives the result a_n, the state of the system immediately after the measurement is the normalized projection, $P_n|\psi>/(<\psi|\psi>)^{1/2}$, of $|\psi>$ onto the eigenspace associated with a_n.

Operators

A linear operator A between vector spaces V and U is defined to be a function A: V $\rightarrow$ U which is linear in its inputs:

$$A \sum_i c_i|v_i> = \sum_i c_i A|v_i>$$

we write A|v> to denote A(|v>).

Examples:

- the identity operator I_V:V$\rightarrow$V, I_V|v> = |v>

- the zero operator 0|v> = 0

Suppose V,U and W are vector spaces, and A:V $\rightarrow$ U and B:V$\rightarrow$W are linear operators, then a composition of B with A, BA:V$\rightarrow$W, is defined as (BA)|v> = B(A|v>) (which we abreviate as BA|v>).

Matrix representation

Let A: V → U be a linear operator between vector spaces V and U, with the basis $\mathcal{B}_V = \{|v_1\rangle, |v_2\rangle, \ldots, |v_m\rangle\}$ and $\mathcal{B}_U = \{|u_1\rangle, |u_2\rangle, \ldots, |u_n\rangle\}$.
then for each j in the range 1,…,m, there exist complex numbers A_{1j} through A_{nj} s.t.

$$A\,|v_j\rangle = \sum_i A_{ij}\,|u_i\rangle$$

The matrix whose entries are A_{ij} is said to form a matrix representation of the operator A.

Examples:
Let V be a vector space with the basis {|0>,|1>} (a qubit!)
• the identity operator in matrix representation is given as

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

• the bit and phase flip operators in matrix representation is

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad iY = XZ = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

X, Y and Z are called Pauli operators/matrices

Eigenvalues and eigenvectors

An *eigenvector* of a linear operator A on a vector space V is a non-zero vector $|v\rangle \in V$ s.t.: $A|v\rangle = v|v\rangle$ where $v \in \mathbb{C}$ is called *eigenvalue*.

The *eigenspace* corresponding to eigenvalue v is the set of vectors which have eigenvalue v. When an eigenspace is more than one dimensional we say that it is degenerate.

A *diagonal representation* for an operator A on a vector space V is a representation:

$$A = \sum_i \lambda_i |i\rangle\langle i|$$

where the vectors $|i\rangle$ form an orthonormal set of eigenvectors for A with corresponsing eigenvalues λ_i. Diagonal representations are also known as *orthonormal decompositions*.

An operator is said to be *diagonalizable* if it has diagonal representation.

Example:

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$$

Adjoint and hermitian operators

Let A be a linear operator on a Hilbert space V, then there exists a unique linear operator called *adjoint* or *Hermitian conjugate* A^+ on V s.t. for all $|v\rangle, |w\rangle \in V$,

$$\langle v|(A|w\rangle) = (\langle v|A^+)w\rangle$$

A acting on $|w\rangle$ $\qquad$ A^+ acting on $\langle v|$

Notes:

$(AB)^+ = B^+A^+$

$(A|v\rangle)^+ = \langle v|A^+$ (we define a bra $\langle v| = |v\rangle^+$)

adjoint is anti-linear:

$$\left(\sum_i a_i A_i\right)^+ = \sum_i a_i^* A_i^+$$

$A^+ = (A^*)^T$ (complex conjugate and transpose)

An operator A is called a *Hermitian* operator if $A=A^+$. These operators represent quantum observables.

Eigenvalues of a Hermitian operator are real numbers.

Example: Projector $P_v = |v\rangle\langle v|$ where $|v\rangle \in V$ is an orthonormal state

$P_v = P_v^+$

$P_v = P_v^2$

Normal, unitary and positive operators

Let A be a linear operator on a Hilbert space V, we say it is *normal* if $A^+A = AA^+$.

Are Hermitian operators normal?

A linear operator (or matrix) U is said to be *unitary* if $U^+U = UU^+ = I$.

Are unitary operators normal?

Unitary operators preserve inner product between vectors:

$$(<v|U^+)(U|w>) = <v|U^+U|w> = <v|I|w> = <v|w>$$

A *outer product representation* for a unitary operator U on a vector space V is a representation:

$$U = \sum_i \lambda_i |w_i><v_i|$$

where the vectors $|v_i>$ form an orthonormal basis. The vectors $|w_i>$ are defined as $U|v_i>$ and thus are also orthonormal vectors.

A hermitian operator (or matrix) A is said to be *positive* if $<v|A|v>$ is a real non-negative number for all $|v>$.

If A is *strictly greater than zero* for all $|v> \neq 0$, we say it is *positive definite.*

Example: density matrices are positive definite.