MP 472 Quantum Information and Computation

http://www.thphys.may.ie/staff/jvala/MP472.htm

Outline

Deductive structure of physical sciences
- postulates of quantum mechanics

Quantum bits
- postulate of quantum mechanics regarding quantum states
- essential properties of Hilbert spaces

Quantum operations

Quantum measurement

Postulates of quantum mechanics

Quantum state

- At a fixed time t, the state of a physical system is defined by specifying a ket $|\psi(t)>$ belonging to the state space $\mathcal{H}$.

Quantum observable

- Every measurable physical quantity $\mathcal{A}$ is described by an operator **A** acting on $\mathcal{H}$; this operator is an observable.

Quantum Dynamics

- The time evolution of the state vector $|\psi\ (t)>$ is governed by the Schroedinger equation $i\hbar\ d|\psi(t)>/dt = \mathbf{H}(t)|\psi(t)>$ where **H**(t) is the observable associated with the total energy of the system.

Measurement

- The only possible result of the measurement of a physical quantity $\mathcal{A}$ is one of the eigenvalues of the corresponding observable **A**.

- When the physical quantity $\mathcal{A}$ is measured on a system in the normalized state $|\psi>$, the probability $p(a_n)$ of obtaining the (non-degenerate) eigenvalue a_n of the corresponding observable **A** is: $P(a_n) = |<u_n|\psi>|^2$, where $|u_n>$ is the normalized eigenvector of **A** with the eigenvalue a_n.

- If the measurement of the physical quantity $\mathcal{A}$ on the system in the state $|\psi>$ gives the result a_n, the state of the system immediately after the measurement is the normalized projection, $P_n|\psi>/(<\psi|\psi>)^{1/2}$, of $|\psi>$ onto the eigenspace associated with a_n.

Vector space

A linear space or vector space $\mathcal{V}$ (over a field of complex numbers) is a set of elements, called vectors, with an operation of *addition*, which for each pair of vectors $|\psi>$ and $|\phi>$ specifies a vector $|\psi> + |\phi>$, and an operation of *scalar multiplication*, which for each vector $|\psi>$ and number a ($\in \mathbb{C}$) specifies a vector $a|\psi>$ s.t.

1) $|\psi> + |\phi> = |\phi> + |\psi>$;
2) $|\psi> + (|\phi> + |\chi>) = (|\psi> + |\phi>) + |\chi>$;
3) there is a unique zero vector s.t. $|\psi> + 0 = |\psi>$;
4) $a(|\psi> + |\phi>) = a|\psi> + a|\phi>$;
5) $(a + b)|\psi> = a|\psi> + b|\psi>$;
6) $a(b|\psi>) = (ab)|\psi>$;
7) $1 . |\psi> = |\psi>$;
8) $0 . |\psi> = 0$;

for any vectors $|\psi>$, $|\phi>$ and $|\chi> \in \mathcal{V}$ and numbers a and b ($\in \mathbb{C}$).

Example: N-dimensional space of complex numbers $\mathbb{C}^N$:
a set of N-tuples of complex numbers with addition of two vectors $|\psi>$ and $|\phi>$ and multiplication by a scalar $a \in \mathbb{C}$ are defined as

$$|\psi> = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_N \end{pmatrix} \qquad |\phi> = \begin{pmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_N \end{pmatrix} \qquad |\psi> + |\phi> = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \cdot \\ \cdot \\ \cdot \\ x_N + y_N \end{pmatrix} \qquad a|\psi> = \begin{pmatrix} a x_1 \\ a x_2 \\ \cdot \\ \cdot \\ \cdot \\ a x_N \end{pmatrix}$$

Linear independence

A set of vectors $|\psi_1>, |\psi_2>, \ldots, |\psi_n>$ is *linearly independent* if $\sum_{k=0}^{n} a_k |\psi_k> = 0$ is possible only for scalars $a_1 = a_2 = \ldots = a_n = 0$.

An infinite set of vectors is *linearly independent* if every finite subset is linearly independent.

A vector space is *n-dimensional* if it contains n linearly independent vectors for every positive integer n.

A vector space is *infinite dimensional* if it contains n linearly independent vectors for every $n \in \mathbb{Z}_+$.

A set of vectors $|\psi_1>, |\psi_2>, \ldots, |\psi_n>$ *spans* a vector space if each vector in the space is a linear combination $a_1 |\psi_1> + a_2 |\psi_2> + \ldots + a_n |\psi_n>$ of the vectors $|\psi_k>$ with scalars a_k for k=1,2,…n.

A set of vectors $|\psi_1>, |\psi_2>, \ldots, |\psi_n>$ is a *basis* for a vector space if it is a linearly independent set and spans the space.

Theorem: A vector space is n-dimensional iff it has a basis of n vectors.

Example: a basis of N-dimensional space of complex numbers $\mathbb{C}^N$ (over the field $\mathbb{C}$)

$$|\psi_1> = \begin{pmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix} \qquad |\psi_2> = \begin{pmatrix} 0 \\ 1 \\ . \\ . \\ . \\ 0 \end{pmatrix} \qquad \cdots \qquad |\psi_N> = \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 1 \end{pmatrix}$$

Inner product

An *inner product* or *scalar product* for a vector space is an assignment to each pair of vectors $|\psi\rangle$ and $|\phi\rangle$ of a scalar $\langle\psi|\phi\rangle$ called an inner or scalar product of $|\psi\rangle$ and $|\phi\rangle$, with the properties that for any vectors $|\psi\rangle$,$|\phi\rangle$ and $|\chi\rangle$ and scalar a

1) $\langle\psi|\phi + \chi\rangle = \langle\psi|\phi\rangle + \langle\psi|\chi\rangle$;

2) $\langle\psi| a\phi\rangle = a\langle\psi|\phi\rangle$;

3) $\langle\psi| \phi\rangle = \langle\phi|\psi\rangle^*$;

4) $\langle\psi| \psi\rangle \geqq 0$ with the equality holding only if $|\psi\rangle = 0$.

Theorem (Schwarz's inequality): $|\langle\psi| \phi\rangle| \leq ||\psi|| + ||\phi||$
with equality holding only if $|\psi\rangle$ and $|\phi\rangle$ are linearly dependent

Example:N-dimensional space of complex numbers $\mathbb{C}^N$:

$$\langle\psi|\phi\rangle = \begin{pmatrix} x_1^* & x_2^* & . & . & . & x_N^* \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_N \end{pmatrix} = x_1^*y_1 + x_2^* y_2 + \dots + x_n^* y_n$$

Two vectors are *orthogonal* if there inner product is zero.

Mutually orthogonal vectors of unit length (i.e. norm) are called *orthonormal*.

A vector space with inner product is called *pre-Hilbert space*.

Norm and metric

The non-negative number $|| \psi || = <\psi| \psi>^{1/2}$ is called the *norm* or the *length* on the vector $|\psi>$:

1) $||\psi|| \geqq 0$ with the equality holding only if $|\psi> = 0$;
2) $||a\psi|| = |a| \, ||\psi||$;
3) $||\psi + \phi|| \leqq ||\psi|| + ||\phi||$.

Two vectors are *orthogonal* if their inner product is zero.

Mutually orthogonal vectors of unit length (i.e. norm) are called *orthonormal*.

A vector space with a norm is called *normed space*.

A *metric* on a vector space is a map which assigns to each pair of vectors $|\psi>$ and $|\phi>$ a scalar $\rho \in [0,\infty]$ s.t.

1) $\rho(|\psi>,|\phi>) = 0$ iff $|\psi>=|\phi>$;
2) $\rho(|\psi>,|\phi>) = \rho(|\phi>,|\psi>)$;
3) $\rho(|\psi>,|\chi>) \leqq \rho(|\psi>,|\phi>) + \rho(|\phi>,|\chi>)$ (triangular inequality).

Metric space can be constructed from normed space if we put $\rho(|\psi>,|\phi>) = || \, |\psi>-|\phi> \, ||$;

- we say that the metric is induced by norm.

A vector space with a metric is called *metric space*.

Infinite dimensional spaces

A Hilbert space for a given system (e.g. a harmonic oscillator) can be infinite dimensional, so we need to know how to handle a linear combination of an infinite number of vectors:

Let us first look at a sum $w = \sum_{k=1}^{\infty} z_k$ of an infinite series of complex numbers z_k. The sum is the complex number w if the sequence of partial sums $w_n = \sum_{k=1}^{n} z_k$ converges to w as $n \to \infty$; i.e. $|w - w_n| \to 0$ as $n \to \infty$.

Similarly, convergence of a sequence of vectors $|\psi_n\rangle$ to a limit vector $|\psi\rangle$, i.e. $|\psi_n\rangle \to |\psi\rangle$, means that $\| \, |\psi\rangle - |\psi_n\rangle \, \| \to 0$ as $n \to \infty$.

An infinite linear combination $\sum_{k=1}^{\infty} a_k|\phi_k\rangle$ is defined if the sequence of partial sums $|\psi_n\rangle = \sum_{k=1}^{n} a_k|\phi_k\rangle$ converges; then $|\psi\rangle = \sum_{k=1}^{\infty} a_k|\phi_k\rangle$ means that $|\psi_n\rangle \to |\psi\rangle$ as $n \to \infty$.

An infinite linear combination can be *added* componentwise just as finite linear combinations:

$$|\psi\rangle = \sum_{k=1}^{\infty} a_k|\phi_k\rangle \qquad |\chi\rangle = \sum_{k=1}^{\infty} b_k|\phi_k\rangle \qquad |\psi + \chi\rangle = \sum_{k=1}^{\infty} (a_k + b_k)|\phi_k\rangle$$

Multiplication of an infinite linear combination by a scalar can also be done componentwise.

$$c|\psi\rangle = \sum_{k=1}^{\infty} c\, a_k|\phi_k\rangle$$

Completeness

A *closed* set (of vectors) has the property that if a sequence of vectors $|\psi_n>$ in the set converges then the limit vector $|\psi>$ is in the set.

A *closed* linear manifold is called a *subspace.* Note

- if a linear manifold contains vectors $|\phi_1>, |\phi_2>, \ldots, |\phi_k>, \ldots$ then it contains all finite linear combinations $\sum_{k=1}^{n} a_k|\phi_k>$
- if the linear manifold is a subspace, then it also contains all infinite linear combinations since it contains the limits of sequences of partial sums.

Questions: is there a limit vector in the space for every sequence that should converge or do some sequences fail to converge just because their limit vectors have been left out?

A sequence of vectors $|\psi_n>$ is called a Cauchy sequence if $\| \, |\psi_n> - |\psi_m> \| \rightarrow 0$ as $m,n \rightarrow \infty$.

Every sequence of vectors which converges to a limit is a Cauchy sequence:

$$\| \, |\psi_n> - |\psi_m> \| = \| \, |\psi_n> - |\psi> + \ |\psi> - |\psi_m> \| \leqq \| \, |\psi_n> - |\psi> \| + \| \, |\psi> - |\psi_m> \| \rightarrow 0$$

A space is said to be *complete* if every Cauchy sequence of vectors converges to a limit vector in the space.

A normed space which is *complete* with respect to the metric induced by a norm is called *Banach space*.

Hilbert space

i.e. a vector space with an inner product

A Hilbert space is a pre-Hilbert vector space which is complete with respect to the metric induced by an inner product.

i.e. a Banach space (special case)