MP 472 Quantum Information and Computation

http://www.thphys.may.ie/staff/jvala/MP472.htm

Outline

Open quantum systems

The density operator

Quantum noise (decoherence)

Quantum error correction

- Shor code

Fault-tolerant quantum computation

Shor code

This code which protects against arbitrary error on a single qubit is a combination of the thee qubit bit flip code and three qubit phase flip code.

Encoding

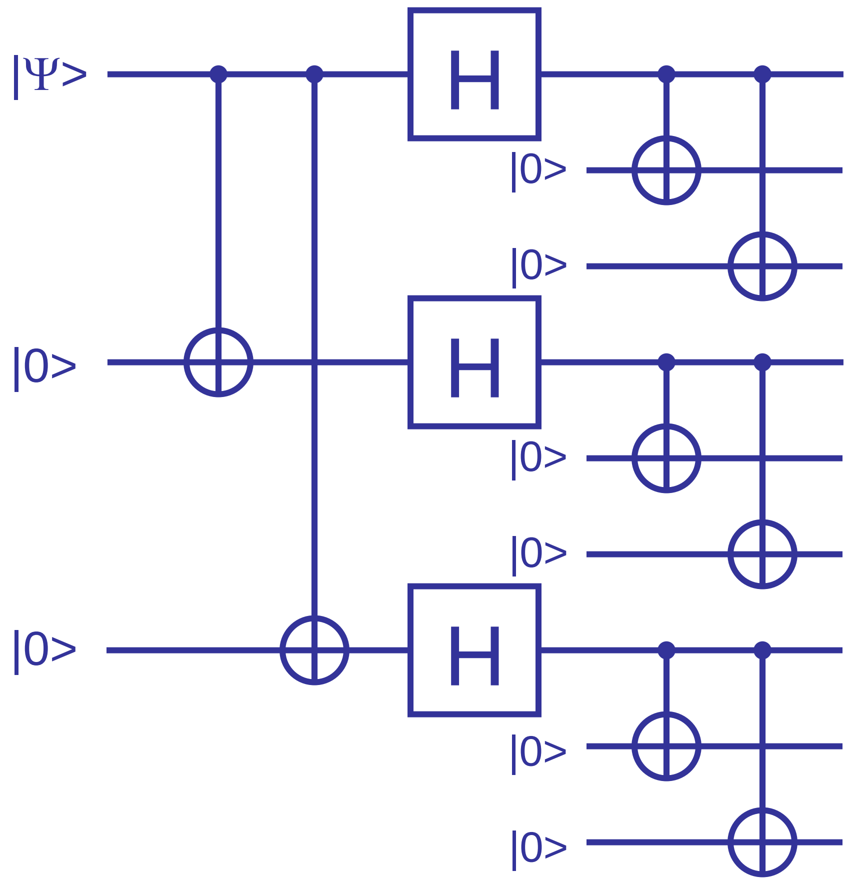
$$|0\rangle \rightarrow |0_L\rangle = (1/2)^{3/2}[(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)]$$

$$|1\rangle \rightarrow |1_L\rangle = (1/2)^{3/2}[(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)]$$

The qubit is first encoded using the phase flip code and then is encoded using the bit flip code. The result is the nine-qubit Shor code

This method of encoding which uses hierarchy of levels is known as

concatenation.



Shor code: bit flip error

Initial state

$|\psi\rangle = (c_0/2)^{3/2}[(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)] + (c_1/2)^{3/2}[(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)]$

Error

Let us assume that the bit flip error happens on the 4^{th} qubit, so the resulting state (after the syndrome measurement) would be

$|\psi'\rangle = (c_0/2)^{3/2}[(|000\rangle+|111\rangle)(|100\rangle+|011\rangle)(|000\rangle+|111\rangle)] + (c_1/2)^{3/2}[(|000\rangle-|111\rangle)(|100\rangle-|011\rangle)(|000\rangle-|111\rangle)]$

Error syndrome measurement

The full set of the bit flip syndromes is obtained by measuring the following six observables:

$Z_1Z_2 \quad Z_2Z_3 \qquad Z_4Z_5 \quad Z_5Z_6 \qquad Z_7Z_8 \quad Z_8Z_9$

which detect whether the neighboring qubits on the 1st, 2nd and 3rd three-qubit block have the same or different value. The result in our specific example is

$+1 \quad +1 \qquad -1 \quad +1 \qquad +1 \quad +1$

and indicates that the bit flip error happened on the fourth qubit (the first qubit of the second block).

Error recovery

The state is now easy to fix by applying X operation on the fourth qubit, i.e. X_4.

Shor code: phase flip error

Initial state

$|\psi\rangle = (c_0/2)^{3/2}[(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)] + (c_1/2)^{3/2}[(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)]$

Error

Let us assume that the phase flip error happens on the 4^{th} qubit, so the resulting state (after the syndrome measurement) would be

$|\psi'\rangle = (c_0/2)^{3/2}[(|000\rangle+|111\rangle)(|000\rangle-|111\rangle)(|000\rangle+|111\rangle)] + (c_1/2)^{3/2}[(|000\rangle-|111\rangle)(|000\rangle+|111\rangle)(|000\rangle-|111\rangle)]$

Error syndrome measurement

Since now the syndrom measurement has to identify on which three-qubit block the phase flip happened, the full set of the phase flip syndromes is obtained by measuring the following two observables:

$$X_1X_2X_3X_4X_5X_6 \qquad X_4X_5X_6X_7X_8X_9$$

which together detect whether the phase flip occurs on the 1^{st}, 2^{nd} or 3^{rd} three-qubit block.
The result in our specific example is

$$-1 \qquad -1$$

and indicates that the phase flip error happened on the 2^{nd} three-qubit block.

Error recovery

The state can now be fixed by applying Z operation on each qubit of the second block, i.e. by applying

$$Z_4Z_5Z_6$$

Classical linear codes

A linear code C encoding k bits of information into a n bit code space is specified by n-by-k generator matrix G whose entries are elements of $\mathbb{Z}_2 = \{0,1\}$

message $\quad x \longrightarrow$ encoded message $y = G(x) = Gx$

(all operations are mod2)

A code which uses n bits to encode k bits of information is an [n,k] code.

Example

a) three-bit repetition code, i.e. [3,1] code

$$G = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad G(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = (0, 0, 0) \qquad G(1) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = (1, 1, 1)$$

(0, 0 ,0): notation for the codeword 000

(1, 1, 1): notation for the codeword 111

b) [6,2] code

$$G = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$

G(0,0) = (0,0,0,0,0,0) $\qquad$ G(0,1) = (0,0,0,1,1,1)

G(1,0) = (1,1,1,0,0,0) $\qquad$ G(1,1) = (1,1,1,1,1,1)

The set of possible codewords for the code corresponds to the vector space spanned by the columns of G, that is, the columns of G have to be linearly independent.

Classical linear codes

Advantage of linear codes

A general code [n,k] requires 2^k codewords each of length n to specify the encoding, i.e. $n2^k$ bits are needed to describe the code.

A linear code [n,k] requires only kn bits of the generator matrix G, and thus exponentially saves the memory required.

Encoding

$$y = G x \qquad \# \text{ of operation} = O(kn)$$

Error correction

Introducing **parity check matrix H:**

In this definition, an [n,k] code is defined to consist of all n-element vectors x over $\mathbb{Z}_2$ s.t.

$$H x = 0$$

Where H is an (n-k)-by-n matrix known as parity check matrix, with entries from zeros and ones, i.e. the code is defined by a kernel of H.

Parity check matrix

Connection between G and H

$H \rightarrow G$:
Pick k linearly independent vectors $y_1, \ldots, y_k$ spanning the kernel of H, and set G to have columns $y_1, \ldots, y_k$.

$G \rightarrow H$:
Pick n-k linearly independent vectors $y_1, \ldots, y_{n-k}$ orthogonal to the columns of G, and set the rowns of H to be $y_1^T, \ldots, y_k^T$.

Example

[3,1] repetition code

$$G = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Pick 3-1=2 linearly independent vectors orthogonal to the columns of G, e.g. (1,1,0) and (0,1,1), and define the parity check matrix as

$$H = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Homework:
Check that in the present example Hx=0 only for codewords (0,0,0) and (1,1,1).

Error correction

Lets assume the encoding y = Gx.

Error e however corrupts y giving y' = y + e (bitwise addition).

Because Hy = 0 for all codewords, then Hy' = He

this is the error syndrome !!

Example

[3,1] repetition code: (0) → (0,0,0)

(1) → (1,1,1)

Let us say the output of a noisy channel is y'=(1,0,0) then

$$H\,y' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

all operations are mod2

Homework:
Check this for all other possible corrupted codewords:
(0,0,1),(0,1,0), … ,(1,1,0).