MP 472 Quantum Information and Computation

http://www.thphys.may.ie/staff/jvala/MP472.htm

Outline

Open quantum systems

The density operator

Quantum noise (decoherence)

Quantum error correction
- Three-qubit bit flip code
- Three qubit phase flip code
- Shor code

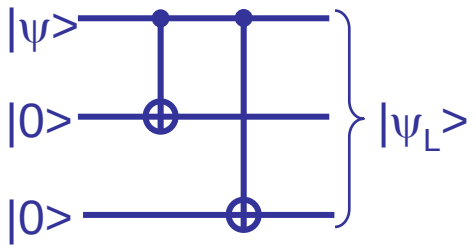
Fault-tolerant quantum computation

Three qubit bit flip code (review)

Let us consider a symmetric binary bit-flip (X) qubit flip channel.

Encoding of a qubit $|\psi\rangle = c_0 |0\rangle + c_1 |1\rangle$ using repetition code:

$$|\psi_L\rangle = c_0 |0_L\rangle + c_1 |1_L\rangle = c_0 |000\rangle + c_1 |111\rangle$$



Error detection and syndrome analysis:

Four error syndroms corresponding to the projectors:

$P_0 = \|000\rangle\langle 000\| + \|111\rangle\langle 111\|$	no error
$P_1 = \|100\rangle\langle 100\| + \|011\rangle\langle 011\|$	bit flip on first qubit
$P_2 = \|010\rangle\langle 010\| + \|101\rangle\langle 101\|$	bit flip on second qubit
$P_3 = \|001\rangle\langle 001\| + \|110\rangle\langle 110\|$	bit flip on third qubit

The result of the measurement is $\langle\psi|P_k|\psi\rangle = 1$ iff the error happened on the qubit k, otherwise it is zero. Measurement do not affect the qubit state!

A different look at syndrome measurement

Instead of measuring P_k's, we perform two bit string parity measurements:

$Z_1Z_2 = Z \otimes Z \otimes I$ $\qquad$ $Z_2Z_3 = I \otimes Z \otimes Z$

Recovery:

Error syndrome is used to recover the original quantum state by applying bit-flip onto the qubit k.

Three qubit phase flip code: encoding

This error, given by the quantum operation

$$\rho \rightarrow (1\text{-}p)\,\rho + p\,Z\rho Z\,,$$

flips the relative phase between |0> and |1> with the probability p.

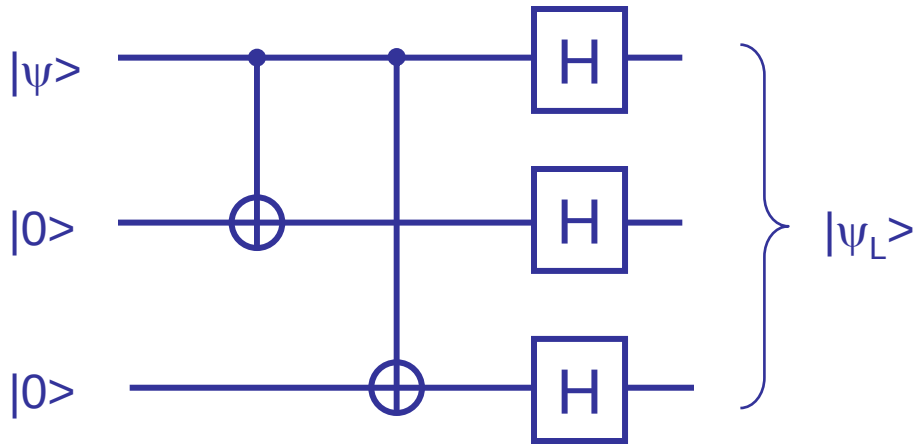
We know that HZH = X (H = Hadamard gate) i.e. the phase flip acts as the bit flip in the basis $|+\rangle = (1/2)^{1/2}(|0\rangle + |1\rangle)$ and $|-\rangle = (1/2)^{1/2}(|0\rangle - |1\rangle)$.

This suggests that the appropriate encoding for the phase flip error is

$|0\rangle \rightarrow |0_L\rangle = |+++\rangle$

$|1\rangle \rightarrow |1_L\rangle = |- - -\rangle$

$|\psi_L\rangle = c_0\,|+++\rangle + c_1\,|- - -\rangle$



Three qubit phase flip code: error correction

Error detection and syndrome analysis:

Error is detected using the same projective measurements as for the bit flip error detection conjugated with Hadamard rotations:

$$P_j' = H^{\otimes 3} P_j H^{\otimes 3}$$

Alternatively, the syndrome measurements can be performed using the observables

$$H^{\otimes 3} Z_1 Z_2 H^{\otimes 3} = X_1 X_2 \qquad H^{\otimes 3} Z_2 Z_3 H^{\otimes 3} = X_2 X_3$$

Measurement of these observables corresponds to comparing the *signs* of qubits, for example X_1X_2 gives eigenvalue +1 for $|{+}{+}{>}\otimes$ (.) and $|{-}{-}{>}\otimes$ (.), and -1 for $|{+}{-}{>}\otimes$ (.) and $|{-}{+}{>}\otimes$ (.).

Recovery:

Error correction is completed with the recovery operation, which is the Hadamard conjugated recovery operation of the bit flip code.

For example, if the phase flip (i.e. flip from |+> and |-> and vice versa) was detected on the second qubit, then the recovery operation is $HX_2H = Z_2$.

Remark:

This code for the phase flip channel obviously has the same characteristics, i.e. the minimum fidelity etc., as the code for the bit flip channel.
These two channels are unitarily equivalent, i.e. are related to each other by a unitary transformation.

Three qubit phase flip code: example

Phase flip error

$$\rho \rightarrow \rho_e = (1\text{-}3p)\,\rho + p\,Z_1\rho Z_1 + p\,Z_2\rho Z_2 + p\,Z_3\rho Z_3$$

creates a state mixed from the original state $\rho = |\psi\rangle\langle\psi|$ with the probability (1-3p), where

$$|\psi\rangle = c_0|+++\rangle + c_1|- - -\rangle$$

and from three erroneous states $\rho_k = Z_k\rho Z_k = |\psi_k\rangle\langle\psi_k|$ with some (small) probability p (for each k=1,2,3), where

$|\psi_1\rangle = c_0|-++\rangle + c_1|+ - -\rangle$ $\quad$ $|\psi_2\rangle = c_0|+-+\rangle + c_1|-+ -\rangle$ $\quad$ $|\psi_3\rangle = c_0|++-\rangle + c_1|--+\rangle$

Error syndrome measurement

By measuring the error syndrom using the observables X_1X_2 and X_2X_3, we obtain the result:

$$X_1X_2 = -1 \text{ and } X_2X_3 = -1$$

the measurement , which revealed the phase flip error on the second qubit, collapsed the mixed state to the pure state (!) given as $\rho' = |\psi'\rangle\langle\psi'|$ where

$$|\psi'\rangle = c_0|+-+\rangle + c_1|- + -\rangle$$

Error recovery

The state is now easy to fix by applying Z operation on the second qubit, i.e. Z_2.

Shor code

This code which protects against arbitrary error on a single qubit is a combination of the thee qubit bit flip code and three qubit phase flip code.

Encoding

$$|0\rangle \rightarrow |0_L\rangle = (1/2)^{3/2}[(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)]$$

$$|1\rangle \rightarrow |1_L\rangle = (1/2)^{3/2}[(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)]$$

The qubit is first encoded using the phase flip code and then is encoded using the bit flip code. The result is the nine-qubit Shor code

This method of encoding which uses hierarchy of levels is known as

concatenation.