

# MP 472 Quantum Information and Computation

<http://www.thphys.may.ie/staff/jvala/MP472.htm>

## Outline

Open quantum systems

The density operator

Quantum noise (decoherence)

### **Quantum error correction**

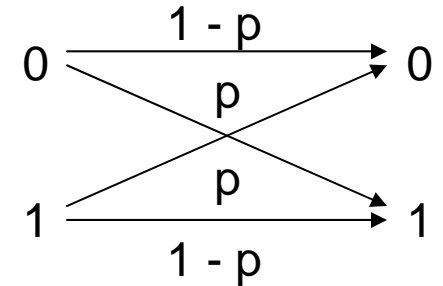
- Three-qubit bit flip code
- Three qubit phase flip code
- Shor code

Fault-tolerant quantum  
computation

# Classical error correction

## Example

Let us consider a bit flip error with probability  $p$  (symmetric binary channel).



If we use one physical bit to represent one bit of information, then the error will destroy the information with probability  $p$ .

But we can encode the information into several physical bits, so the error, occurring with not too high probability  $p$ , will not be able to flip the logical bit even if it flips some of the physical bits of the code.

Encoding using repetition code

$0 \rightarrow 000$	logical bit
$1 \rightarrow 111$	

For example, after sending the logical qubit through the channel, we get 100 as the output. For small  $p$ , we can conclude that the first bit was flipped and that the input bit was 0.

The probability that two or more bits are flipped is

$$p_{\text{error}} = 3 p^2 (1-p) + p^3 = 3 p^3 + 2 p^2$$

If  $p < \frac{1}{2}$ , then the encoded information is transmitted more reliably:  $p_{\text{error}} < p$ .

# Quantum error correction

## Difficulties

Quantum information faces some nontrivial difficulties which have no analog in classical information processing:

1) No-cloning:

duplicating quantum states to implement repetition code is impossible.

2) Errors are continuous:

a continuum of different errors can occur on a single qubit; determining which error occurred in order to correct it would require infinite precision (i.e. resources).

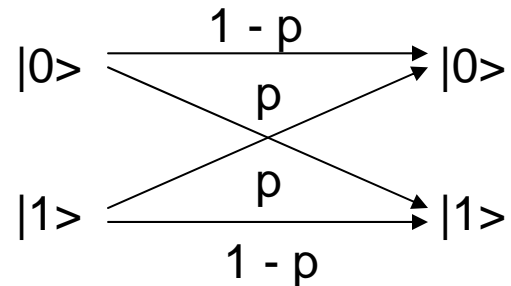
3) Measurement destroys quantum information:

Classical information can be observed without destroying it and then decoded, but quantum information is destroyed by measurement and can not be recovered.

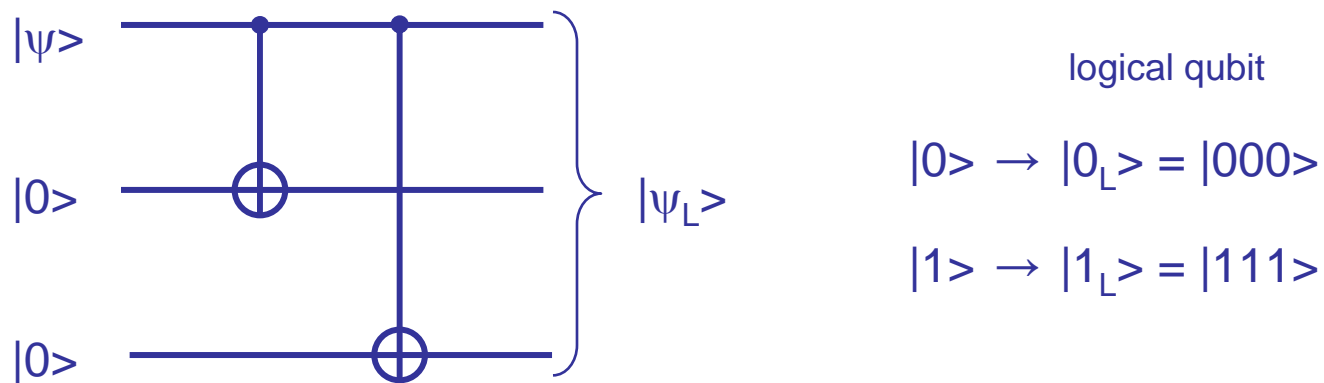
Despite these difficulties, **quantum error correction is possible.**

## Three qubit bit flip code: encoding

Let us consider a qubit flip error (i.e. X) with probability  $p$  (symmetric binary quantum channel):



Encoding of a qubit  $|\psi\rangle = c_0 |0\rangle + c_1 |1\rangle$  using repetition code:



$$|\psi_L\rangle = c_0 |0_L\rangle + c_1 |1_L\rangle = c_0 |000\rangle + c_1 |111\rangle$$

## Three qubit bit flip code: error correction

### Error detection and syndrome analysis:

We need to measure what error occurred on the quantum state, i.e. error syndrom. For bit flip error there are four error syndroms corresponding to the projectors:

$$\begin{aligned} P_0 &= |000\rangle\langle 000| + |111\rangle\langle 111| && \text{no error} \\ P_1 &= |100\rangle\langle 100| + |011\rangle\langle 011| && \text{bit flip on first qubit} \\ P_2 &= |010\rangle\langle 010| + |101\rangle\langle 101| && \text{bit flip on second qubit} \\ P_3 &= |001\rangle\langle 001| + |110\rangle\langle 110| && \text{bit flip on third qubit} \end{aligned}$$

Assuming the error happens on the first qubit, so the corrupted state is

$$|\psi\rangle = c_0 |100\rangle + c_1 |011\rangle$$

then  $\langle\psi|P_1|\psi\rangle = 1$  indicates the bit flip happened on the first qubit. Note that it does not destroy the qubit superposition, i.e. the state after the measurement is the same as the state before the syndrom measurement, i.e.

$$|\psi\rangle = c_0 |100\rangle + c_1 |011\rangle$$

so we learn only about what error occurred but no information about the state itself.

### Recovery:

Error syndrome is used to recover the original quantum state.

For example, the error syndrom 1 (as above), implies we need to apply bit flip on the first qubit to correct the error. Similarly, other syndroms imply different recovery procedure.

## Three qubit bit flip code: fidelity analysis

### Error analysis:

This error correction works perfectly, if bit flips occur on one or fewer of the three Qubits (with the probability  $(1-p)^3 + 3p(1-p)^2$  like for the classical repetition code).

The probability of an error which remains uncorrected is then  $3p^2 - 2p^3$ .

However, the effect of an error on a state depends on the state also. To analyze the errors properly, we use the fidelity.

### Example:

The objective (of the error correction) is to increase the fidelity to its maximum. Suppose the bit flip error channel, and  $|\psi\rangle$  as the state of interest.

**Without using the error correcting code**, the state after the error channel is

$$\rho \rightarrow (1-p) \rho + p X\rho X$$

and the fidelity is  $F = \langle \psi | \rho | \psi \rangle^{1/2} = [(1-p) + p \langle \psi | X | \psi \rangle \langle \psi | X | \psi \rangle]^{1/2}$

and since the second term is nonnegative and equals zero for  $|\psi\rangle = |0\rangle$ , the minimum fidelity is  $F = (1-p)^{1/2}$ .

**Using the three qubit bit flip code**, the state after passing the error channel is

$$\rho \rightarrow [(1-p)^3 + 3p(1-p)^2] \rho + \dots$$

and the fidelity is  $F = \langle \psi | \rho | \psi \rangle^{1/2} \geq [(1-p)^3 + 3p(1-p)^2]^{1/2} = (1 - 3p^2 + 2p^3)^{1/2}$

so the fidelity is improved by using the error correcting code provided  $p < 1/2$ .

## Three qubit bit flip code: towards generalization

### A different look at syndrome measurement

Instead of measuring  $P_0$ ,  $P_1$ ,  $P_2$ , and  $P_3$ , we perform two measurements

$$Z_1Z_2 = Z \otimes Z \otimes I \qquad Z_2Z_3 = I \otimes Z \otimes Z$$

Each of these observables has eigenvalue +1 and -1, so both measurements provide the total of two bits of information (four possible syndromes) without revealing the qubit state (i.e. without collapsing it).

The first measurement,  $Z_1Z_2$ , can be seen as comparing whether the first and second qubit are the same; the spectral decomposition shows that

$$Z_1Z_2 = (|00\rangle\langle 00| + |11\rangle\langle 11|) \otimes I - (|01\rangle\langle 01| + |10\rangle\langle 10|) \otimes I$$

the observable,  $Z_1Z_2$ , corresponds to two projective measurements with eigenvalue +1 if both qubits are the same or -1 if they are different.

Similarly,  $Z_2Z_3$  compares values of the second and third qubit.

By combining both measurements, we can determine whether a bit flip error occurred on the first, second or third qubit:

no error	$Z_1Z_2 = +1$	$Z_2Z_3 = +1$
error on 1 <sup>st</sup>	$Z_1Z_2 = -1$	$Z_2Z_3 = +1$
error on 2 <sup>nd</sup>	$Z_1Z_2 = -1$	$Z_2Z_3 = -1$
error on 3 <sup>rd</sup>	$Z_1Z_2 = +1$	$Z_2Z_3 = -1$

## Three qubit phase flip code: encoding

This error, given by the quantum operation

$$\rho \rightarrow (1-p)\rho + p Z\rho Z ,$$

flips the relative phase between  $|0\rangle$  and  $|1\rangle$  with the probability  $p$ .

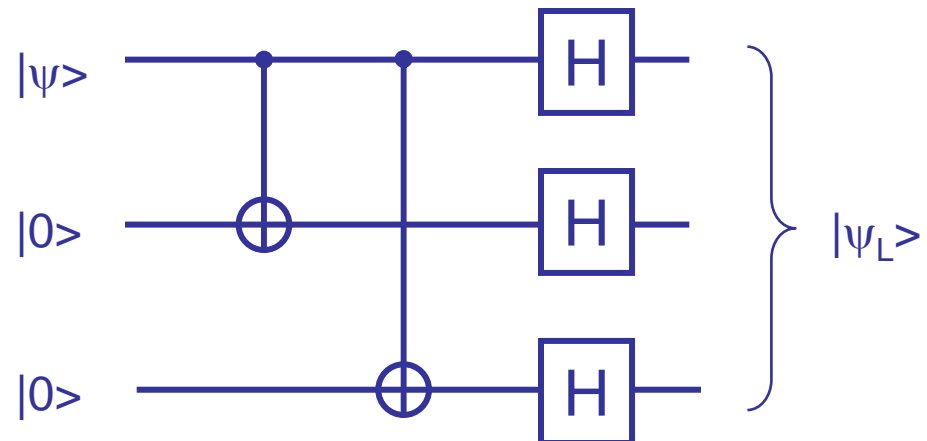
We know that  $HZH = X$  ( $H =$  Hadamard gate) i.e. **the phase flip acts as the bit flip** in the basis  $|+\rangle = (1/2)^{1/2}(|0\rangle + |1\rangle)$  and  $|-\rangle = (1/2)^{1/2}(|0\rangle - |1\rangle)$ .

This suggests that the appropriate encoding for the phase flip error is

$$|0\rangle \rightarrow |0_L\rangle = |+++ \rangle$$

$$|1\rangle \rightarrow |1_L\rangle = |--- \rangle$$

$$|\psi_L\rangle = c_0 |+++ \rangle + c_1 |--- \rangle$$



## Three qubit phase flip code: error correction

### Error detection and syndrome analysis:

Error is detected using the same projective measurements as for the bit flip error detection conjugated with Hadamard rotations:

$$P_j' = H^{\otimes 3} P_j H^{\otimes 3}$$

Alternatively, the syndrome measurements can be performed using the observables

$$H^{\otimes 3} Z_1 Z_2 H^{\otimes 3} = X_1 X_2 \quad H^{\otimes 3} Z_2 Z_3 H^{\otimes 3} = X_2 X_3$$

Measurement of these observables corresponds to comparing the *signs* of qubits, for example  $X_1 X_2$  gives eigenvalue +1 for  $|++\rangle \otimes (.)$  and  $|--\rangle \otimes (.)$ , and -1 for  $|+ -\rangle \otimes (.)$  and  $|- +\rangle \otimes (.)$ .

### Recovery:

Error correction is completed with the recovery operation, which is the Hadamard conjugated recovery operation of the bit flip code.

For example, if the phase flip (i.e. flip from  $|+\rangle$  and  $|-\rangle$  and vice versa) was detected on the second qubit, then the recovery operation is  $H X_2 H = Z_2$ .

### Remark:

This code for the phase flip channel obviously has the same characteristics, i.e. the minimum fidelity etc., as the code for the bit flip channel.

These two channels are unitarily equivalent, i.e. are related to each other by a unitary transformation.

# Shor code

This code which protects against arbitrary error on a single qubit is a combination of the three qubit bit flip code and three qubit phase flip code.

## Encoding

$$|0\rangle \rightarrow |0_L\rangle = (1/2)^{3/2}[(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)]$$

$$|1\rangle \rightarrow |1_L\rangle = (1/2)^{3/2}[(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)]$$

The qubit is first encoded using the phase flip code and then is encoded using the bit flip code.

This method of encoding which uses hierarchy of levels is known as

**concatenation.**

