MP 472 Quantum Information and Computation

http://www.thphys.may.ie/staff/jvala/MP472.htm

Outline

Open quantum systems

The density operator

Quantum noise (decoherence)

Quantum error correction
- Distance measures for quantum information
- Three-qubit bit flip code
- Three qubit phase flip code
- Shor code

Fault-tolerant quantum computation

Overview of distance measures for quantum information

How different are two items of information?

Static measures: how different are two quantum states?

Dynamic measures: how well has the information preserved during dynamics?

Static measures

1) Trace distance

The trace distance is a metric on the space of density operators

$$D(\rho_1,\rho_2) = (1/2)\mathrm{tr}|\rho_1 - \rho_2| \qquad \text{where } |A| = (A^+A)^{1/2}$$

Example: two single qubit states $\rho_1 = (1/2)(I + r_1.\sigma)$, $\rho_2 = (1/2)(I + r_1.\sigma)$

$D(\rho_1,\rho_2) = (1/2)\,\mathrm{tr}|\rho_1 - \rho_2| = (1/4)\,\mathrm{tr}\,|(r_1 - r_2).\sigma|$

$(r_1 - r_2).\sigma$ has the eigenvalues $\pm|r_1 - r_2|$, so $(r_1 - r_2).\sigma$ is $2|r_1 - r_2|$, giving

$D(\rho_1,\rho_2) = (1/2)\,\mathrm{tr}|r_1 - r_2|$

that is, the distance between two single qubit states is equal to one half of of the Euclidean distance between them on the Bloch sphere.

Homework:

What is the trace distance between the density operators:

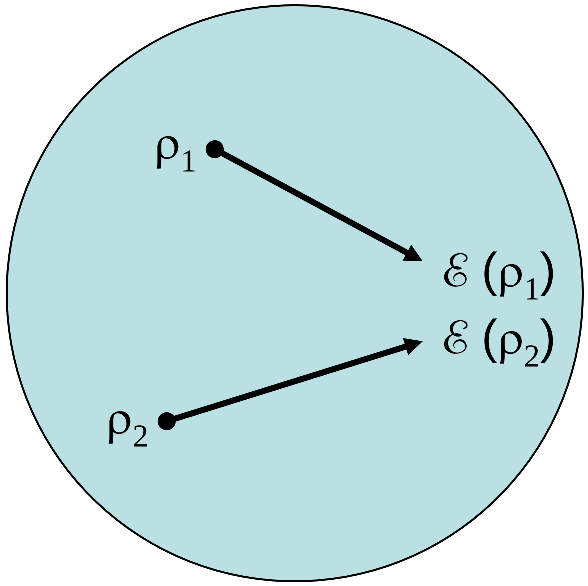
a) (3/4)|0><0| + (1/4)|1><1| and (2/3)|0><0| + (1/3)|1><1|

b) (3/4)|0><0| + (1/4)|1><1| and (2/3)|+><+| + (1/3)|–><–|

where $|\pm\rangle = (1/2)^{1/2}(|0\rangle \pm |1\rangle)$

Overview of distance measures for quantum information

Theorem – trace preserving operations are contractive:
Suppose $\mathcal{E}$ is a trace preserving operation. Let ρ and σ be density operators. Then



$$D(\mathcal{E}(\rho),\mathcal{E}(\sigma)) \leqq D(\rho,\sigma)$$

i.e. distinct states appear closer to each other if only a partial information about them is available.

Remark – unitary invariance of the trace distance:

$$D(U\rho U^+,U\sigma U^+) = D(\rho,\sigma)$$

Theorem – strong convexity of the trace distance:
Let $\{p_i\}$ and $\{q_i\}$ be probability distributions over the same index set, and ρ_i and σ_i be density operators, also with indices from the same set. Then

$$D(\Sigma_i p_i \rho_i, \Sigma_i q_i \sigma_i) \leqq D(p_i,q_i) + \Sigma_i p_i D(\rho_i,\sigma_i)$$

where $D(p_i,q_i)$ is the classical trace distance between the probbaility distributions $\{p_i\}$ and $\{q_i\}$.

Overview of distance measures for quantum information

2) Fidelity

$$F(\rho_1,\rho_2) = tr(\rho_1^{1/2}\rho_2\,\rho_1^{1/2})^{1/2}$$

is not a metric on the space of density operators but it is a good distance measure

Example: fidelity between pure state $|\psi>$ and mixed state ρ

$$F(|\psi><\psi|, \rho) = tr[(|\psi><\psi|)^{1/2}\rho\,(|\psi><\psi|)^{1/2}]^{1/2} = tr(|\psi><\psi|\rho|\psi><\psi|)^{1/2}$$

$$= tr(<\psi|\rho|\psi>|\psi><\psi|)^{1/2} = (<\psi|\rho|\psi>)^{1/2}$$

that is, the fidelity is the square root of the overlap between $|\psi>$ and ρ.

Properties

Unitary invariance:

$$F(U\rho_1U^+,U\rho_2U^+) = F(\rho_1,\rho_2)$$

Symmetry in the inputs:

$$F(\rho_1, \rho_2) = F(\rho_2,\rho_1)$$

Boundedness:

$$0 \leqq F(\rho_2,\rho_1) \leqq 1$$

$F(\rho_1,\rho_2)=0$ iff ρ_1 and ρ_2 have support on orthogonal subspaces (i.e. these are perfectly distinguishable states)

$F(\rho_1,\rho_2)=1$ iff $\rho_1 = \rho_2$

Overview of distance measures for quantum information

Theorem – Monotonicity of the fidelity:
Suppose $\mathcal{E}$ is a trace preserving operation. Let ρ and σ be density operators. Then

$$F(\mathcal{E}(\rho),\mathcal{E}(\sigma)) \geqq F(\rho,\sigma)$$

Remark – unitary invariance of the fidelity:

$$F(U\rho U^+,U\sigma U^+) = F(\rho,\sigma)$$

Theorem – strong concativity of the trace distance:
Let $\{p_i\}$ and $\{q_i\}$ be probability distributions over the same index set, and ρ_i and σ_i be density operators, also with indices from the same set. Then

$$F(\Sigma_i p_i \rho_i, \Sigma_i q_i \sigma_i) \geqq \Sigma_i (p_i q_i)^{1/2} F(\rho_i,\sigma_i)$$

This property is similar (though not strictly analogous) to the strong convexity of the trace distance.

Remark – relation between the trace distance and the fidelity:

$$1 – F(\rho,\sigma) \leqq D(\rho,\sigma) \leqq [1 - F(\rho_i,\sigma_i)^2]^{1/2}$$

Qualitatively, the trace distance and the fidelity are equivalent measures of distance between quantum states.

Overview of distance measures for quantum information

Dynamic measures

How well does a quantum channel preserves information?

Example: phase damping channel $\rho \rightarrow \mathcal{E}(\rho) = (1-p)\rho + pZ\rho Z$

How well the state $|\psi\rangle$ is preserved by the depolarization channel?

$$F(|\psi\rangle\langle\psi|, \mathcal{E}(|\psi\rangle\langle\psi|)) = \{\langle\psi|[(1-p)I|\psi\rangle\langle\psi| + pZ|\psi\rangle\langle\psi|Z]|\psi\rangle\}^{1/2}$$

$$= [(1 - p) + p\langle\psi|Z|\psi\rangle^2]^{1/2}$$

The higher the probability of depolarizing, the lower the fidelity.

In reality, we do not know the initial state of the system in advance, so we have to quantify the worst case scenario

$$F_{min}(\mathcal{E}) = \min_{|\psi\rangle} F(|\psi\rangle\langle\psi|, \mathcal{E}(|\psi\rangle\langle\psi|))$$

For the phase damping channel, the second term is non-negative and equals to zero when $|\psi\rangle = (1/2)^{1/2}(|0\rangle + |1\rangle)$, so the minimal fidelity for this channel is

$$F_{min}(\mathcal{E}) = (1-p)^{1/2}$$

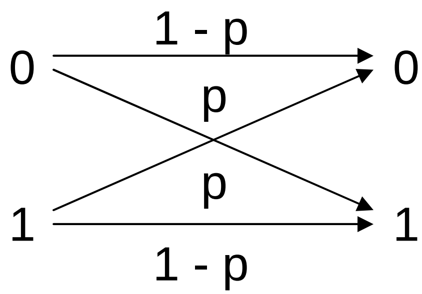
Remark: allowing mixed states as initial states does not change F_{min} – a consequence of strong concativity:

$$F(\rho, \mathcal{E}(\rho)) = F(\Sigma_i \lambda_i |i\rangle\langle i|, \Sigma_i \lambda_i \mathcal{E}(|i\rangle\langle i|)) \geqq \Sigma_i \lambda_i F(|i\rangle\langle i|, \mathcal{E}|i\rangle\langle i|)$$

Classical error correction

Example

Let us consider a bit flip error with probability p (symmetric binary channel).



If we use one physical bit to represent one bit of information, then the error will destroy the information with probability p.

But we can encode the information into several physical bits, so the error, occurring with not too high probability p, will not be able to flip the logical bit even if it flips some of the physical bits of the code.

Encoding using repetition code $0 \rightarrow 000$, $1 \rightarrow 111$ (logical bit)

For example, after sending the logical qubit through the channel, we get 100 as the output. For small p, we can conclude that the first bit was flipped and that the input bit was 0.

The probability that two or more bits are flipped is

$$p_{error} = 3\,p^2\,(1-p) + p^3 = 3\,p^3 + 2\,p^2$$

If $p < ½$, then the encoded information is transmitted more reliably: $p_{error} < p$.

Quantum error correction

Difficulties

Quantum information faces some nontrivial difficulties which have no analog in classical information processing:

1) No-cloning:
duplicating quantum states to implement repetition code is impossible.

2) Errors are continuous:
a continuum of different errors can occur on a single qubit; determining which error occurred in order to correct it would require infinite precision (i.e. resources).

3) Measurement destroys quantum information:
Classical information can be observed without destroying it and then decoded, but quantum information is destroyed by measurement and can not be recovered.

Despite these difficulties, quantum error correction is possible.