MP 472 Quantum Information and Computation

http://www.thphys.may.ie/staff/jvala/MP472.htm

Outline

Open quantum systems

The density operator

Quantum noise (decoherence)
- Applications of quantum operations
  - Dynamics of open quantum systems
  - Quantum process tomography

Quantum error correction

Fault-tolerant quantum computation

Main approaches to dynamics of open quantum systems

Master equation approaches (density matrix dynamics)

Liouville-von Neumann equation (generalized Schroedinger equation) with for example

- Lindblad superoperator;
- Redfield superoperator;
- etc.

Quantum trajectory approaches (stochastic wavefunction dynamics)

Stochastic dynamics whose average (over trajectories) reproduces the system dynamics as given by master equation

- Stratonovich approach (to integration of stochastic equations);
- Ito approach.

Hamiltonian approaches

Full system-environment dynamics with complete or effective description of the environment

- system-environment Hamiltonian dynamics (works for small environment);
- surrogate Hamiltonian approach (effective bath, works for short time).

Liouville-von Neumann equation

a generatized Schroedinger equation, a master equation that properly describes non-unitary dynamics of an open quantum system

$$d\rho/dt = -(i/\hbar)\,[H, \rho] + \mathcal{L}(\rho)$$

unitary or
coherent term

dissipative term

where $\mathcal{L}(\rho)$ is a dissipative super-operator (an “operator” that acts on operators), H is the Hamiltonian, ρ is a density matrix of the system.

Generally, $\mathcal{L}(\rho)$ can be given in the Lindblad form

$$\mathcal{L}(\rho) = \Sigma_k\, \lambda_k\, [2L_k\rho L_k^{+} - \{L_k^{+}L_k, \rho\}]$$

where $\{x,y\} = xy + yx$ is an anticommutator, and L_k are the Lindblad operators, generators of dissipative dynamics representing system-environment interaction, and λ is a given rate constant.

Liouville-von Neumann equation: example

Gaussian pure dephasing (phase flip error) of a harmonic oscillator

We make the following choice: $L_k = L = H = H^+ = \hbar\omega(a^+a + 1/2)$ (Gaussian process)

$$d\rho/dt = -(i/\hbar)\,[H, \rho] + \lambda[2H\rho H + \{HH,\rho\}] = -(i/\hbar)\,[H, \rho] - \lambda\,[H,[H, \rho]]$$

the evolution operator is given as ($H = a^+a + 1/2$ is not explicitely time dependent):

$$\rho(t) = \exp\left(\int_0^t \{-(i/\hbar)\,[H, \rho_0] - \underbrace{\lambda\,[H,[H, \rho_0]]}_{\text{Gaussian term}}\}\,dt \right) = \exp(\{-(i/\hbar)\,[H, \rho_0] - \lambda\,[H,[H, \rho_0]]\}\,t)$$

In the energy representation, the propagator becomes (for the components of ρ)

$$\rho_{nm}(t) = \exp[-\,(i/\hbar)\,\omega_{mn} t - \lambda\,\omega^2_{mn}\,t]\,\rho_{mn}(0) \qquad \omega_{mn} = \omega\,(m-n)$$

- diagonal elements of ρ, i.e. populations ρ_{mm}, are constant with time;
- phase of off diagonal elements of ρ, i.e. coherences, oscillate with time due to coherent dynamics; the higher is the energy difference between the levels (m-n), the linearly faster are the oscillations;
- moduli of coherences decay with time due to dephasing, the higher is the energy difference between the levels (m-n), the quadratically faster is the decay.

Quantum state tomography of a single qubit

Experimental determining an unknown quantum state (of a single qubit) ρ:

- using a single copy of ρ it is impossible to characterize the state (recall measurement of non-orthogonal states)
- with many copies (from repeated preparation procedure), it is possible to estimate ρ as follows:

Using the set $I/2^{1/2}$,$X/2^{1/2}$, $Y/2^{1/2}$, $Z/2^{1/2}$ which forms an orthonormal set of matrices (with respect to the Hilbert-Schmidt norm $(A,B)=\mathrm{tr}(A^{+}B)$), ρ can be expanded as

$$\rho = (1/2^{1/2})[\mathrm{tr}(\rho) + \mathrm{tr}(X\rho)X + \mathrm{tr}(Y\rho)Y + \mathrm{tr}(Z\rho)Z]$$

where the quantities $\mathrm{tr}(X\rho)$, $\mathrm{tr}(Y\rho)$, and $\mathrm{tr}(Z\rho)$ have interpretation of the average value of observable X, Y, and Z respectively. To get estimates of these quantities, the measurements of X, Y and Z need to be performed (with increasing number of measurements m, the uncertainty of the result is decreasing as $1/m^{1/2}$ via the central limit theorem, so we need a large number of copies of ρ).

The density matrix can then be reconstructed from the measurement results.

Quantum process tomography

Experimental identification of the dynamics of quantum systems.

In general, for d dimensional quantum system (i.e. $\dim(\mathcal{H})=d$),

- we choose d^2 pure quantum states $\{|\psi_k\rangle\}$, chosen so that the corresponding density matrices $\{|\psi_k\rangle\langle\psi_k|\}$ form a basis for the space of matrices;
- then we subject the state to the process we wish to characterize;
- after completion of this process, we run quantum state tomography to determine the state $\mathcal{E}(|\psi_j\rangle\langle\psi_j|)$ output from the process.

A way of determining useful representation of $\mathcal{E}$: (χ matrix representation):

$$\mathcal{E}(\rho) = \Sigma_k E_k \rho E_k^+$$

To determine the E_k from measurable parameters, we can consider using a fixed set of E'_k, which form a basis for the set of operators on the Hilbert space:

$$E_k = \Sigma_m e_{km} E'_m$$

where e_{km} are complex numbers. The quantum operation is then given as

$$\mathcal{E}(\rho) = \Sigma_m E'_m \rho E'^{+}_m \chi_{mn} \qquad \chi_{mn} = \Sigma_i e_{im} e^*_{in}$$

The χ matrix completely describes $\mathcal{E}(\rho)$ once the set of operators E'_k has been fixed.

Quantum process tomography

Let ρ_k (for k=1, … ,d^2) be fixed linearly independent basis for the space of d-by-d matrices. A convenient choice of operators is |n><m|. Experimentally, the output state $\mathcal{E}(|n\rangle\langle m|)$ may be determined by preparing the input states |n>, |m>, $|+\rangle = (|n\rangle + |m\rangle)/2^{1/2}$, and $|+\rangle = (|n\rangle + i|m\rangle)/2^{1/2}$ and forming linear combinations

$$\mathcal{E}(|n\rangle\langle m|) = \mathcal{E}(|+\rangle\langle +|) + i\mathcal{E}(|-\rangle\langle -| - (1+i)\,\mathcal{E}(|n\rangle\langle n|)/2 - (1+i)\,\mathcal{E}(|m\rangle\langle m|)/2$$

Thus it is possible to determine $\mathcal{E}(\rho_k)$ by state tomography, for each ρ_k.

Furthermore, each $\mathcal{E}(\rho_k)$ may be expressed as linear combination of the basis states,

$$\mathcal{E}(\rho_k) = \sum_j \lambda_{kj}\rho_j$$

and since $\mathcal{E}(\rho_k)$ is known from the state tomography, λ_{kj} can be determined through linear algebra. We may write

$$E'_m\rho_k E'^{+}_n = \sum_j \beta_{kj}^{mn}\rho_j$$

where β_{kj}^{mn} are complex numbers which can be determined using linear algebra given the E'_m and ρ_k operators. Combining the last two expressions, we have

$$\sum_j\sum_{mn} \chi_{mn}\beta_{kj}^{mn}\rho_j = \sum_j \lambda_{kj}\rho_j$$

From the linear independence of ρ_k, it follows that for each k,

$$\sum_{mn} \beta_{kj}^{mn}\chi_{mn} = \lambda_{kj}$$

This relation is necessary and sufficient condition for the matrix χ to give the correct quantum operation $\mathcal{E}$.

Quantum process tomography

Having determined the matrix χ, operator sum representation for $\mathcal{E}$ is obtained as follows. Let the unitary matrix U^+ diagonalize χ

$$\chi_{mn}= \Sigma_{xy} U_{mx}d_x\delta_{xy}U^*_{ny}$$

from which it can be verified that

$$E_i = (d_i^{1/2}) \Sigma_j U_{ji} E'_j$$

are operational elements for $\mathcal{E}$.

Summary of the algorithm:

- λ is experimentally determined using state tomography;
- χ is determined from the equation $\beta\chi = \lambda$
- χ then provides complete description of $\mathcal{E}$, including the set of operational elements E_i.

Quantum process tomography of a single qubit

In the case of one qubit, we fix the following set of operators

$E'_0 = I$ $\quad$ $E'_1 = X$ $\quad$ $E'_2 = -iY$ $\quad$ $E'_3 = Z$

There are 12 parameters, given by χ, which determine an arbitrary single qubit quantum operation $\mathcal{E}$. These parameters may be measured by four sets of Experiments, for example:

Suppose the input states $|0\rangle$, $|1\rangle$, $|+\rangle = 2^{-1/2}(|0\rangle + |1\rangle)$, $|-\rangle = 2^{-1/2}(|0\rangle + i|1\rangle)$ are prepared, and the four matrices

$$\rho'_1 = \mathcal{E}(|0\rangle\langle 0|) \qquad \rho'_3 = \mathcal{E}(|+\rangle\langle +|) - i\mathcal{E}(|-\rangle\langle -| - (1-i)(\rho'_1 + \rho'_4)/2$$

$$\rho'_2 = \mathcal{E}(|1\rangle\langle 1|) \qquad \rho'_4 = \mathcal{E}(|+\rangle\langle +|) + i\mathcal{E}(|-\rangle\langle -| - (1+i)(\rho'_1 + \rho'_4)/2$$

are determined by quantum state tomography. These correspond to $\rho'_j = \mathcal{E}(\rho_j)$ where

$$\rho_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \rho_2 = \rho_1 X \qquad \rho_3 = X\rho_1 \qquad \rho_4 = X\rho_1 X$$

Due to the particular choice of basis, and the Pauli representation of E'_i, we may

express $\beta = \Lambda \otimes \Lambda$, where $\Lambda = (1/2)\begin{pmatrix} I & X \\ X & -I \end{pmatrix}$ so that $\chi = \Lambda \begin{pmatrix} \rho'_1 & \rho'_2 \\ \rho'_3 & \rho'_4 \end{pmatrix} \Lambda$.