MP 472 Quantum Information and Computation

http://www.thphys.may.ie/staff/jvala/MP472.htm

Outline

Open quantum systems

The density operator

Quantum noise (decoherence)
- Examples of quantum noise and operations
  - Bit flip
  - Phase flip
  - Bit-phase flip channel
  - Depolarizing channel
  - Amplitude damping
  - Phase damping

Quantum error correction

Fault-tolerant quantum computation

Operator sum representation (review)

Let $|e_k>$ be the orthonormal basis for the (finite dimensional) Hilbert space of the environment, and let $\rho = |e_0><e_0|$ be the initial (pure) state of the environment. Then we can express quantum operations as

$$\mathcal{E}(\rho) = \sum_k <e_k|U(\rho \otimes |e_0><e_0|)U^+|e_k> = \sum_k E_k \rho E_k^+$$

where $E_k = <e_k|U|e_0>$ are Kraus operators on the Hilbert space of the (principal) system.

We focus on trace preserving maps, i.e. the Kraus operators satisfy the completeness relation

$$\sum_k E_k^+ E_k = I$$

Bloch sphere for mixed states

$$\rho = \Sigma_i \, p_i \, |\psi_i\rangle\langle\psi_i|$$

An arbitrary single qubit density matrix can be written as

$$\rho = (1/2)\,(I + r.\sigma)$$

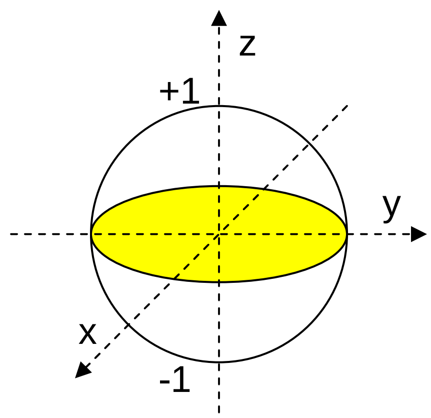
where r is a three dimensional vector s.t. $||r|| \leqslant 1$, known as the Bloch vector. $\sigma = (X,Y,Z)$ is the vector of Pauli matrices.

Bloch representation for a pure state of quantum bit

$$\rho = |\phi\rangle\langle\phi| = |c_0|^2|0\rangle\langle 0| + c_0{}^*c_1|0\rangle\langle 1| + c_1{}^*c_0|1\rangle\langle 0| + |c_1|^2|1\rangle\langle 1| =$$

$$= \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} \begin{pmatrix} c_0{}^* & c_1{}^* \end{pmatrix} = \begin{pmatrix} |c_0|^2 & c_0 c_1{}^* \\ c_0{}^* c_1 & |c_1|^2 \end{pmatrix} = (1/2)(\, I + r_x \sigma_x + r_y \sigma_y + r_z \sigma_z \,)$$

$$= (1/2) \begin{pmatrix} 1+r_z & r_x - i r_y \\ r_x + i r_y & 1-r_z \end{pmatrix}$$



$$r_x = 2\mathrm{Re}(c_0{}^*c_1)$$
$$r_y = 2\mathrm{Im}(c_0{}^*c_1)$$
$$r_z = |c_0|^2 - |c_1|^2$$

Phase flip error channel

$\rho \rightarrow \mathcal{E}(\rho) = ?$

$E_0 = (1-p)^{1/2}\, I \qquad E_1 = (p)^{1/2}\, Z$

$\rho \rightarrow E_0\, \rho\, E_0^+ + E_1\, \rho\, E_1^+ = (1-p)\, \rho + p\, Z\rho Z = (1/2)[(1-p)(I + r.\sigma) + p\, Z(I + r.\sigma)Z] =$

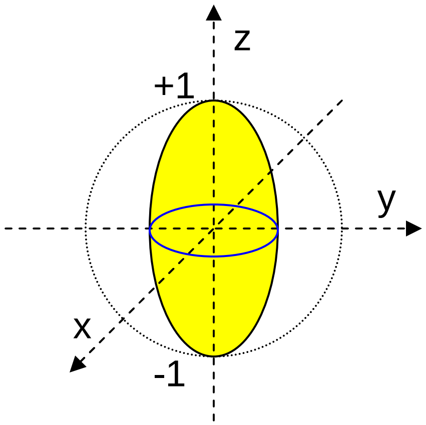
$= (1/2)[(1-p)I + (1-p)\,(r_x.X + r_y.Y + r_z.Z) + p\, I + p\,(r_x.ZXZ + r_y.ZYZ + r_z.Z^3)] =$

$= (1/2)[\, I + (1-p)\,(r_x.X + r_y.Y + r_z.Z) + p\,(- r_x.X - r_y.Y + r_z.Z)] =$

$= (1/2)[\, I + (1-2p)\, r_x.X + (1-2p)\, r_y.Y + r_z.Z]$

<u>Phase flip contracts Bloch sphere in the x-y plane</u>

- relative phase between qubit basis states is being lost
- coherences (off-diagonal elements of ρ) decay
- populations (diagonal elements of ρ) do not change



Bit flip error channel

$\rho \rightarrow \mathcal{E}(\rho) = ?$

$$E_0 = (1\text{-}p)^{1/2}\, I \qquad E_1 = (p)^{1/2}\, X$$

$$\rho \rightarrow E_0\, \rho\, E_0^{+} + E_1\, \rho\, E_1^{+} = (1\text{-}p)\, \rho + p\, X\rho X = (1/2)[(1\text{-}p)(I + r.\sigma) + p\, X(I + r.\sigma)X] =$$

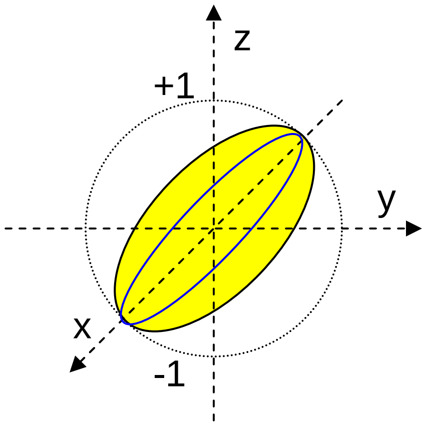
$$= (1/2)[(1\text{-}p)I + (1\text{-}p)\,(r_x.X + r_y.Y + r_z.Z) + p\, I + p\,(r_x.X^3 + r_y.XYX + r_z.XZX)] =$$

$$= (1/2)[\, I + (1\text{-}p)\,(r_x.X + r_y.Y + r_z.Z) + p\,(r_x.X - r_y.Y - r_z.Z)] =$$

$$= (1/2)[\, I + r_x.X + (1\text{-}2p)\, r_y.Y + (1\text{-}2p)\, r_z.Z]$$

<u>Bit flip contracts Bloch sphere in the y-z plane</u>

- populations flip stochastically under this error process, difference between both populations is being reduced
- imaginary part of coherences is being lost



Bit-phase flip error channel

$\rho \rightarrow \mathcal{E}(\rho) = ?$

$E_0 = (1-p)^{1/2}\, I \qquad E_1 = (p)^{1/2}\, Y$

$\rho \rightarrow E_0\, \rho\, E_0^{+} + E_1\, \rho\, E_1^{+} = (1-p)\, \rho + p\, Z\rho Z = (1/2)[(1-p)(I + r.\sigma) + p\, Y(I + r.\sigma)Y] =$

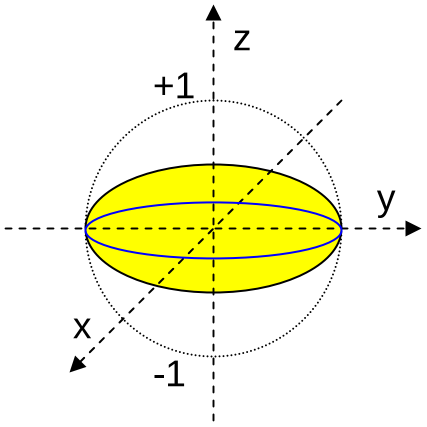
$= (1/2)[(1-p)I + (1-p)\,(r_x.X + r_y.Y + r_z.Z) + p\, I + p\,(r_x.YXY + r_y.Y^3 + r_z.YZY)] =$

$= (1/2)[\, I + (1-p)\,(r_x.X + r_y.Y + r_z.Z) + p\,(- r_x.X + r_y.Y - r_z.Z)] =$

$= (1/2)[\, I + (1-2p)\, r_x.X + r_y.Y + (1-2p)\, r_z.Z]$

Bit-phase flip contracts Bloch sphere in the x-z plane

- both populations and phase flip stochastically, difference between both populations is being reduced
- real part of coherences is being lost

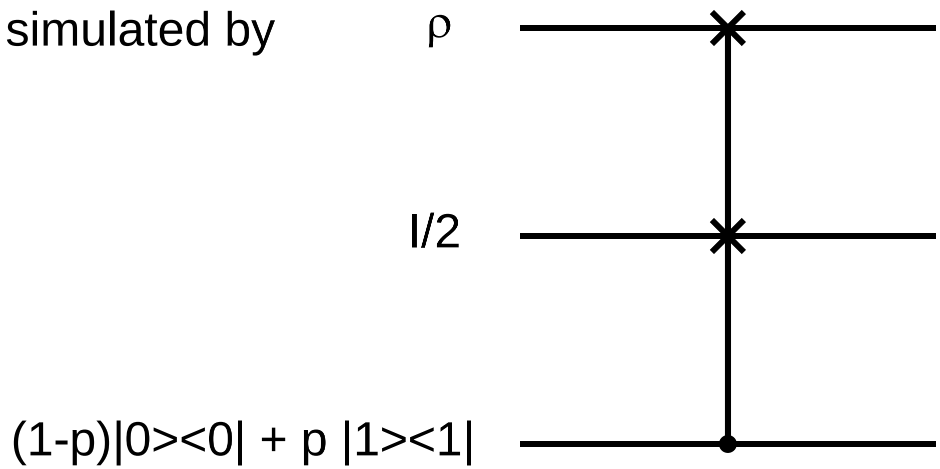


Depolarizing channel

$\rho \rightarrow \mathcal{E}(\rho) = (1/2)\, p\, I + (1-p)\, \rho$

with probability p the qubit state is depolarized, i.e. replaced by the completely mixed state I/2.

This process can be simulated by the following circuit:

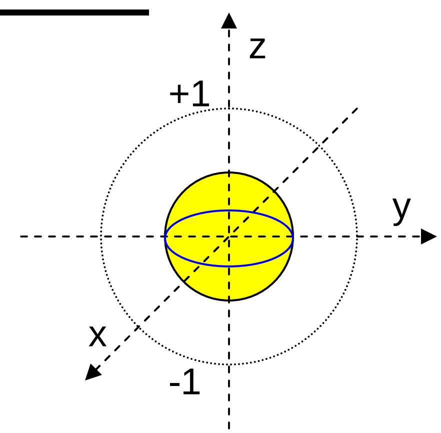


Note that $(1/2)\, I = (1/4)(\rho + X\rho X + Y\rho Y + Z\rho Z$, then

$$\mathcal{E}(\rho) = (1 - 3p/4)\, \rho + (p/4)\, (X\rho X + Y\rho Y + Z\rho Z)$$

which can be conveniently simplified using the parametrization $q=3p/4$ as

$$\mathcal{E}(\rho) = (1 - q)\, \rho + (q/3)\, (X\rho X + Y\rho Y + Z\rho Z)$$



Homework: What are the operational elements of the depolarizing channel?
Derive the effect of the depolarizing channel on the Bloch sphere.

Amplitude damping

Description of energy dissipation (loss of energy from the system, e.g. spontaneous emission)

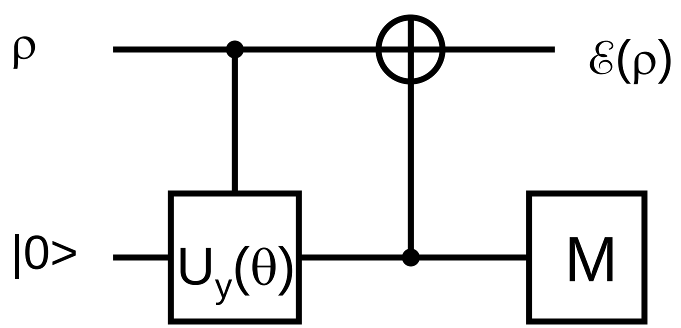
$\rho \rightarrow \mathcal{E}(\rho) = E_0 \rho E_0^{+} + E_1 \rho E_1^{+}$

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & (1-\gamma)^{1/2} \end{pmatrix} \qquad E_1 = \begin{pmatrix} 1 & \gamma^{1/2} \\ 0 & 0 \end{pmatrix}$$

The effect of amplitude damping on the Bloch sphere:

$$(r_x, r_y, r_z) \rightarrow (r_x(1-\gamma)^{1/2}, r_y(1-\gamma)^{1/2}, \gamma + r_z(1-\gamma))$$

This process can be simulated by the circuit:



Homework: Calculate the effect of amplitude damping on the Bloch sphere.
Show that the circuit models the amplitude damping with $\sin^2(\theta/2) = \gamma$.

Phase damping

Loss of quantum information without loss of energy
(dephasing or T_2 process, caused for example by elastic scattering)

$\rho \rightarrow \mathcal{E}(\rho) = E_0 \rho E_0^+ + E_1 \rho E_1^+$

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & (1-\lambda)^{1/2} \end{pmatrix} \qquad E_1 = \begin{pmatrix} 0 & 0 \\ 0 & \lambda^{1/2} \end{pmatrix} \qquad \lambda = 1-\cos^2(\chi\Delta t/2)$$

Using unitary freedom of quantum operations, a new operations elements can be obtained:

$$E'_0 = \alpha^{1/2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad E'_1 = (1-\alpha)^{1/2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \alpha = (1-(1-\lambda)^{1/2})/2$$

which reveal that:
the phase damping is exactly the same as the phase shift channel.

The effect of amplitude damping on the Bloch sphere is therefore:

$(r_x, r_y, r_z) \rightarrow (r_x(1-\gamma)^{1/2}, r_y(1-\gamma)^{1/2}, r_z)$

The circuit: