MP 472 Quantum Information Processing

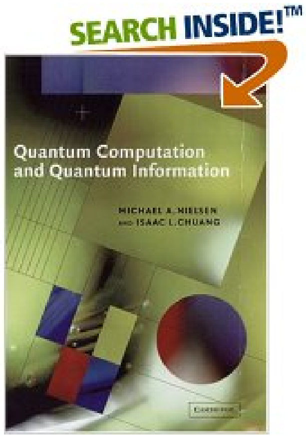
Requirements

- Exam 80%
- Continuous assessment (~3 quizzes) 20%

Essential information/resources:

- Lectures: Tuesday 3 pm and Friday 11 am in MAH, Logic House
- Tutorials: Friday 10 am in MAH

http://www.thphys.may.ie/staff/jvala/MP472.htm



Additional reading (not required)

- Michael Nielsen, and Isaac Chuang,
  Quantum Computation and Quantum Information
  Cambridge 2000
- Umesh Vazirani, Michael Crommie, Birgitta Whaley,
  Lecture Notes – CS/Physics/Chem 191, UC Berkeley
  http://inst.eecs.berkeley.edu/~cs191/sp05/
- John Preskill,
  Lecture Notes – Physics 219, Caltech
  http://www.theory.caltech.edu/~preskill/ph219/index.html#lecture
- A. Yu. Kitaev, A.H. Shen, M.N. Vyalyi
  Classical and Quantum Computation, AMS 2002

Introduction

Classical information
- classical bit
- Boolean function
- Boolean circuit

Quantum information
- quantum bit(s)
- quantum operations
- quantum state measurement
- quantum circuit
- example: quantum entangler

Examples of quantum information processing
- teleportation

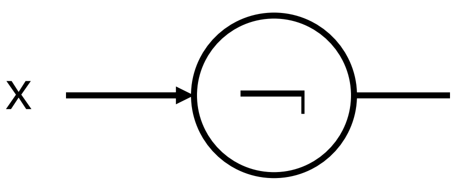
Classical information and its processing

• an elementary unit of (classical) information is classical bit $\mathbb{B} = \{0,1\}$ whose values, 0 or 1, correspond to two distinct values (states) of an appropriate physical quantity (e.g. electric potential, etc.)

• Boolean function on n variables $F(x_1,..,x_n): \mathbb{B}^n \to \mathbb{B}^k$

• Examples of simple Boolean functions:

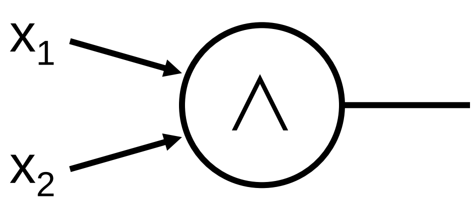
negation $\neg x$
(bit flip)



x		$\neg x$
0	→	1
1	→	0

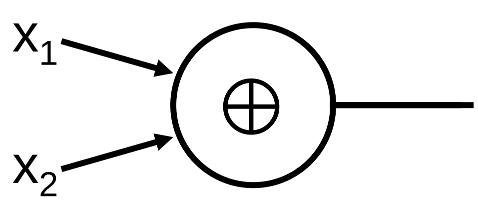
(bit flip)

conjunction $x_1 \wedge x_2$



x_1	x_2		$x_1 \wedge x_2$
0	0	→	0
0	1	→	0
1	0	→	0
1	1	→	1

addition mod(2) $x_1 \oplus x_2$



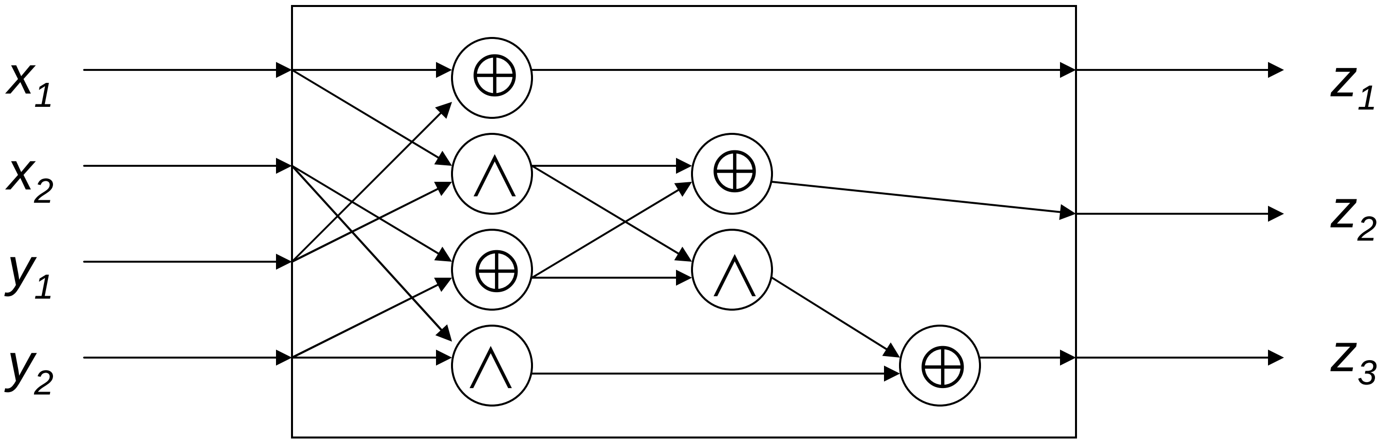
x_1	x_2		$x_1 \oplus x_2$
0	0	→	0
0	1	→	1
1	0	→	1
1	1	→	0

Classical information and its processing

• a Boolean circuit is a representation of a Boolean function as a composition of other Boolean functions $\mathcal{B}$ (e.g. $\mathcal{B}=\{\wedge,\oplus\}$);

• a circuit over $\mathcal{B}$ is a sequence of assignments involving n input variables $\{x_1, \ldots, x_n\}$ and several auxiliary variables $\{y_1, \ldots, y_m\}$ where $y_k = f_k(u_1, \ldots, u_r)$ and each of the variables $u_1,\ldots,u_r$ are either input variables or auxiliary variables preceding y_k;

• example – addition of two 2-digit numbers (Kitaev et al.)



• a basis $\mathcal{B}$ is called complete, if for any Boolean function f, there is a circuit over $\mathcal{B}$ that computes f. Example: $\mathcal{B}=\{\wedge,\oplus\}$.

Quantum information - qubit

An elementary unit of quantum information is quantum bit, qubit, which is physically represented by a quantum two-level system:

a basis of a two-level system is formed by two states (vectors) $\mathcal{B} = \{|0\rangle, |1\rangle\}$ which are **orthonormal**, *i.e.* orthogonal and normalized:

$\langle i|j\rangle = \delta_{ij}$

scalar product

Kronecker delta: $\delta_{ij} = 1$ iff $i = j$

$\delta_{ij} = 0$ iff $i \neq j$

Dirac bra-ket notation

ket: $|\phi\rangle = c_0|0\rangle + c_1|1\rangle$

bra: $\langle\phi| = c_0^*\langle 0| + c_1^*\langle 1|$

The qubit can in general be in a **superposition** of basis states (vectors) $|0\rangle, |1\rangle$:

No classical analog !!!

$$|\phi\rangle = c_0|0\rangle + c_1|1\rangle$$

$c_k \in \mathbb{C}$, for $k = 0, 1$, s.t. $|c_0|^2 + |c_1|^2 = 1$

Quantum information - qubit

Quantum states of a qubit form a vector space known as Hilbert space (will be defined rigorously later in the class).

The qubit Hilbert space is two dimensional and is isomorphic to a two-dimensional space of complex numbers

$$|\phi\rangle = c_0|0\rangle + c_1|1\rangle \in \mathcal{H}^2 \simeq \mathbb{C}^2$$

Physical realization of a qubit can *for example* be a spin ½ particle:

$$|0\rangle = |\uparrow\rangle \,,\ |1\rangle = |\downarrow\rangle: \quad |\phi\rangle = c_0|\uparrow\rangle + c_1|\downarrow\rangle$$

or two energy levels of an atom or ion,
or opposite superconducting fluxes in a Josephson junction,
or ....

Quantum logical operations are rotations of a quantum state vector in a Hilbert space, i.e. they are unitary (and **reversible**) operations.

- classical computation processing can be made reversible.

Quantum information - qubits

a quantum state of n qubits is a vector in 2^n-dimensional Hilbert space (isomorphic to 2^n-dim. space $\underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2}_{\text{n-times}} = \mathbb{C}^{2^n}$)

Examples of composite quantum states:

a product state, e.g.

$$|\psi> = |0>\otimes|0> = |0>|0> = |00>$$

or

$$|\psi> = (c_0|0>+c_1|1>)|0>= c_{00}|00> + c_{01}|10>$$

$c_{00}= c_0$, $c_{11}= c_1$

or an entangled (strongly correlated) state, e.g.

No classical analog !!!

$$|\psi> = c_{00}|00> + c_{11}|11>$$

Quantum information cannot be cloned (copied):

No classical analog !!!

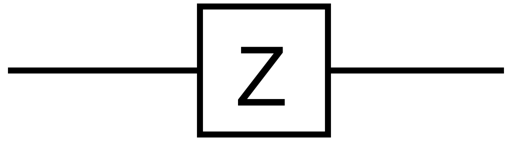
assume there is a cloning operator $\mathcal{O}$ s.t.

$$\mathcal{O}|0> = |0>|0> \text{ and } \mathcal{O}|1> = |1>|1>$$

but $\mathcal{O}|\phi> \neq |\phi>|\phi>$ then contradicts the assumtion

Examples: single-qubit operations
Phase flip

$|q\rangle$



$|\psi\rangle = ?$

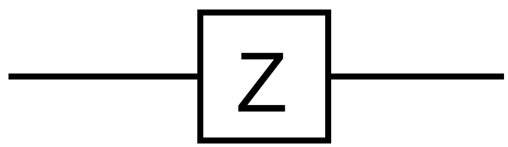
$|0\rangle \longrightarrow |0\rangle$

$|1\rangle \longrightarrow -|1\rangle$

No classical analog !!!

General state of a qubit:

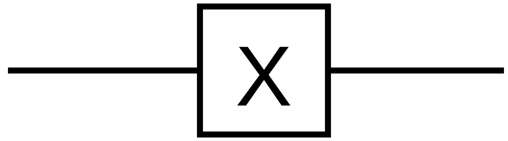
$|\phi\rangle = c_0|0\rangle + c_1|1\rangle$



$|\psi\rangle = c_0\,|0\rangle - c_1|1\rangle$

Homework: show that $|\phi\rangle$ and $|\psi\rangle$ are orthogonal if $c_0 = c_1 = 2^{-1/2}$. Why?

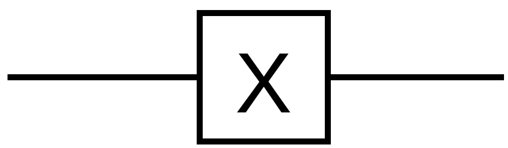
Bit flip

$|q\rangle$  $|\psi\rangle = ?$

$|0\rangle \longrightarrow |1\rangle$

$|1\rangle \longrightarrow |0\rangle$

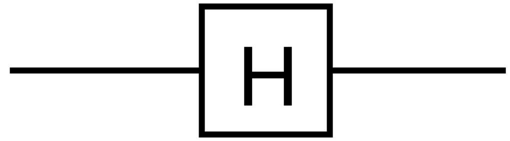
General state of a qubit:

$|\phi\rangle = c_0|0\rangle + c_1|1\rangle$  $|\psi\rangle = c_1\,|0\rangle + c_0|1\rangle$

Homework: For what c_0 and c_1 are $|\phi\rangle$ and $|\psi\rangle$ orthogonal?

Hadamard gate

No classical analog !!!

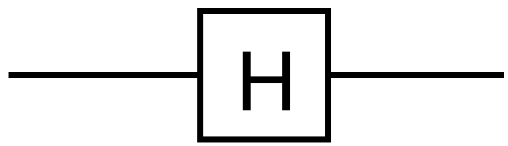
$|q\rangle$ 

$|\psi\rangle = ?$

$|0\rangle \longrightarrow 2^{-1/2} (|0\rangle + |1\rangle)$

$|1\rangle \longrightarrow 2^{-1/2} (|0\rangle - |1\rangle)$

General state of a qubit:

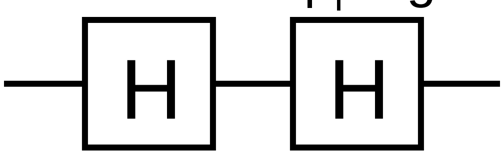
$|\phi\rangle = c_0|0\rangle + c_1|1\rangle$ 

$|\psi\rangle =$

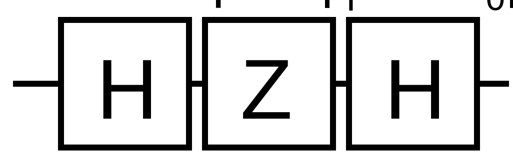
$$= 2^{-1/2} (c_0|0\rangle + c_0|1\rangle + c_1|0\rangle - c_1|1\rangle)$$

$$= 2^{-1/2} [(c_0 + c_1) |0\rangle + (c_0 - c_1) |1\rangle]$$

Homework: What $|\psi\rangle$ give these circuits for the input $|\phi\rangle = c_0|0\rangle + c_1|1\rangle$?

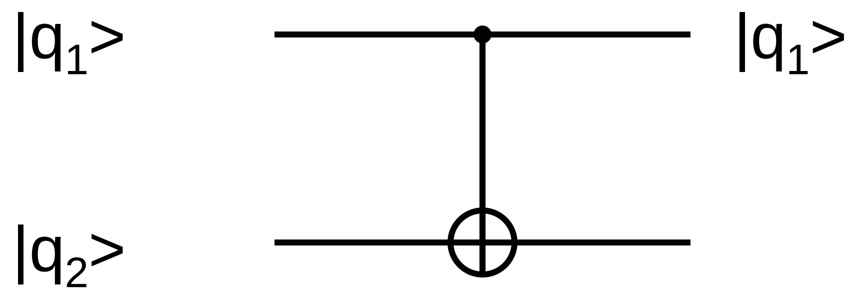
A) 

$|\psi\rangle = ?$

B) 

$|\psi\rangle = ?$

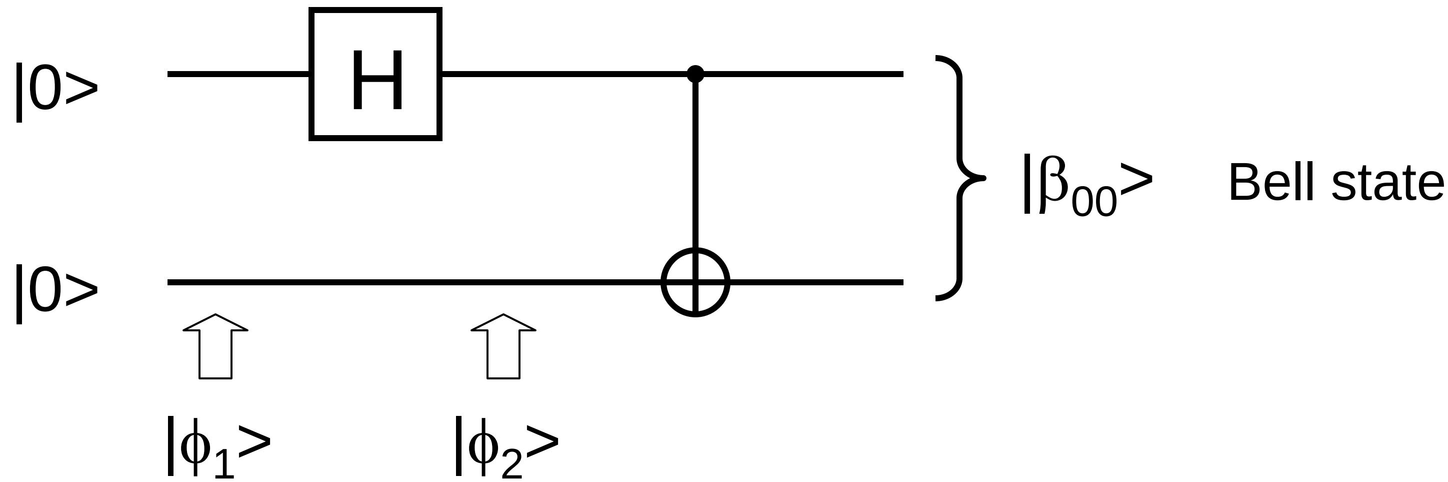
Example of a two-qubit operation: CNOT gate



$\|q_1q_2\rangle$		
$\|00\rangle$	$\rightarrow$	$\|00\rangle$
$\|01\rangle$	$\rightarrow$	$\|01\rangle$
$\|10\rangle$	$\rightarrow$	$\|11\rangle$
$\|11\rangle$	$\rightarrow$	$\|10\rangle$

Note that iff $|q_1\rangle = |1\rangle$, then $|q_2\rangle$ is flipped!!!

Example: a Bell state generator



$|\phi_1\rangle = |0\rangle \otimes |0\rangle = |0\rangle|0\rangle = |00\rangle$

$|\phi_2\rangle = 2^{-1/2}(|0\rangle + |1\rangle)|0\rangle = 2^{-1/2}(|00\rangle + |10\rangle)$

$|\beta_{00}\rangle = 2^{-1/2}(|00\rangle + |11\rangle)$

Homework: Construct circuits to generate the other Bell states from |00>:

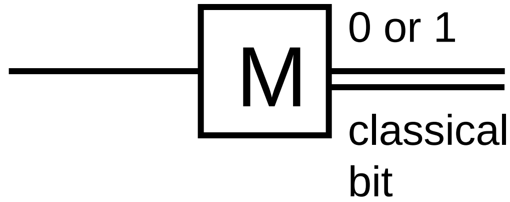
$|\beta_{01}\rangle = 2^{-1/2}(|00\rangle - |11\rangle)$

$|\beta_{10}\rangle = 2^{-1/2}(|01\rangle + |10\rangle)$

$|\beta_{11}\rangle = 2^{-1/2}(|01\rangle - |10\rangle)$

Measurement

$|\phi\rangle = c_0|0\rangle + c_1|1\rangle$



$|\psi\rangle = ?$

Measurement of one qubit (in the basis {|0>,|1>}) gives a classical bit of information:

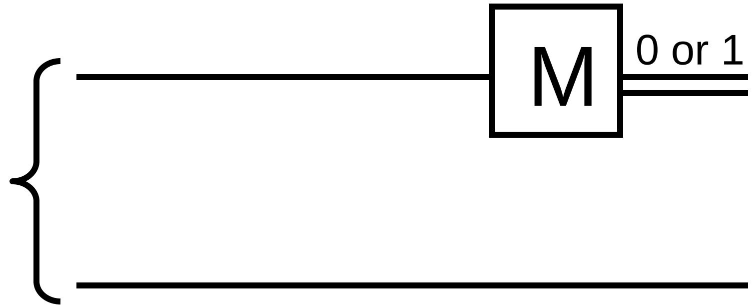
• with a probability $|c_0|^2$, the measurement gives the result M = 0;
the quantum state after the measurement is collapsed to $|\psi\rangle = |0\rangle$;

No classical analog !!!

• with probability $|c_1|^2$,the result is M = 1 and $|\psi\rangle = |1\rangle$.

Measurement of an entangled state – EPR paradox

$|\phi\rangle = c_0|00\rangle + c_1|11\rangle$



$|\psi\rangle = ?$

Measurement of qubit 1:

M = 0 is measured with probability $|c_0|^2$: $|\psi\rangle = |00\rangle = |0\rangle \otimes |0\rangle$

M = 1 is measured with probability $|c_1|^2$: $|\psi\rangle = |11\rangle = |1\rangle \otimes |1\rangle$

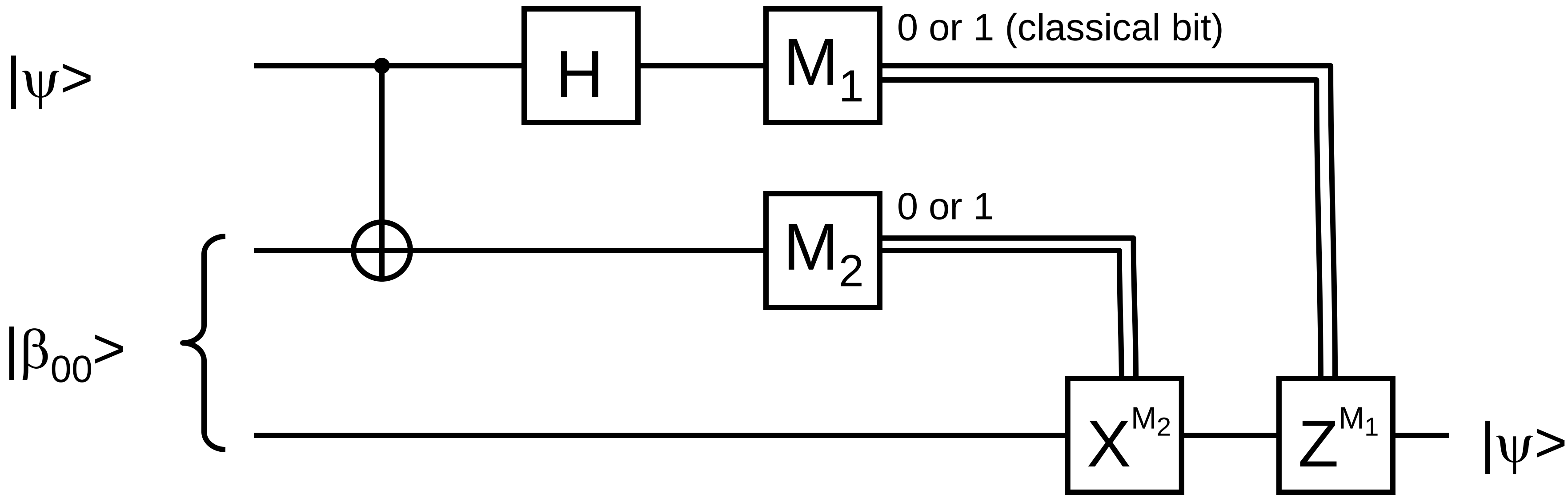
if the state of (two) qubits is entangled, measurement of the first qubit completely determines the state of the second qubit after the measurement even if both qubits are spatially separated and cannot communicate (interact)

Einstein, Podolsky, Rosen paradox

No classical analog !!!


No classical analog !!!
TELEPORTATION

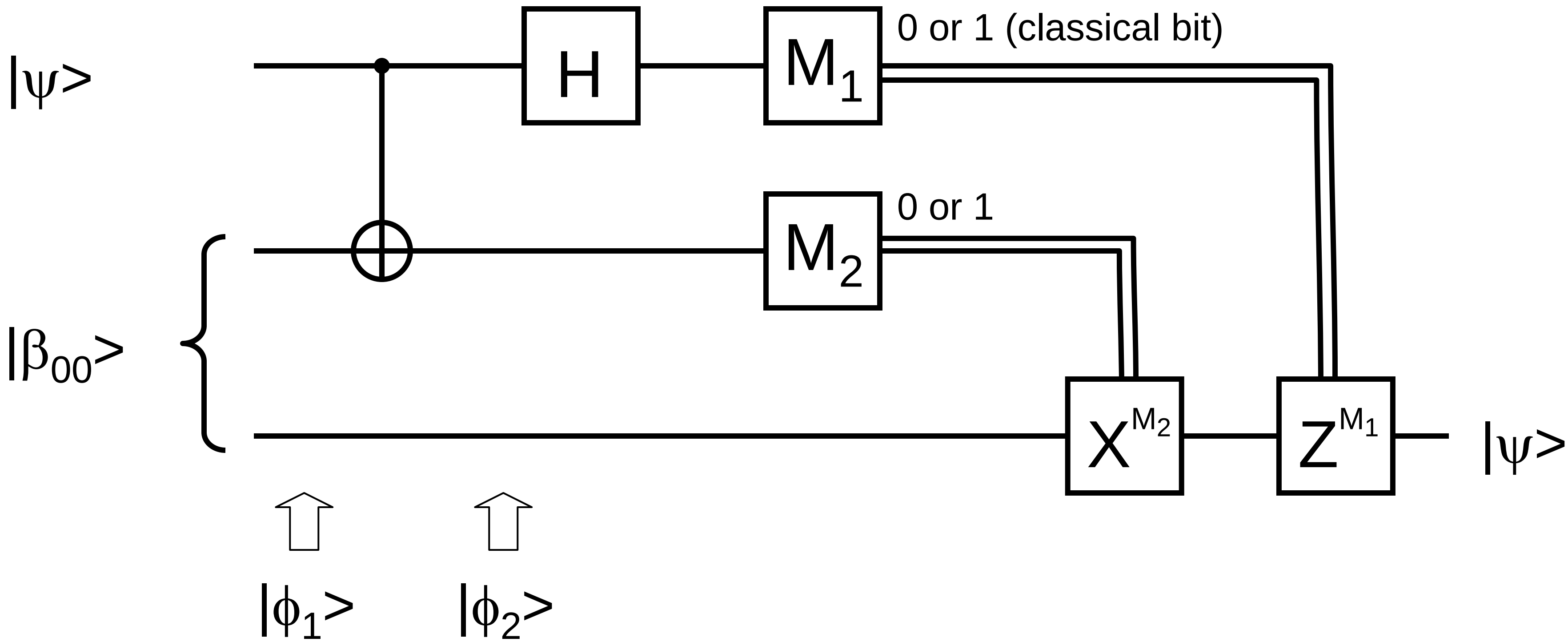
Teleportation



Task:

Teleport an unknown qubit state $|\psi>$ using one Bell state and single qubit and two qubit operations and local mesurements.

Teleportation circuit and initial state

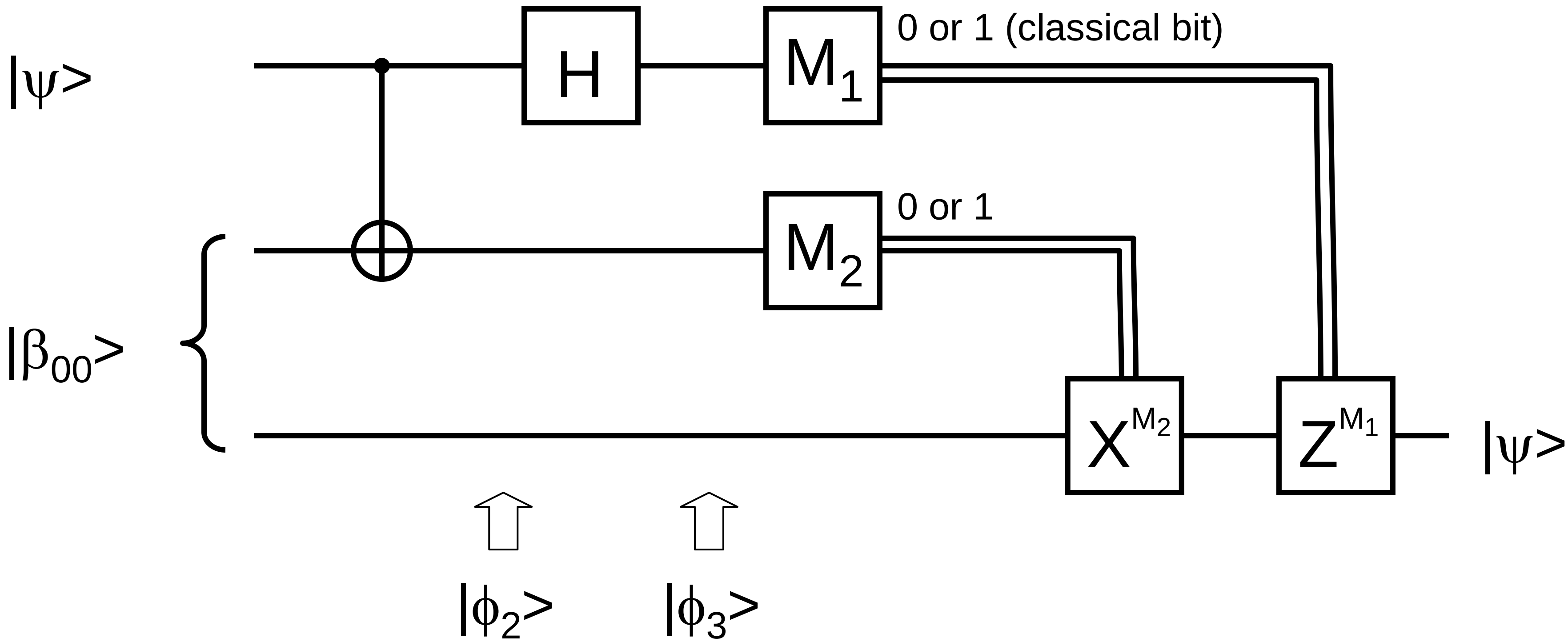


$|\psi\rangle = c_0|0\rangle + c_1|1\rangle$ $\qquad$ $|\beta_{00}\rangle = 2^{-1/2}(|00\rangle + |11\rangle)$

$|\phi_1\rangle = |\psi\rangle|\beta_{00}\rangle = 2^{-1/2}(c_0|000\rangle + c_0|011\rangle + c_1|100\rangle + c_1|111\rangle)$

$|\phi_2\rangle = 2^{-1/2}(c_0|000\rangle + c_0|011\rangle + c_1|110\rangle + c_1|101\rangle)$

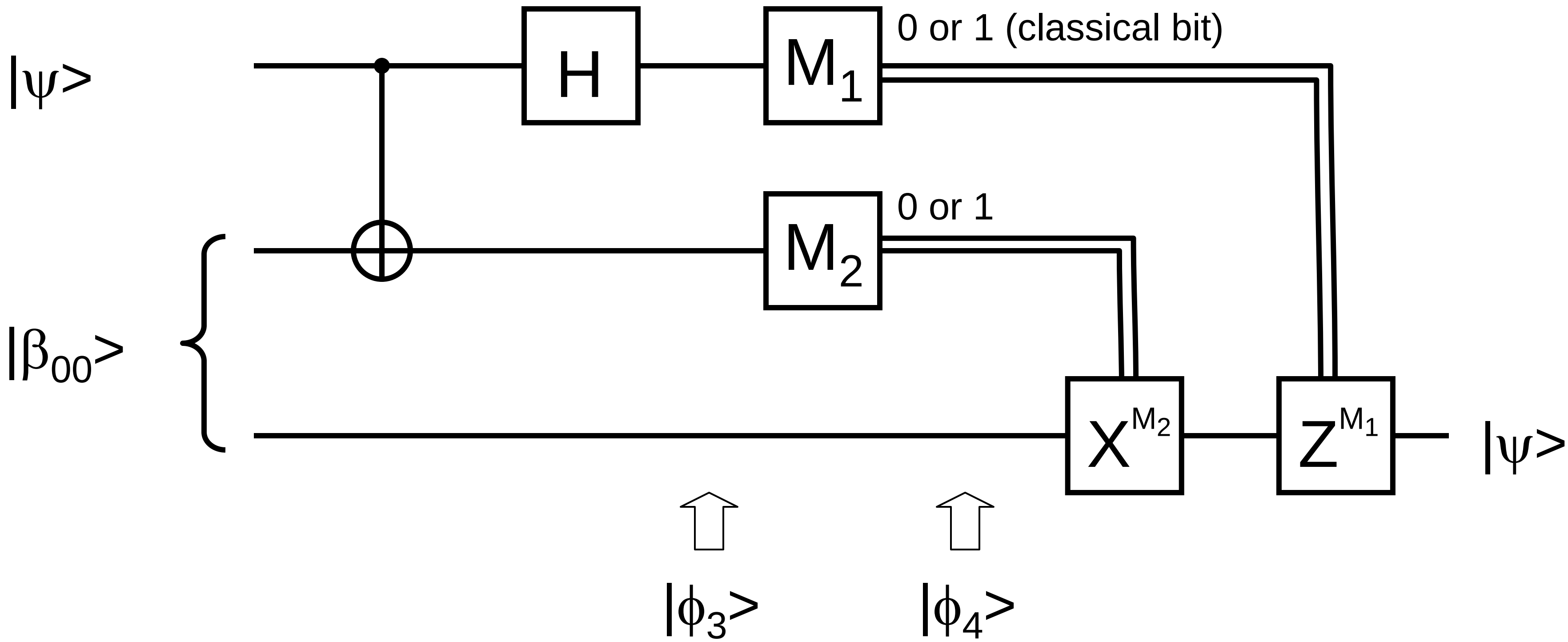
Teleportation – state before measurement



$$|\phi_2\rangle = 2^{-1/2}(c_0|000\rangle + c_0|011\rangle + c_1|110\rangle + c_1|101\rangle)$$

$$|\phi_3\rangle = 2^{-1}\ (c_0|000\rangle + c_0|100\rangle + c_0|011\rangle + c_0|111\rangle + c_1|010\rangle - c_1|110\rangle + c_1|001\rangle - c_1|101\rangle)$$

Teleportation - measurement



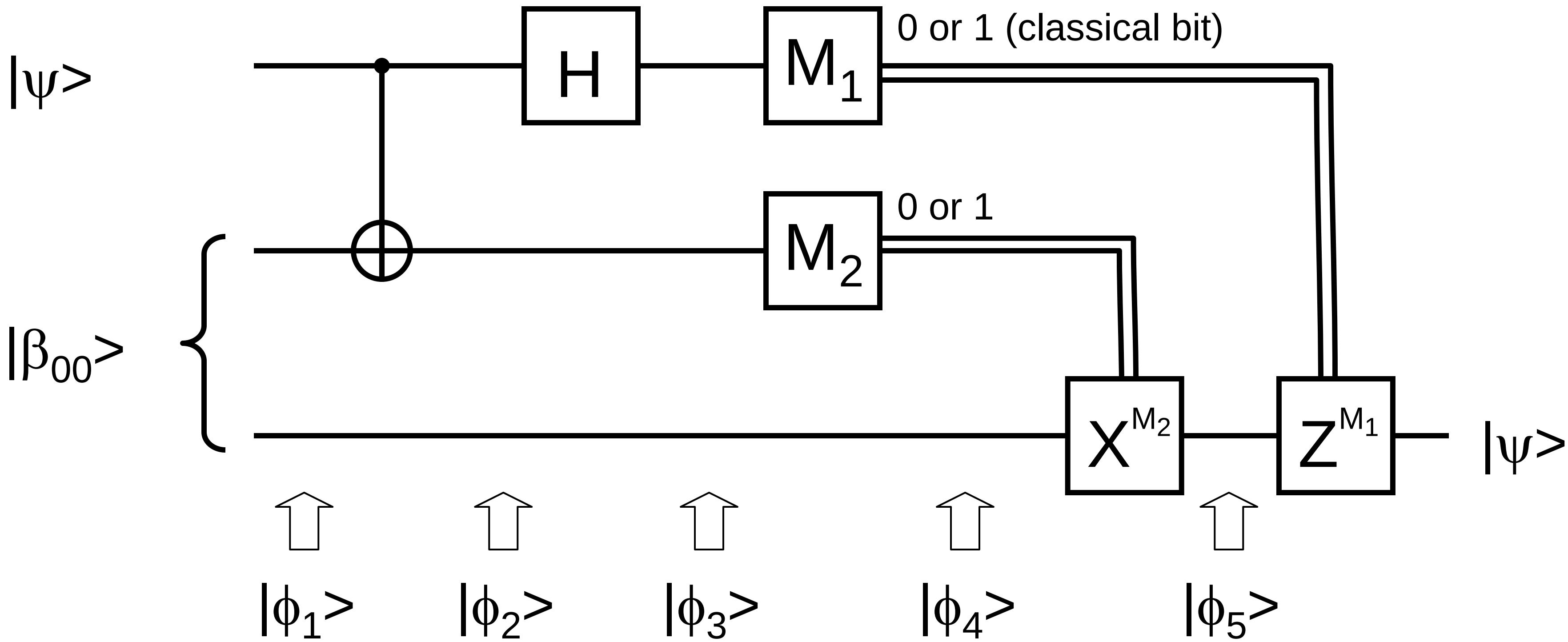
$$|\phi_3\rangle = 2^{-1}(c_0|000\rangle + c_0|100\rangle + c_0|011\rangle + c_0|111\rangle + c_1|010\rangle - c_1|110\rangle + c_1|001\rangle - c_1|101\rangle)$$

Four possible results of the measurements of 1st and 2nd qubit:

$M_1 = 0, M_2 = 0$ $\quad$ $M_1 = 0, M_2 = 1$

$M_1 = 1, M_2 = 0$ $\quad$ $M_1 = 1, M_2 = 1$

Teleportation – measurement results $M_1 = 0$, $M_2 = 0$



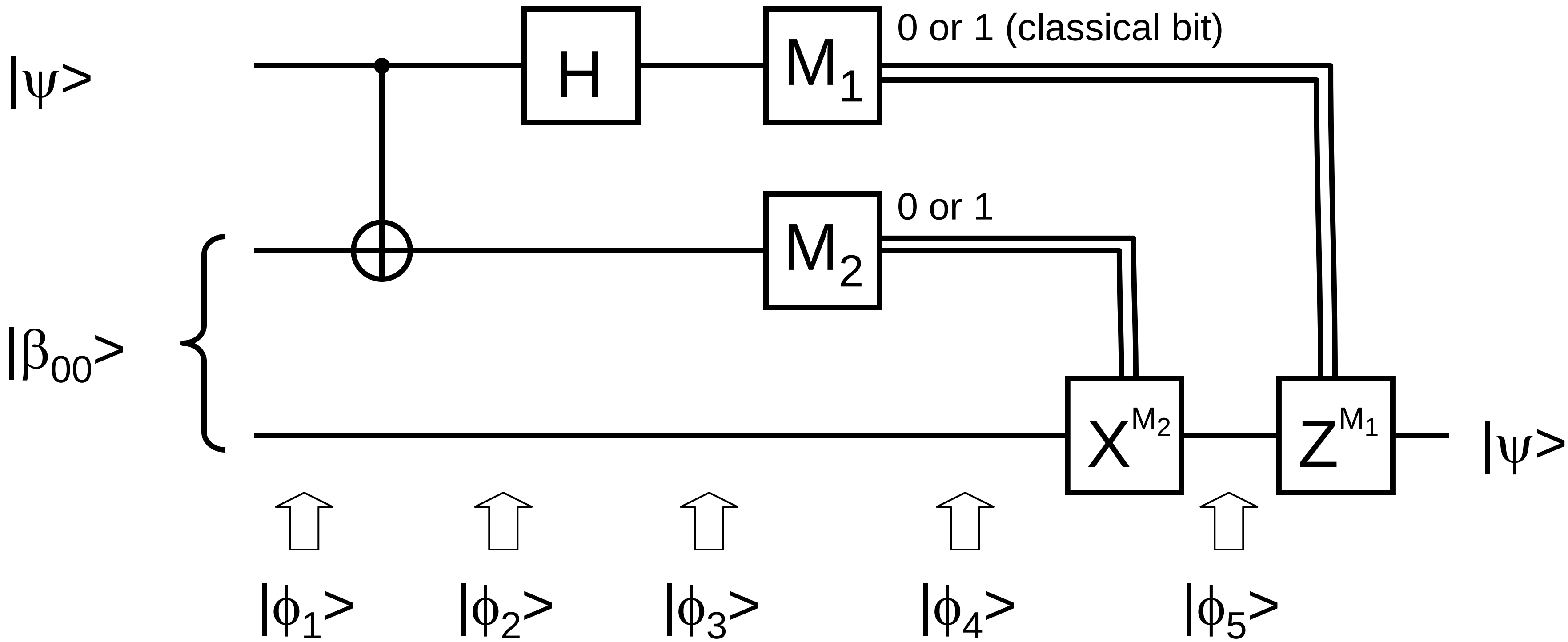
$$|\phi_3\rangle = 2^{-1}(c_0|000\rangle + c_0|100\rangle + c_0|011\rangle + c_0|111\rangle + c_1|010\rangle - c_1|110\rangle + c_1|001\rangle - c_1|101\rangle)$$

$M_1 = 0$, $M_2 = 0$:

$$|\phi_4\rangle = c_0|000\rangle + c_1|001\rangle = |00\rangle(c_0|0\rangle + c_1|1\rangle)$$

$$|\phi_4\rangle \xrightarrow{X^0 = \mathbb{1}} |\phi_5\rangle = |\phi_4\rangle \xrightarrow{Z^0 = \mathbb{1}} |\psi\rangle = |\phi_5\rangle = |\phi_4\rangle$$

Teleportation – measurement results $M_1 = 1$, $M_2 = 0$



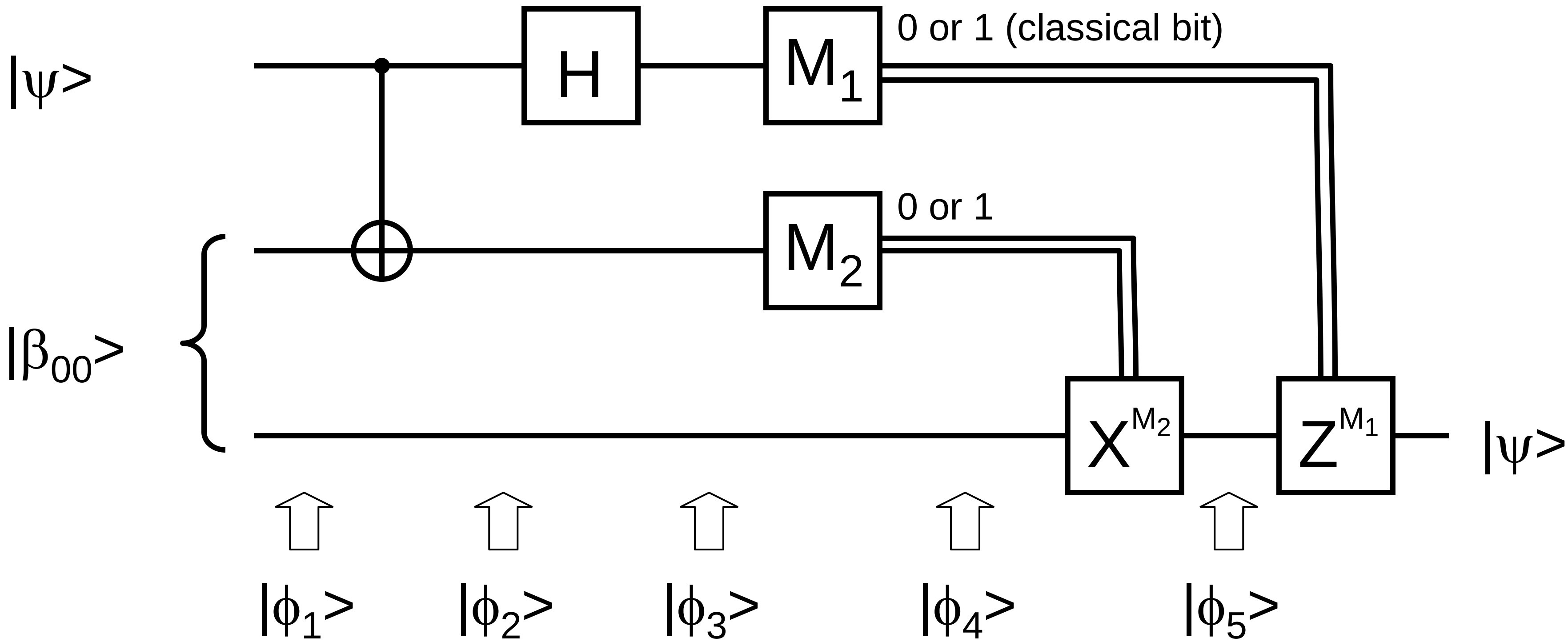
$$|\phi_3\rangle = 2^{-1}(c_0|000\rangle + c_0|100\rangle + c_0|011\rangle + c_0|111\rangle + c_1|010\rangle - c_1|110\rangle + c_1|001\rangle - c_1|101\rangle)$$

$M_1 = 1$, $M_2 = 0$:

$$|\phi_4\rangle = c_0|100\rangle - c_1|101\rangle = |10\rangle(c_0|0\rangle - c_1|1\rangle)$$

$$|\phi_4\rangle \xrightarrow{X^0 = \mathbb{1}} |\phi_5\rangle = |\phi_4\rangle \xrightarrow{Z^1 = Z} |\psi\rangle = c_0|0\rangle + c_1|1\rangle$$

Teleportation – measurement results $M_1 = 0$, $M_2 = 1$



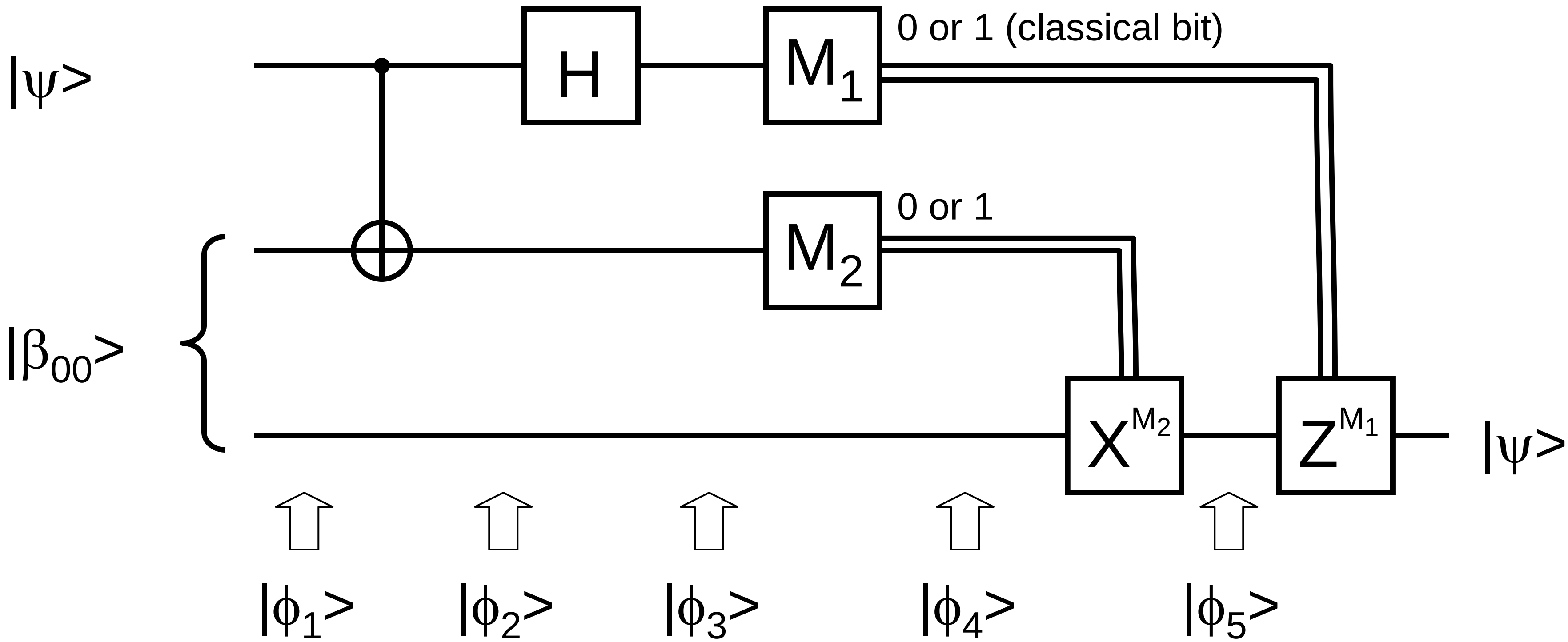
$$|\phi_3\rangle = 2^{-1}(c_0|000\rangle + c_0|100\rangle + c_0|011\rangle + c_0|111\rangle + c_1|010\rangle - c_1|110\rangle + c_1|001\rangle - c_1|101\rangle)$$

$M_1 = 0$, $M_2 = 1$:

$$|\phi_4\rangle = c_0|011\rangle + c_1|010\rangle = |01\rangle(c_0|1\rangle + c_1|0\rangle)$$

$$|\phi_4\rangle \xrightarrow{X^1 = X} |\phi_5\rangle = c_0|0\rangle + c_1|1\rangle \xrightarrow{Z^0 = \mathbb{1}} |\psi\rangle = |\phi_5\rangle$$

Teleportation – measurement results $M_1 = 1$, $M_2 = 1$



$$|\phi_3\rangle = 2^{-1}(c_0|000\rangle + c_0|100\rangle + c_0|011\rangle + c_0|111\rangle + c_1|010\rangle - c_1|110\rangle + c_1|001\rangle - c_1|101\rangle)$$

$M_1 = 1$, $M_2 = 1$:

$$|\phi_4\rangle = c_0|111\rangle - c_1|110\rangle = |11\rangle(c_0|1\rangle - c_1|0\rangle)$$

$$|\phi_4\rangle \xrightarrow{X^1 = X} |\phi_5\rangle = c_0|0\rangle - c_1|1\rangle \xrightarrow{Z^1 = Z} |\psi\rangle = c_0|0\rangle + c_1|1\rangle$$