The Kitaev Honeycomb lattice

and some numerical observations

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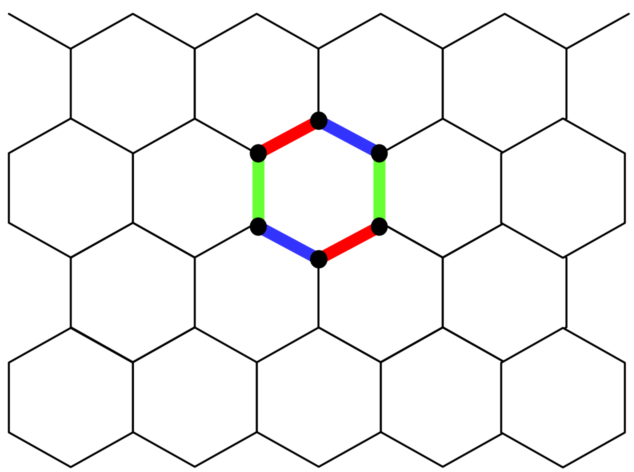
Overview

- Theory
  - The model
  - Vortices and Fermions
  - Spectral Properties – phase diagram
- Numerical
  - Sparse representations
- Results
  - Toral hexagonal lattice structures
  - Spectral Phase diagrams
- Future work
  - ICHEC
  - Larger Lattice systems

The Model

- Hexagonal lattice system with spin exchange parameter dependent on direction

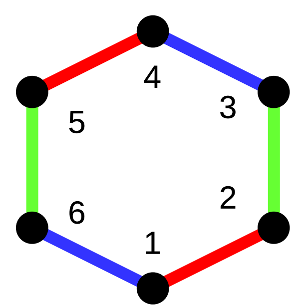
$$H = -J_{xx} \sum_{xx-links} \sigma_j^x \sigma_k^x - J_{yy} \sum_{yy-links} \sigma_j^y \sigma_k^y - J_{zz} \sum_{zz-links} \sigma_j^z \sigma_k^z$$



xx spin exchange

yy spin exchange

zz spin exchange



Anyons in an exactly solvable model and beyond, A. Kitaev, cond-mat/0506438

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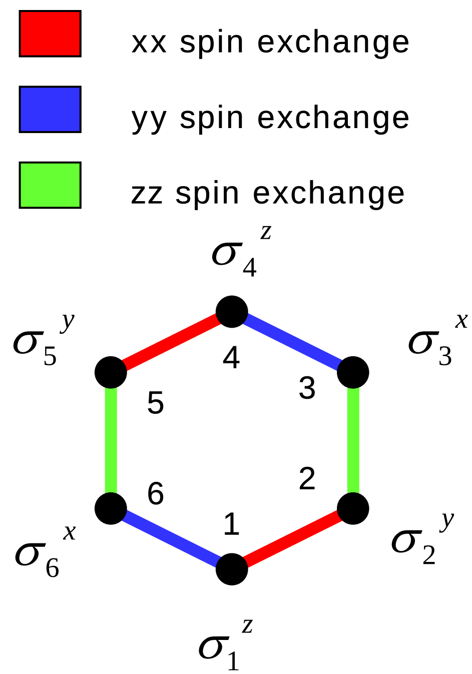
- Plaquette operators

$$W_p = \sigma_1^z\sigma_2^y\sigma_3^x\sigma_4^z\sigma_5^y\sigma_6^x$$

with eigenvalues [-1,1].



- In the quasi-particle description to be described later we will call the -1 state a vortex and the +1 state vortex free. The vortices display anyonic statistics.

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Vortices

- Commutes with Hamiltonian and therefore can be simultaneously diagonalised

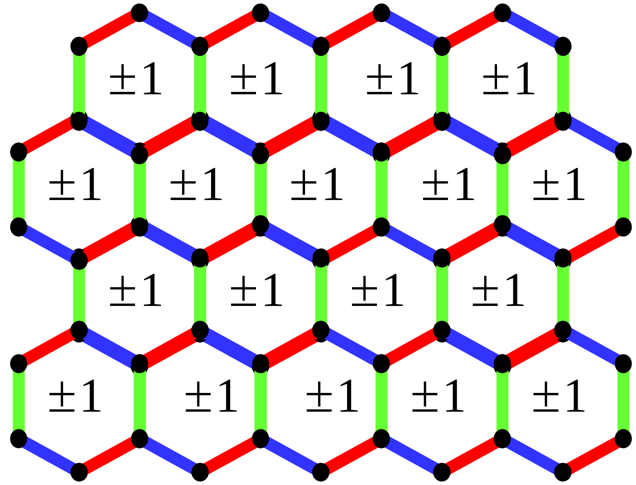
$$[W_p, H] = 0$$

- Any Energy eigenstate has expectation value

$$w_p = \langle E_n | W_p | E_n \rangle = \pm 1$$

- This allows us to split the Hilbert space into sectors labelled by a fixed configuration of plaquette eigenvalues.

$$L = \bigoplus_{w_{1,\ldots\ldots,}w_m} L_{w_{1,\ldots,}w_m}$$



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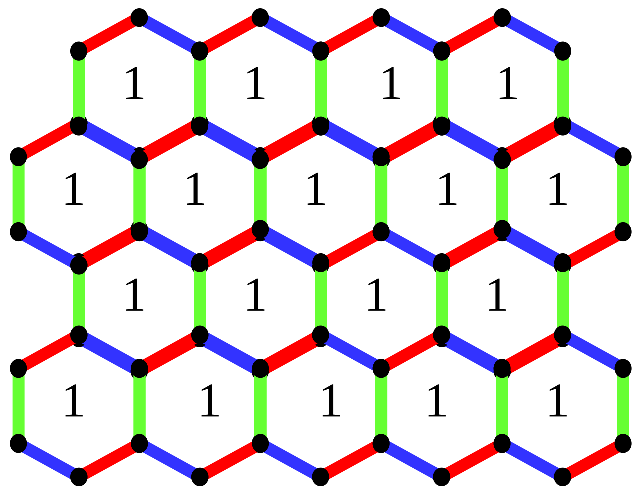

Vortex free sector

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The Fermionic picture

- It is possible to map this spin ½ model to a free majorana fermionic picture. (Jordan-Wigner Transformation)

- Represent each spin with 2 fermionic modes a_1 & a_2 so that the no-fermion state represents a spin up and 2-fermion state represents a spin down.

- Majorana fermions are defined as the ‘real’ and ‘imaginary’ parts of the fermionic creation and annihilation operators in the following way

$$c_{2k-1} \equiv a_k + a_k^+ \qquad c_{2k} \equiv \frac{a_k - a_k^+}{i}$$

- They are Hermitian and satisfy the relations

$$c_i = c_i^+ \qquad \{c_i, c_j\} = 2\delta_{ij}$$

The Fermionic picture

- There are 4 Majorana operators corresponding to each spin

$$c_1 = a_1 + a_1^+ = b^x \qquad c_3 = a_2 + a_2^+ = b^z$$

$$c_2 = \frac{a_1 - a_1^+}{i} = b^y \qquad c_4 = \frac{a_2 - a_2^+}{i} = c$$

- We need to be able to project the full 4-D space of states $\tilde{L}$ of the 2 fermions back to the physical space of the spin states L .

- This is done with the 'projection' operator

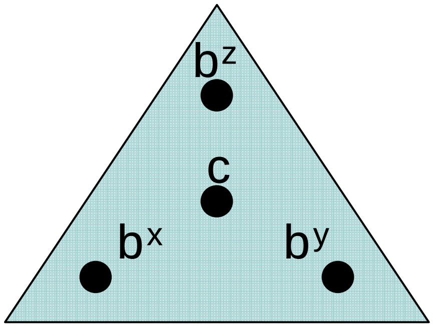
$$D = -b^x b^y b^z c \quad \text{where} \quad |\psi\rangle \in L \Leftrightarrow D|\psi\rangle = |\psi\rangle$$

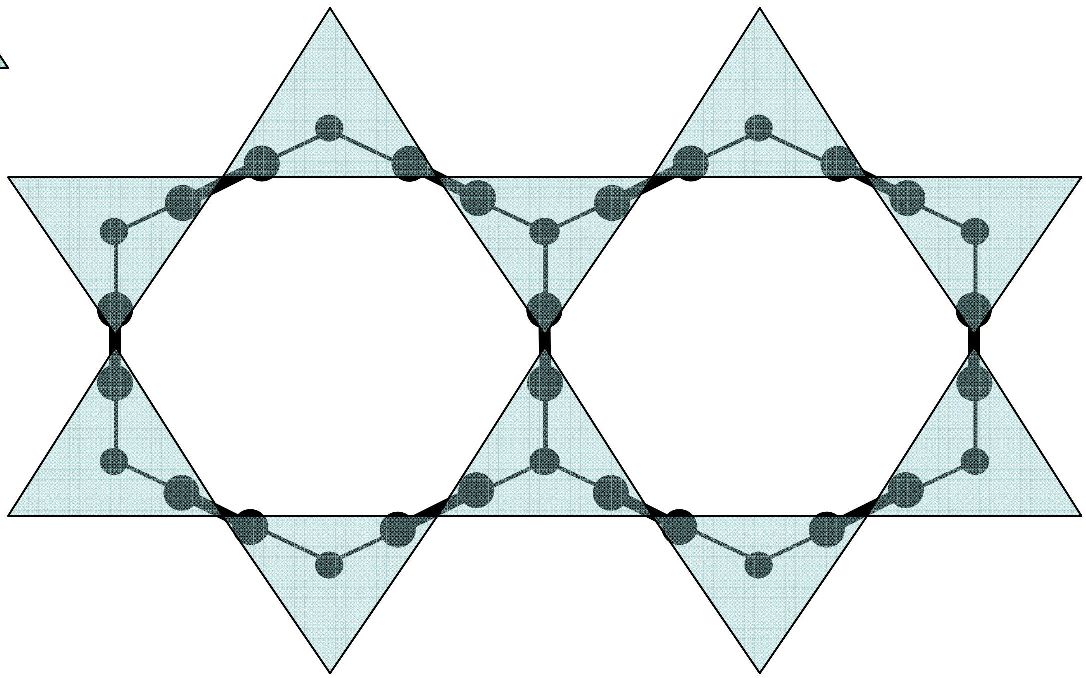
and

$$\tilde{\sigma}^x = ib^x c \qquad \tilde{\sigma}^y = ib^y c \qquad \tilde{\sigma}^z = ib^z c$$

represent the Pauli spin operators on this extended space

Graphical Representation





The Fermionic picture

Matrix representation:

- Fermionic annihilation ops $a_1 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $a_2 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

- 4 Majorana operators b^x, b^y, b^z, c:

$$b^x = -\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad b^y = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & i \\ 0 & -i & 0 & 0 \end{bmatrix} \quad b^z = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad c = \begin{bmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}$$

- Stabilizer

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Quadratic Hamiltonian

- Recall that

$$H = -J_x \sum_{xx-links} \sigma_j{}^x \sigma_k{}^x - J_y \sum_{yy-links} \sigma_j{}^y \sigma_k{}^y - J_z \sum_{zz-links} \sigma_j{}^z \sigma_k{}^z$$

- The diagonalization of the Hamiltonian is re-expressed as the diagonalization of

$$H(\sigma_j^\alpha) = H(\{ib_j^\alpha c_j\})$$

accompanied by the constraint $D_j|\psi\rangle = |\psi\rangle$.

- Using this transformation we have $\sigma_j{}^\alpha \sigma_k{}^\alpha = -i\hat{u}_{jk} c_j c_k$ with $\hat{u}_{jk} = ib_j^\alpha b_k^\alpha$ to finally obtain

$$\tilde{H} = \frac{i}{4} \sum_{j,k} \hat{A}_{j,k} c_j c_k \quad \text{where} \quad \hat{A}_{j,k} = \begin{cases} 2J_\alpha \hat{u}_{j,k}, & \text{if j and k are connected,} \\ 0, & \text{otherwise.} \end{cases}$$

Quadratic Hamiltonian

- The operators $\hat{u}_{jk}$ commute with the Hamiltonian and with each other.
- Similarly to before we use these to break the space up into sectors

$$\tilde{L} = \bigoplus_u \tilde{L}_u$$

- To restrict the Hamiltonian to the sector we want we replace the operators by the eigenvalues by 'removing hats' to get

$$\tilde{H} = \frac{i}{4} \sum_{j,k} A_{j,k} c_j c_k$$

- This type of Hamiltonian can be solved exactly. In fact, so can any Hamiltonian that can be written as

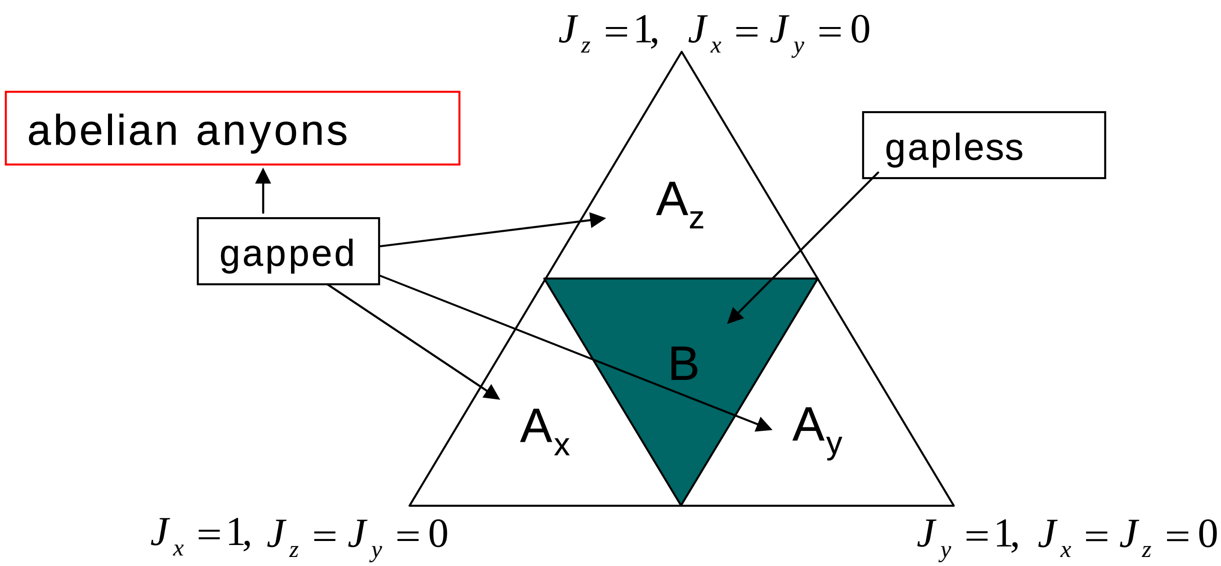
$$H = \sum_{jk} \left(\alpha_{jk} a_j a_k^+ + \alpha_{jk}^* a_j^+ a_k + \beta_{jk} a_j a_k + \beta_{jk}^* a_j^+ a_k^+ \right)$$

Lieb, Schultz and Mattis, Annals of Physics, **16**, pages 407-466 (1961)

Spectral Properties

- Discrete or gapped spectrum implies local excitations

$$H = -J_x \sum_{xx-links} \sigma_j^x \sigma_k^x - J_y \sum_{yy-links} \sigma_j^y \sigma_k^y - J_z \sum_{zz-links} \sigma_j^z \sigma_k^z$$



Spectral Properties

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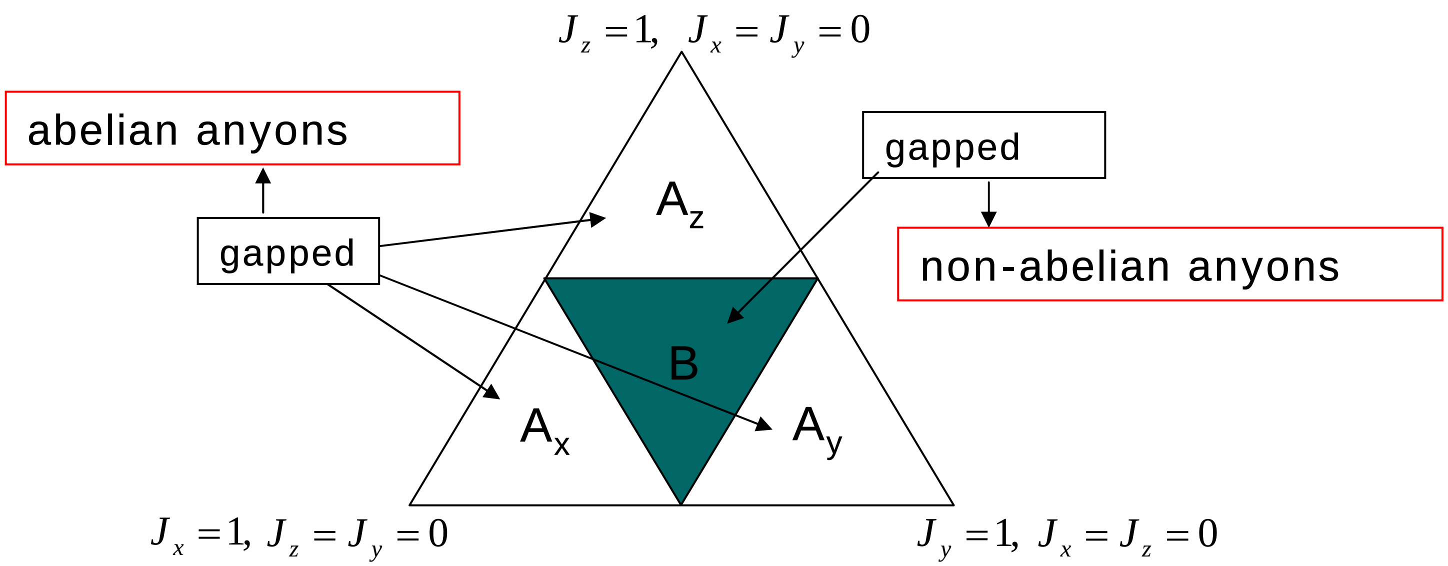
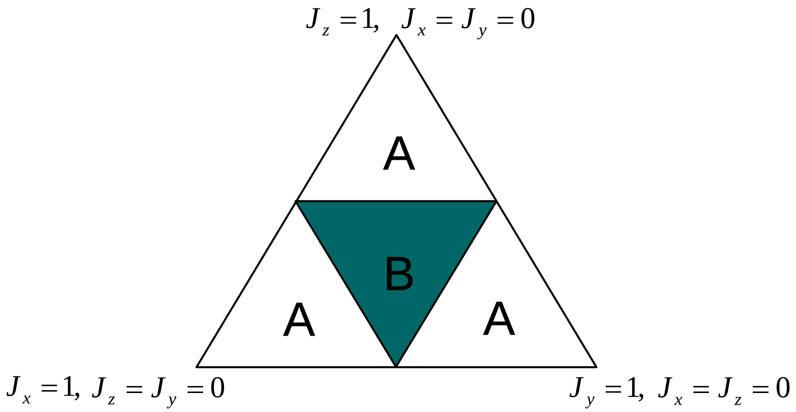
External Magnetic Field

$$+ \sum_n B_x \sigma_n^x + B_y \sigma_n^y + B_z \sigma_n^z$$



Our approach

- Examine finite size effects of the spectrum of the system without external magnetic field.



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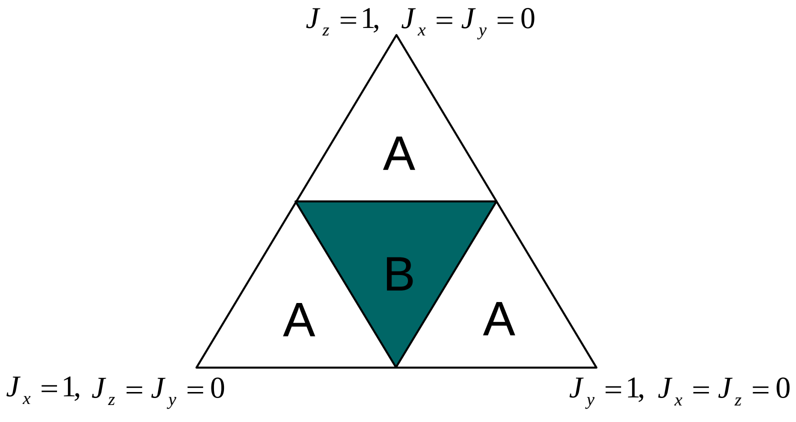
- Examine finite size effects of the spectrum of the system without external magnetic field.
- Study (B) phase with external magnetic field beyond the perturbative limit.



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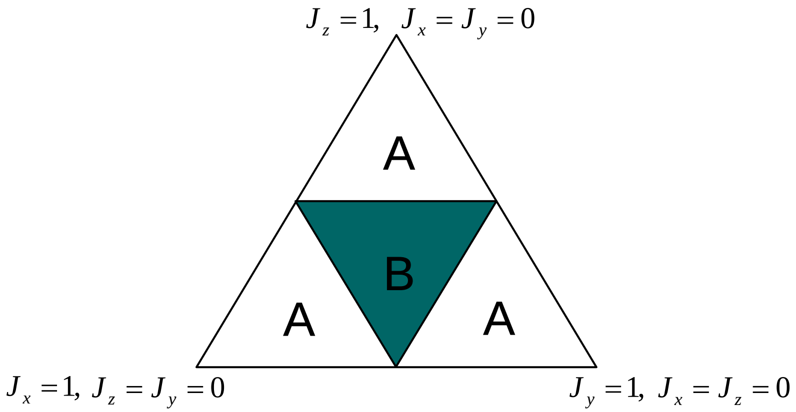
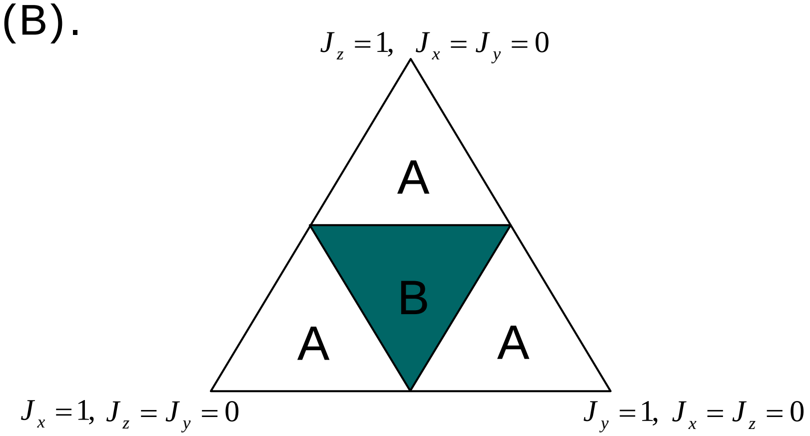
- Isolate and examine non-abelian anyons.



Our approach

- Examine finite size effects of the spectrum of the system without external magnetic field.
- Study (B) phase with external magnetic field beyond the perturbative limit.
- Isolate and examine non-abelian anyons.
- Examine phase-transition between abelian (A) and non-abelian phase (B).



Numerical Model

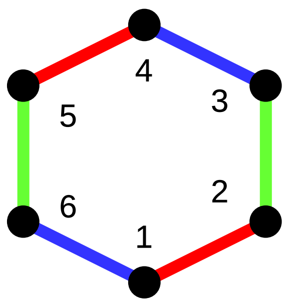
$$H = -J_x \sum_{xx-links} \sigma_j{}^x \sigma_k{}^x - J_y \sum_{yy-links} \sigma_j{}^y \sigma_k{}^y - J_z \sum_{zz-links} \sigma_j{}^z \sigma_k{}^z$$

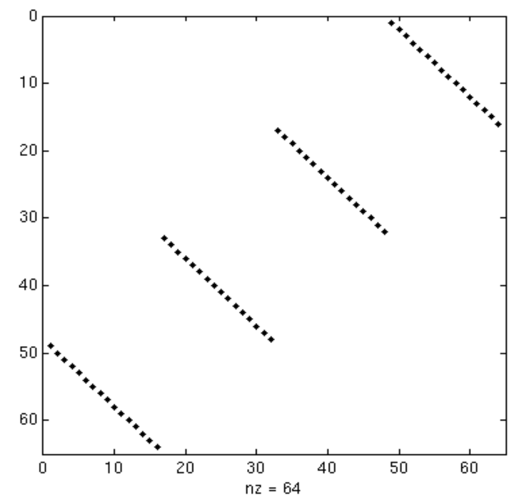
External Magnetic Field

$$+ \sum_n B_x \sigma_n{}^x + B_y \sigma_n{}^y + B_z \sigma_n{}^z$$

- Spin Hamiltonian can be represented as sparse Matrix, e.g.

$$\sigma_1{}^x \sigma_2{}^x = \sigma^x \otimes \sigma^x \otimes I \otimes I \otimes I \otimes I$$

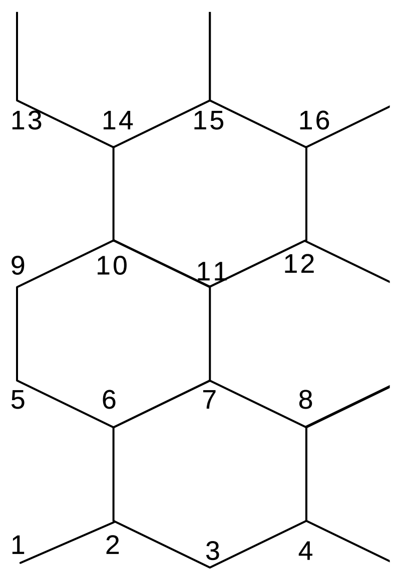


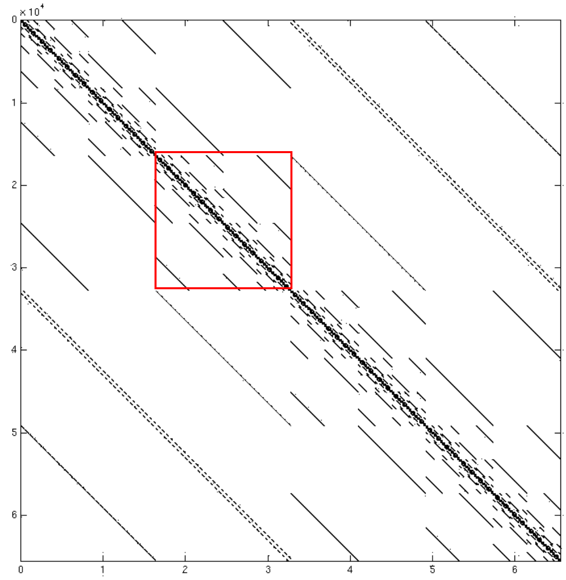


- Lower energy eigensolutions via the ARPACK library with LAW (Linear Algebra Wrapper) library to perform matrix multiplication

Numerical Model

- Spin Hamiltonian can be represented as sparse Matrix
- Choose valid lattice configuration and label lattice sites





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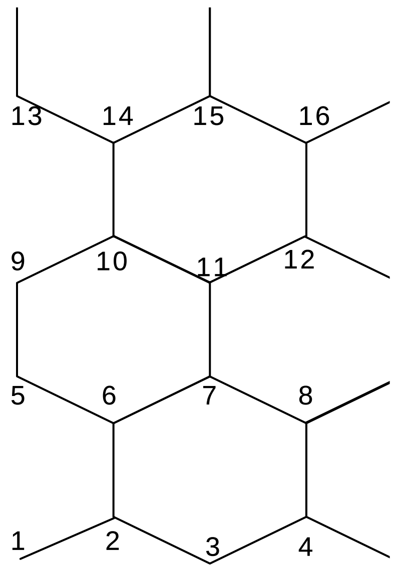

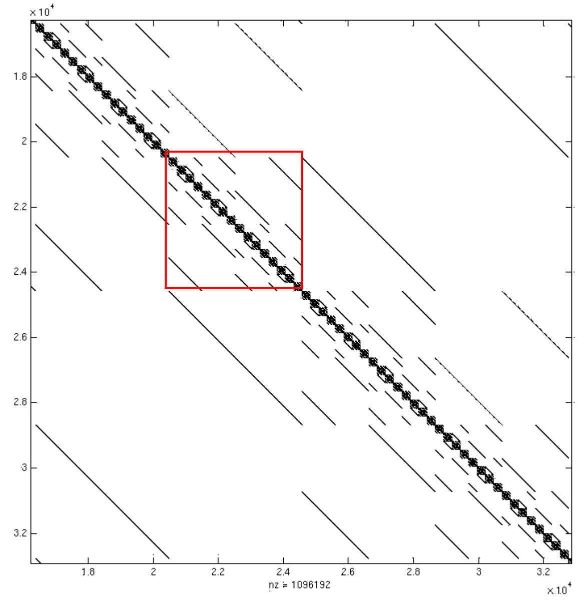

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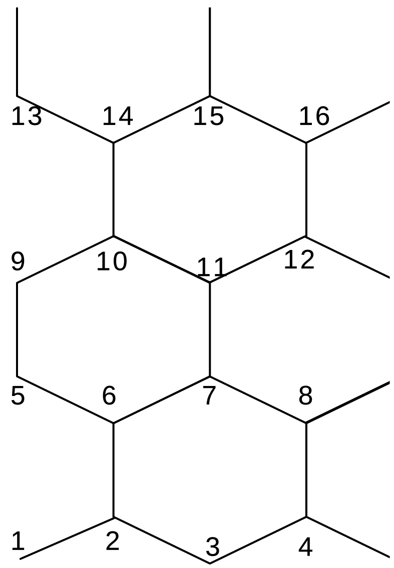

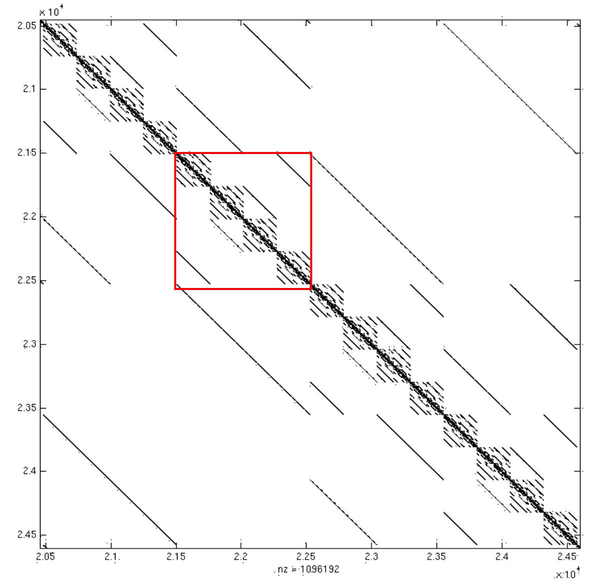

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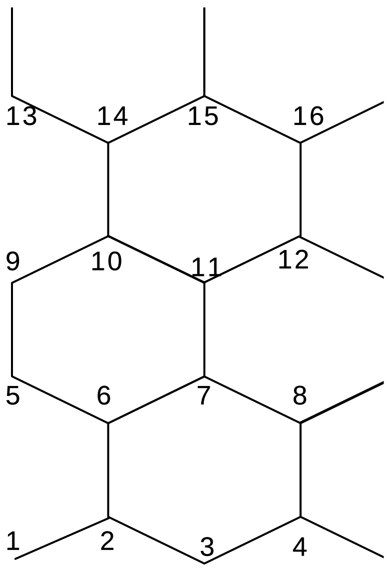

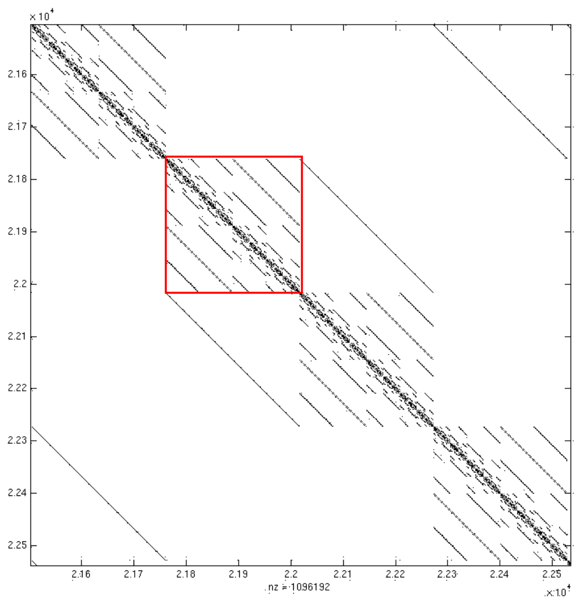

Numerical Model

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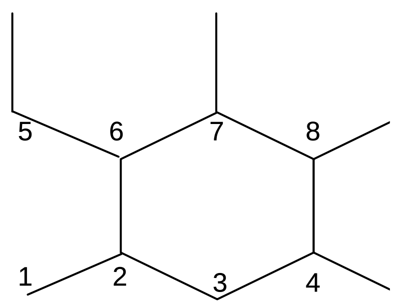
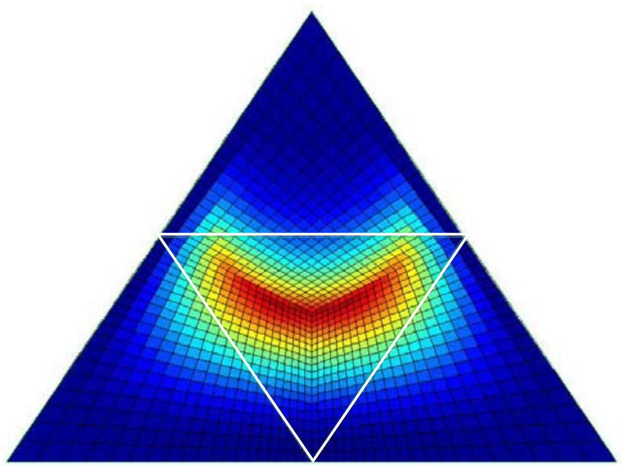
- Configurations examined





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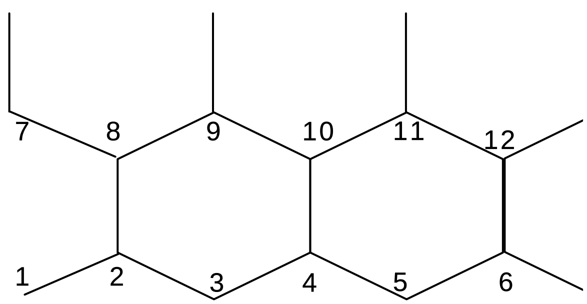

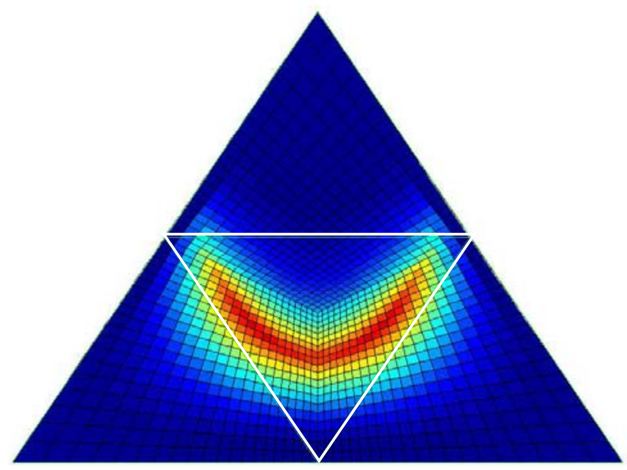

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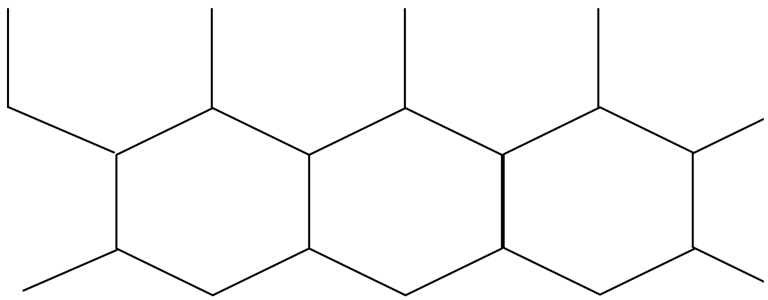

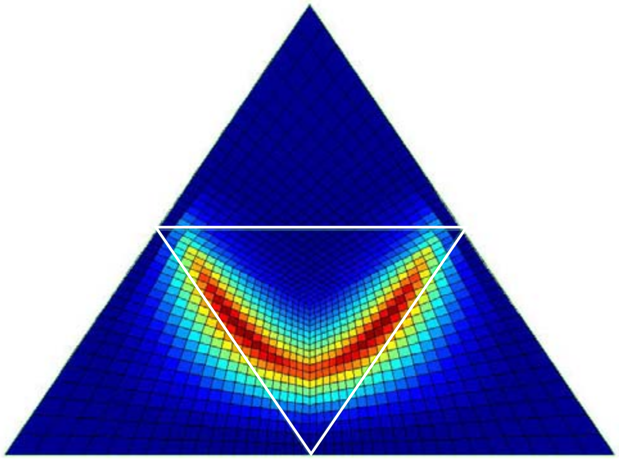

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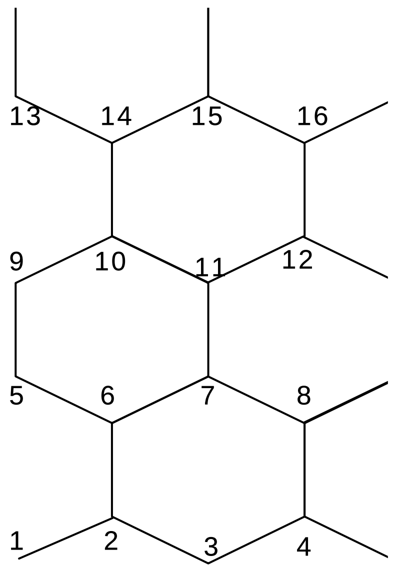

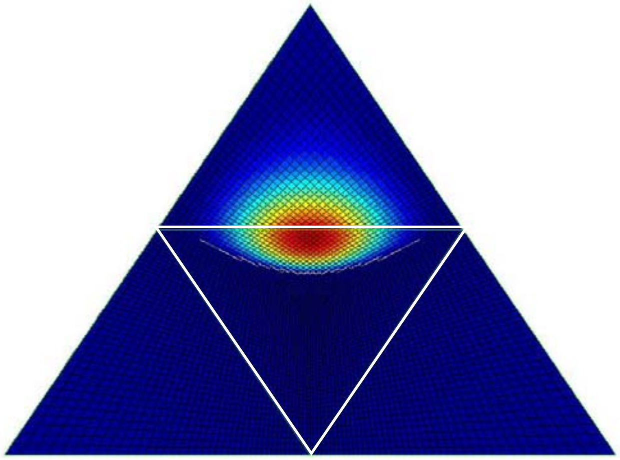

Hardware

- Currently calculating jobs for 20 and 24 and 28 spin lattice configurations

- Hamilton is a shared memory machine belonging to ICHEC (Irish Centre for High-End Computing),

- 32 Intel Itanium 2 processors and 256GB of RAM.

Hardware

- Plans to write code that will run on a distributed memory cluster.
- Walton has
  - 948 AMD Opteron cpus.
  - 2.1 TB memory.
  - 474 separate servers connected with gigabit Ethernet.
- This would allow > 32 spins to be analysed.

THANK YOU FOR YOUR ATTENTION