Anyon models – Theory, Interferometry, Bose condensation

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Based in part on work with

Parsa Bonderson, Kirill Shtengel, Sander Bais, Bernd Schroers

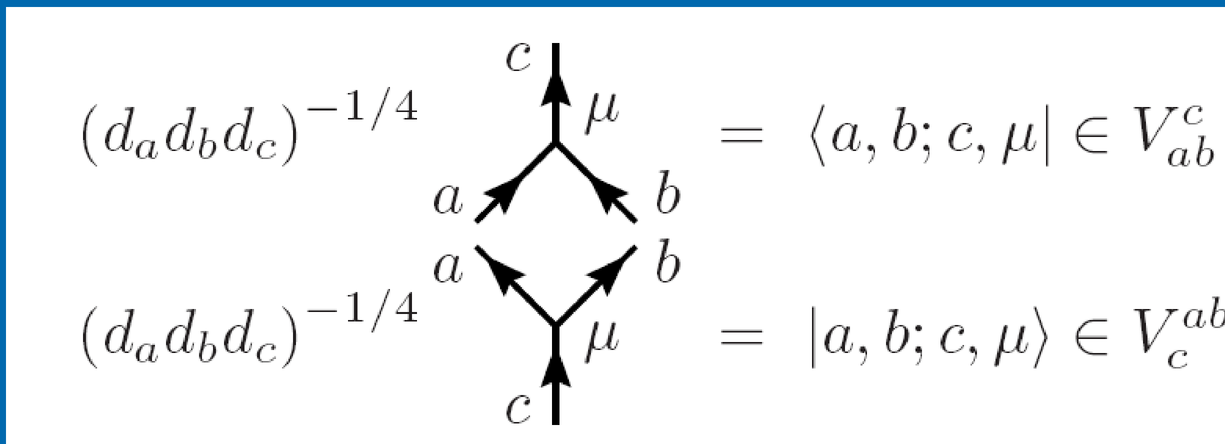
- Bonderson, Kitaev, Shtengel. PRL 96, 016803 (2006)
- Bonderson, JKS, Shtengel. PRL 97, 016401 (2006)
- Bonderson, JKS, Shtengel. Quant-ph/0608119, tbp in PRL
- Bais, Schroers, JKS. PRL 89:18601, 2002
- Bais, Schroers, JKS. JHEP 05:068, 2003
- Bais, Mathy, cond-mat/0602101, cond-mat/0602109, cond-mat/0602115

Part I: Some Theory of Anyon Models

- Want: algorithmic calculation of topological transition amplitudes (for example for Quantum Gate/Algorithm design)
- Need: Information that characterizes the model
  (input for the algorithm)
  Here F-symbols and R-symbols
- Get these from much more basic information (fusion rules), using consistency (pentagon and hexagon equations)

Fusion Theory

Fusion/splitting histories and States, bra vs. ket,
can build up multiparticle states, inner products, operators ("computations") etc.



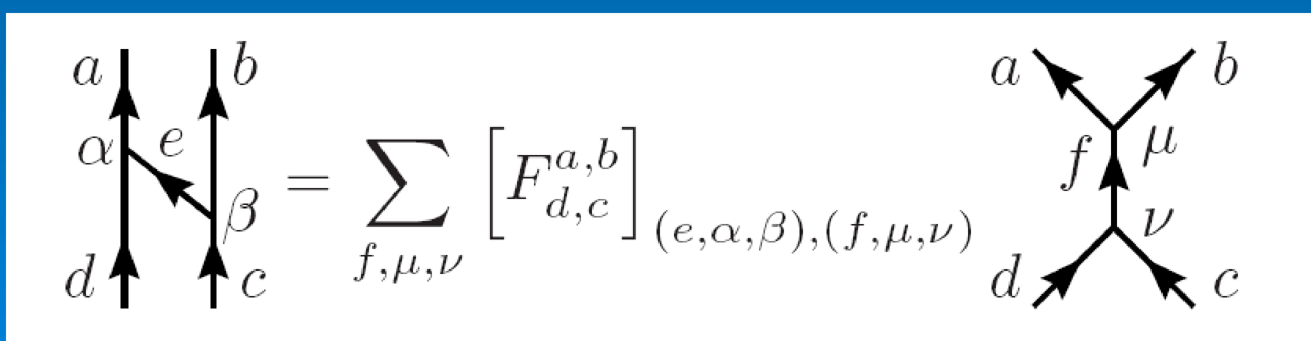
Dimensions of these spaces:

N_c^{ab}

Fusion rules:

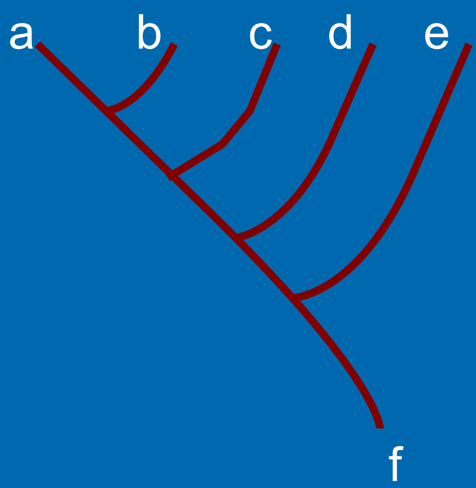
$a \times b = \sum_c N_c^{ab} c$

Recoupling, F-matrix / F-symbols
Needed a.o. for reduction to a standard (computational) basis

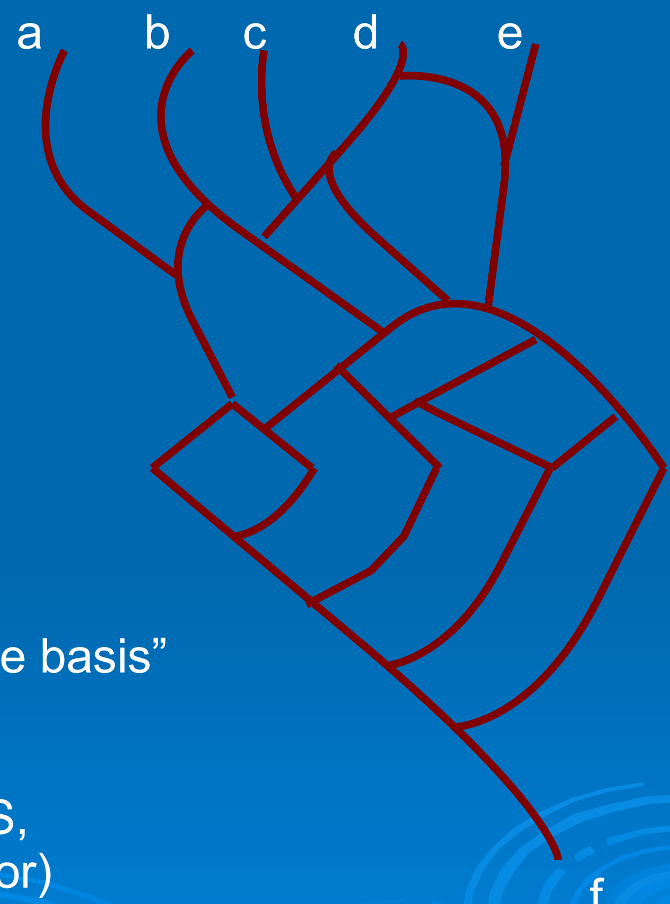


Standard vs. Nonstandard States

Standard



No Longer Standard

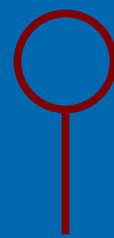
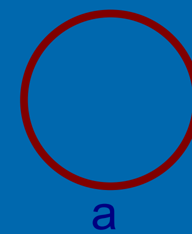


LHS gives an “intermediate charge basis” for the abcde-f Hilbert space

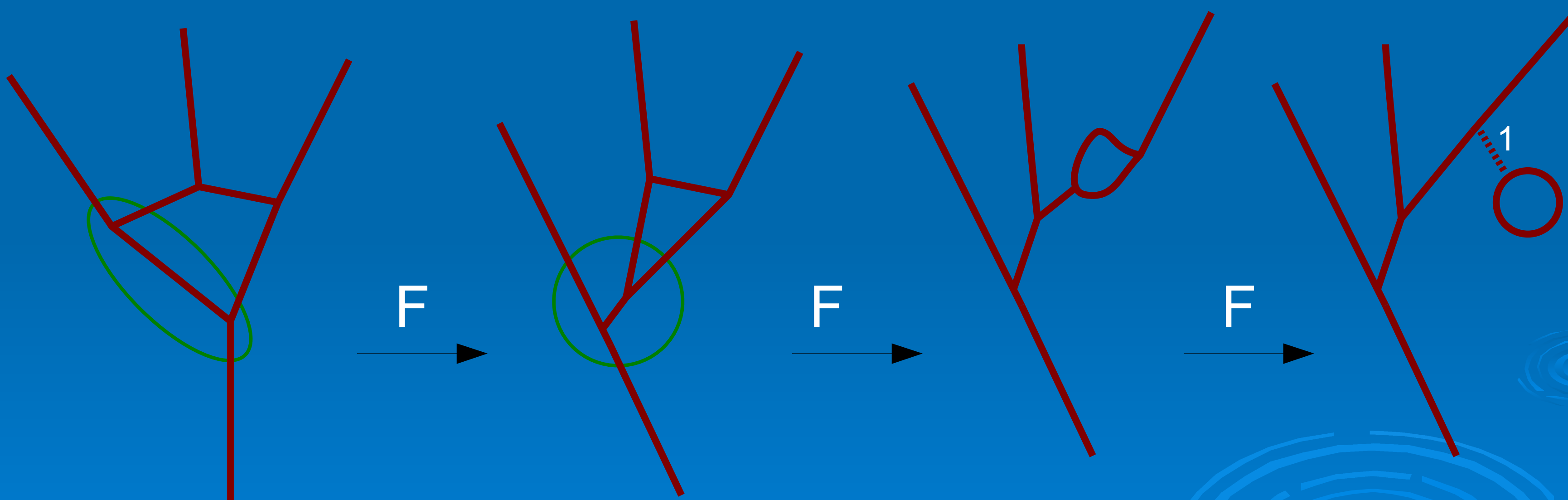
RHS could be an evolution of LHS,
(top half can be viewed as operator)
This is some superposition of basis states

Reduction Strategy

Strategy: Remove loops (surplus vertices) to reach a tree.
Then move branches around to get the chosen standard tree

Note: tadpoles vanish  $= 0$ but 

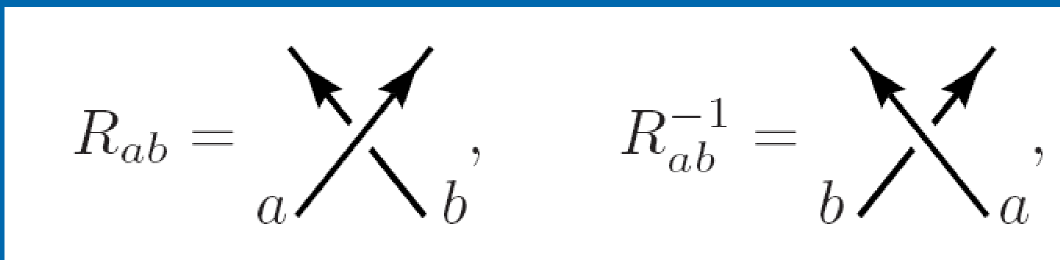
$= d_a$



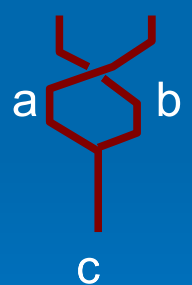
Two vertices removed at the end (only the term with a disconnected loop survives);
We can build an algorithm for complete reduction out of this.

Braiding, R-symbols, Twist

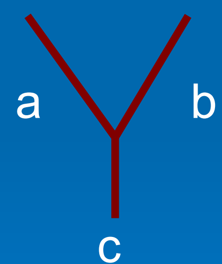
Braiding, R-matrix



Braiding acting on splitting spaces: R-symbols



$$= R_c^{ab}$$

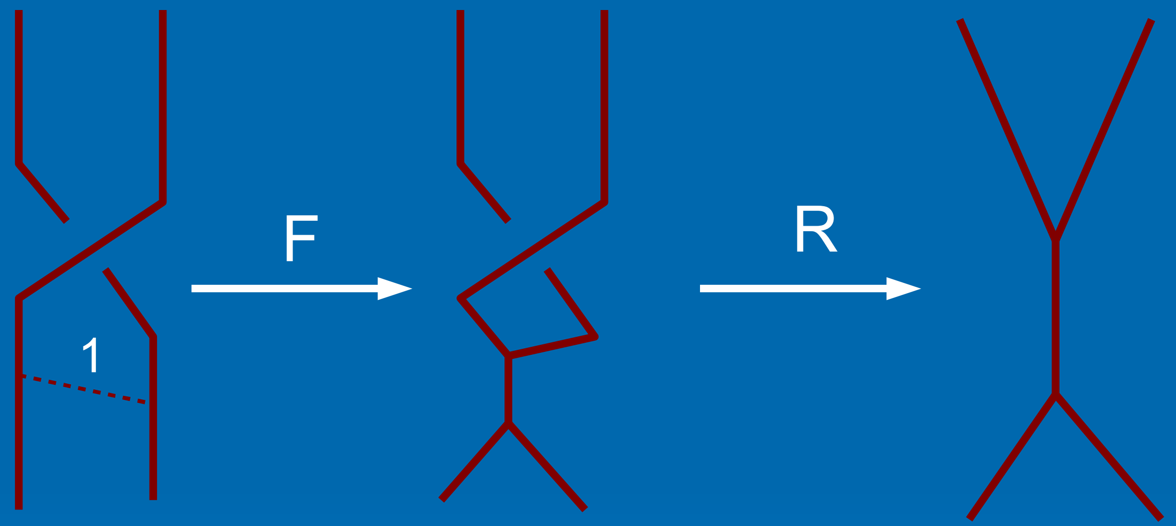


Twists, spin factors

$$= e^{2\pi i h_a}$$

Removing Crossings

Need F-Symbols and R-symbols for this

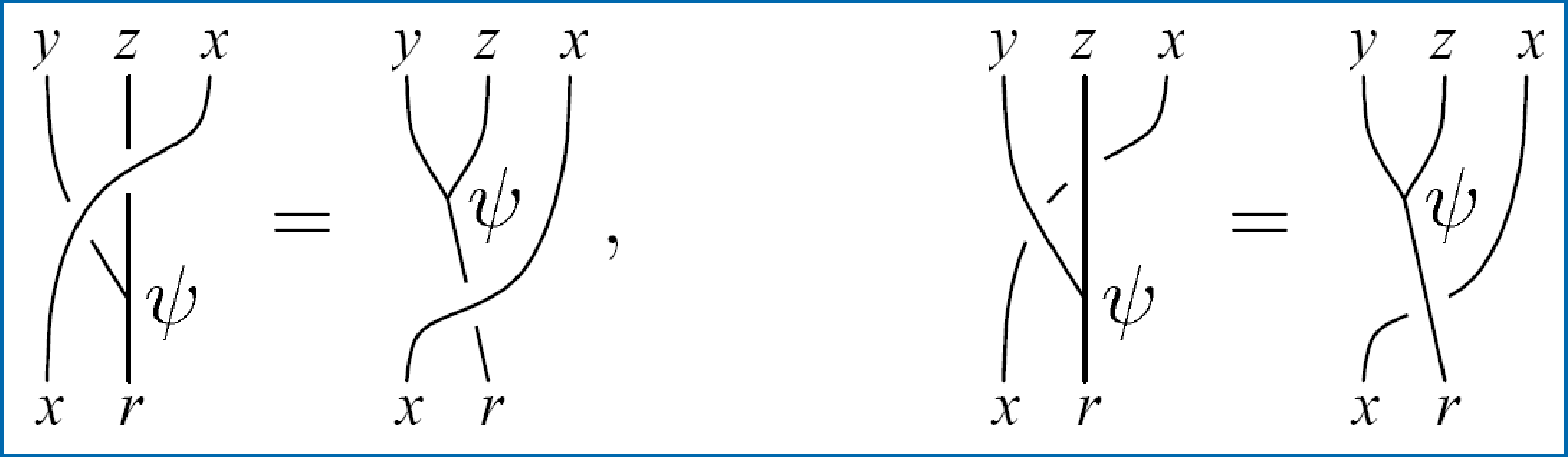


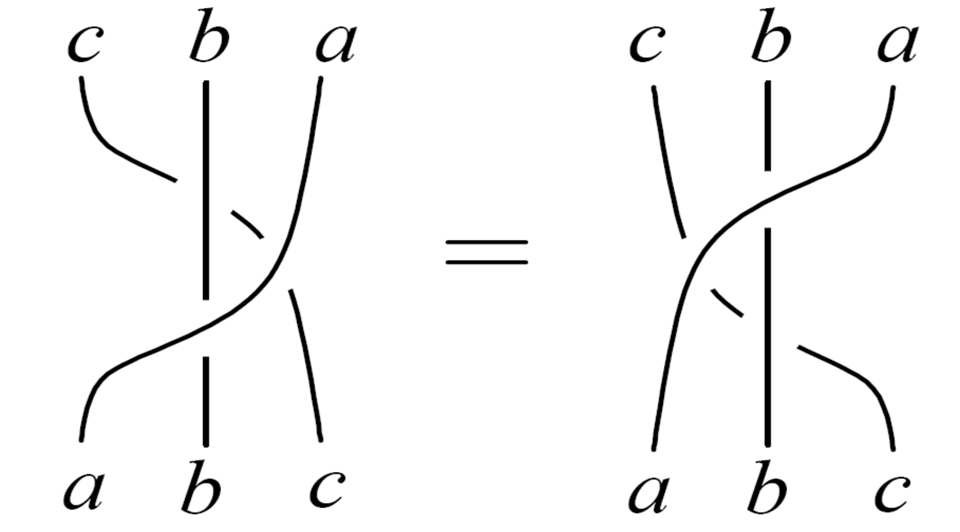
The two extra crossings can be removed by F-moves later on.

The suggested algorithm is very slow in general
(the number of terms generated is likely exponential in vertices/crossings)

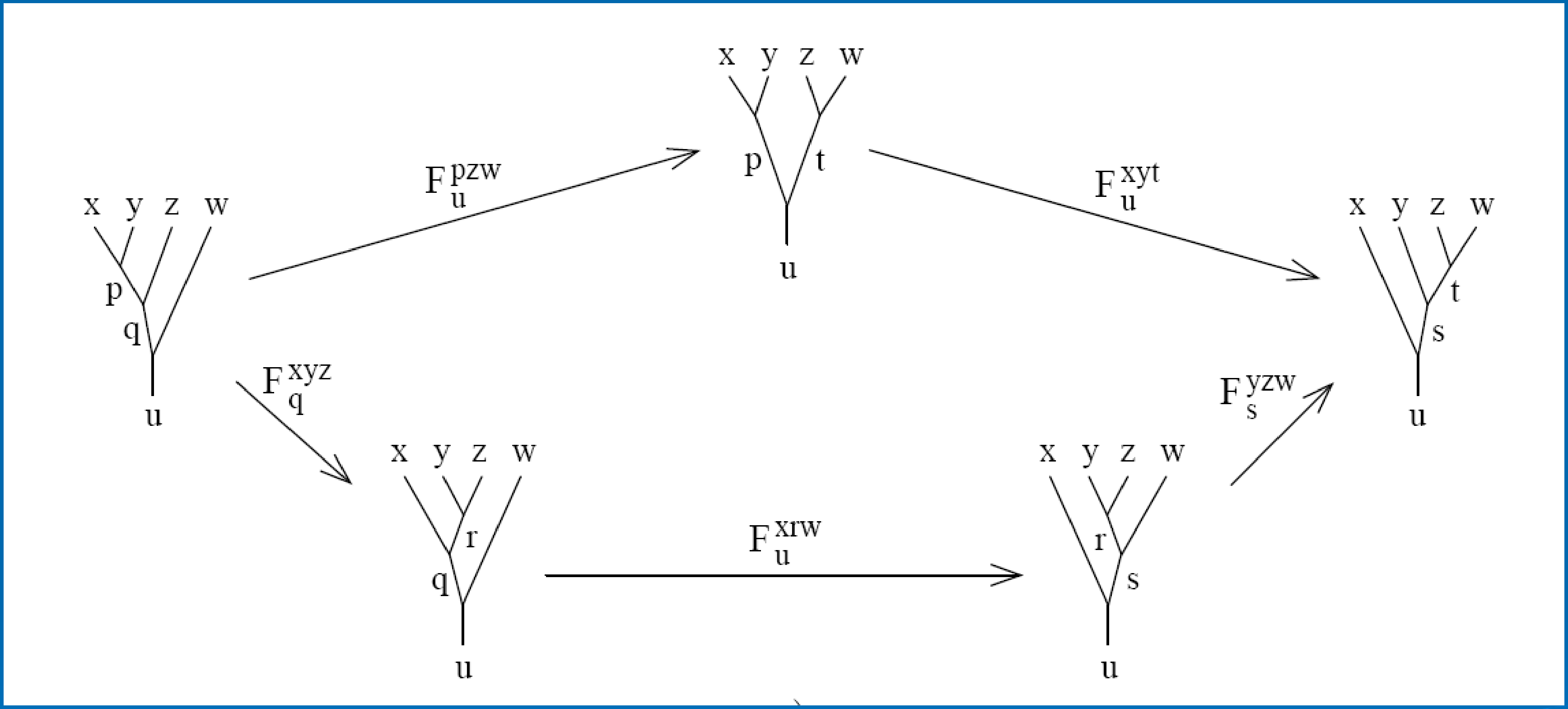
But that's why we can have/need the quantum computer :)

Braiding and Fusion Requirements



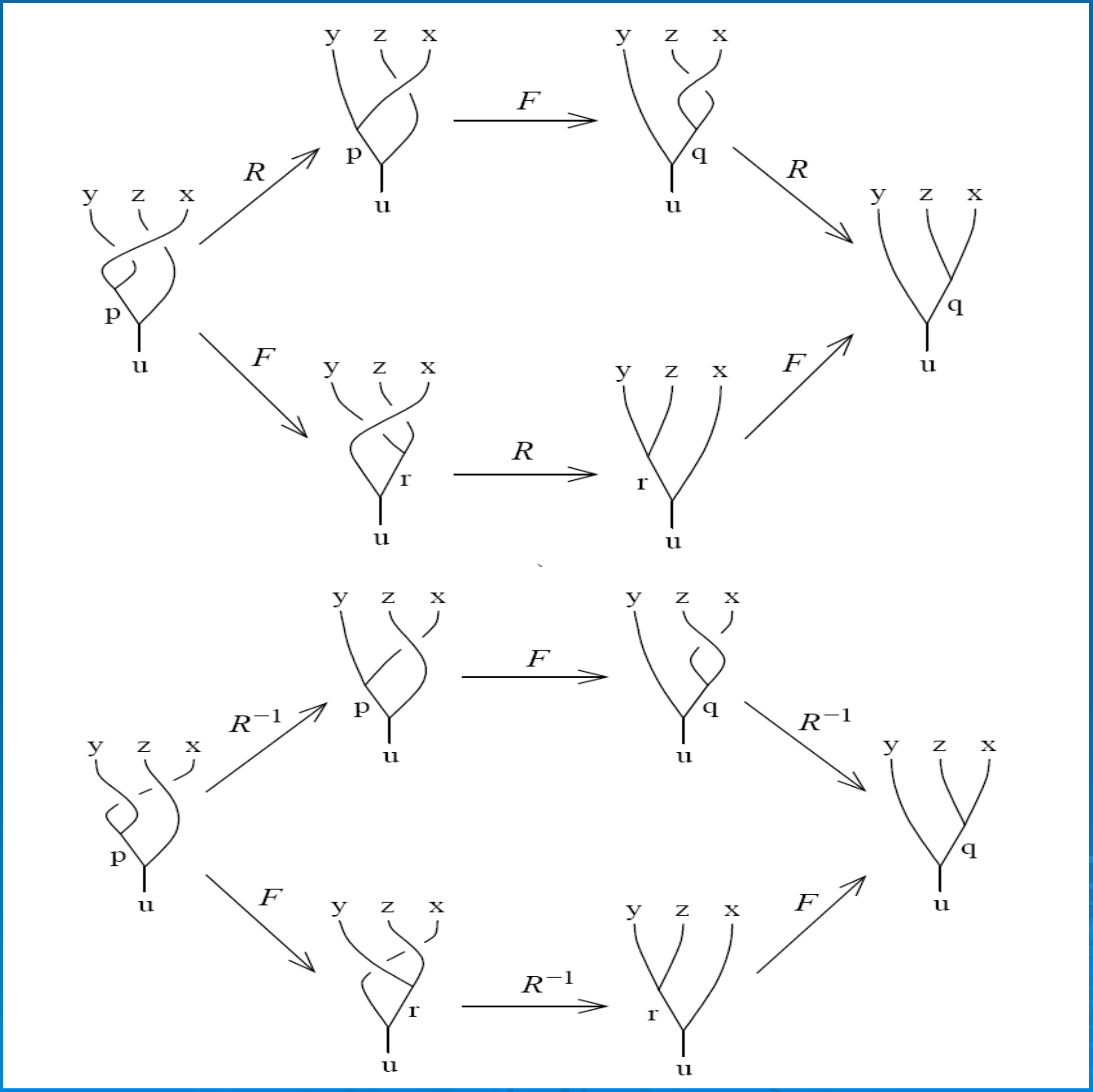


Consistency of Fusion: The Pentagon



Taken from Alexei Kitaev's notes

More Consistency of Braiding and Fusion: the Hexagons



Properties of the pentagon and hexagon equations

- Third order polynomial equations in many variables
- Many more equations than variables: solutions do not always exist
- "Gauge" freedom (basis choices) gives parameter families of equivalent solutions

- **Ocneanu rigidity**:
  only discrete solutions can exist modulo gauge freedom

- Once gauge freedom is fixed the equations can (in principle)
  be solved algorithmically, using Groebner bases (uses rigidity)
  (scales very badly with the number of variables)

- Number of variables can be drastically reduced by noting that
  many equations are linear in at least one variable

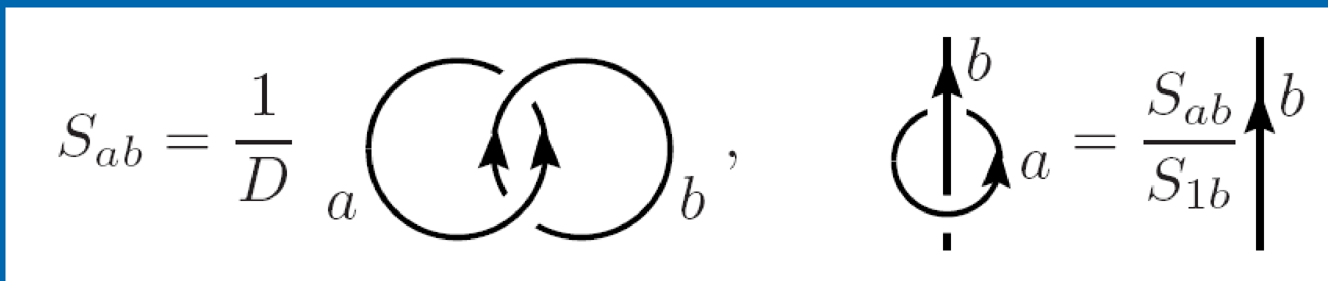
A program to solve the pentagon/hexagon using these observations
and automated gauging can solve theories up to 6 particles (and many more),
provided that the gauge can be fixed (Parsa Bonderson, JKS, in preparation)

The “periodic system” of TQFTs (Zhenghan Wang)

	1		A=abelian N=non-abelian U=universal for anyonic QC #=number of UNITARY theories	
	A 2 **$SU(2)_1$** Semion=Z_2	N 2 **$SO(3)_3$** Fib U2		
A 2 **Z_3** (ν=1/3)	N 8 **$SU(2)_2$** **Ising** (ν=5/2)	N 2 **$SO(3)_5$** U4		
A 4 **Z_4**	N 4 **$SU(2)_3$** (ν=12/5) U	N 2 **$SO(3)_7$** U6	N 4 **Fib×Fib** U	A 6 **$Z_2 \times Z_2$**

S-matrix etc.

Trace and ‘S-matrix’



S-matrix gives fusion by Verlinde's formula:

$$N_{ab}^{c} = \sum_{x} \frac{S_{ax} S_{bx} S_{\bar{c}x}}{S_{1x}}$$

Using Ocneanu rigidity, S almost determines the theory!

Part II: Quantum Hall Interferometry

- Quantum Hall systems are closest to experimental realisation of nonabelions
- Have proposals to detect these by interferometry, experiments underway

- What will we need from TQFT to describe the experiments?
- Conversely will we learn about the Hall TQFTs?

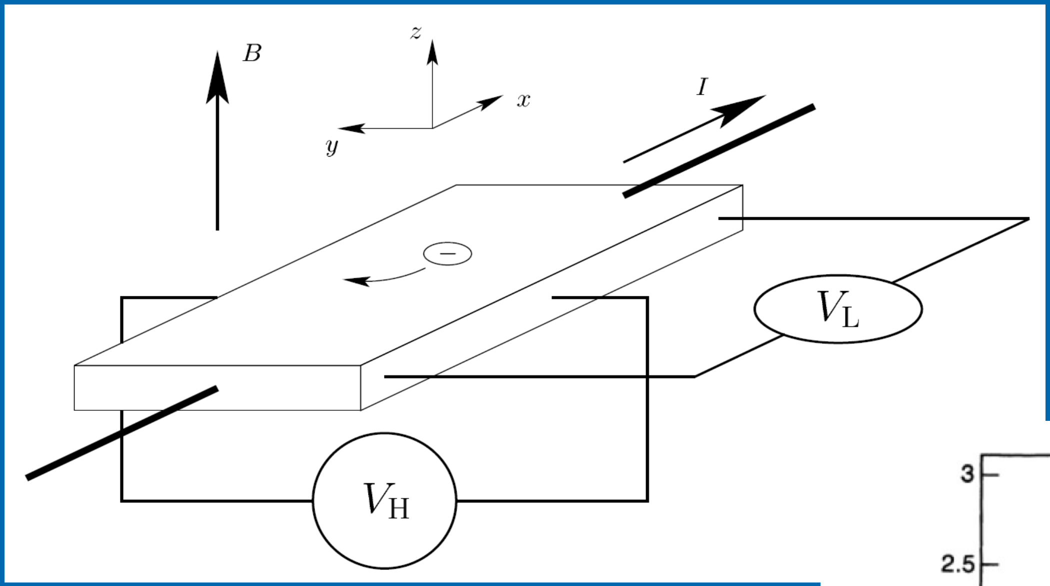
Turns out:
Only the normalized monodromy matrix M is important for Hall interferometry:

$$M_{ab} = \frac{S_{ab} S_{11}}{S_{a1} S_{b1}}$$

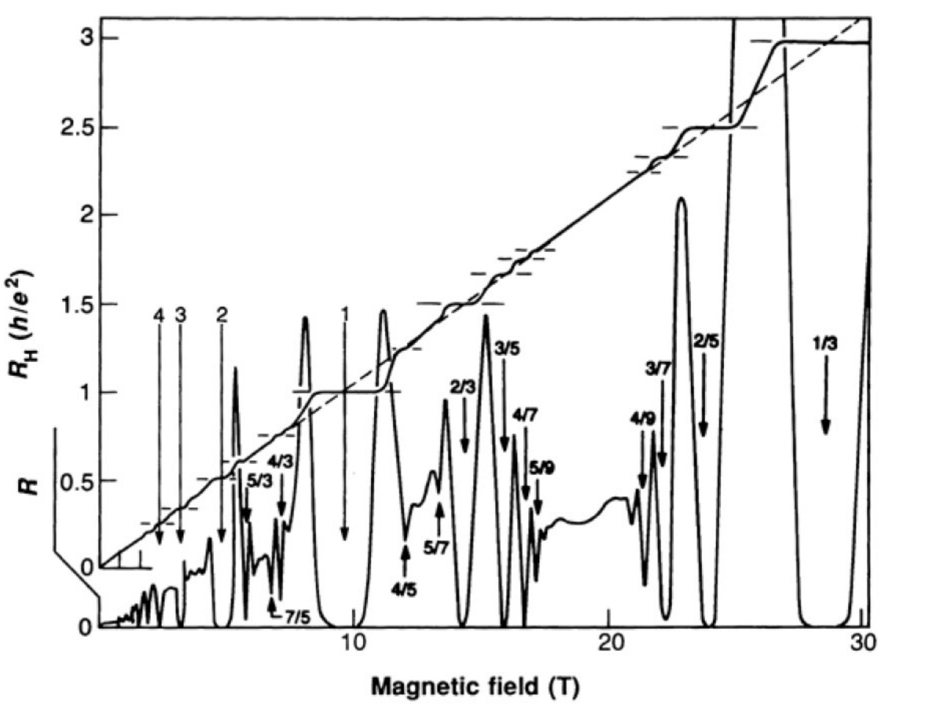
Note $|M_{ab}| \leq 1$

$M_{ab} = 1$ signals trivial monodromy

The Quantum Hall Effect



B ~ 10 Tesla
T ~ 10 mK



Eisenstein, Stormer, Science 248, 1990

On the Plateaus:

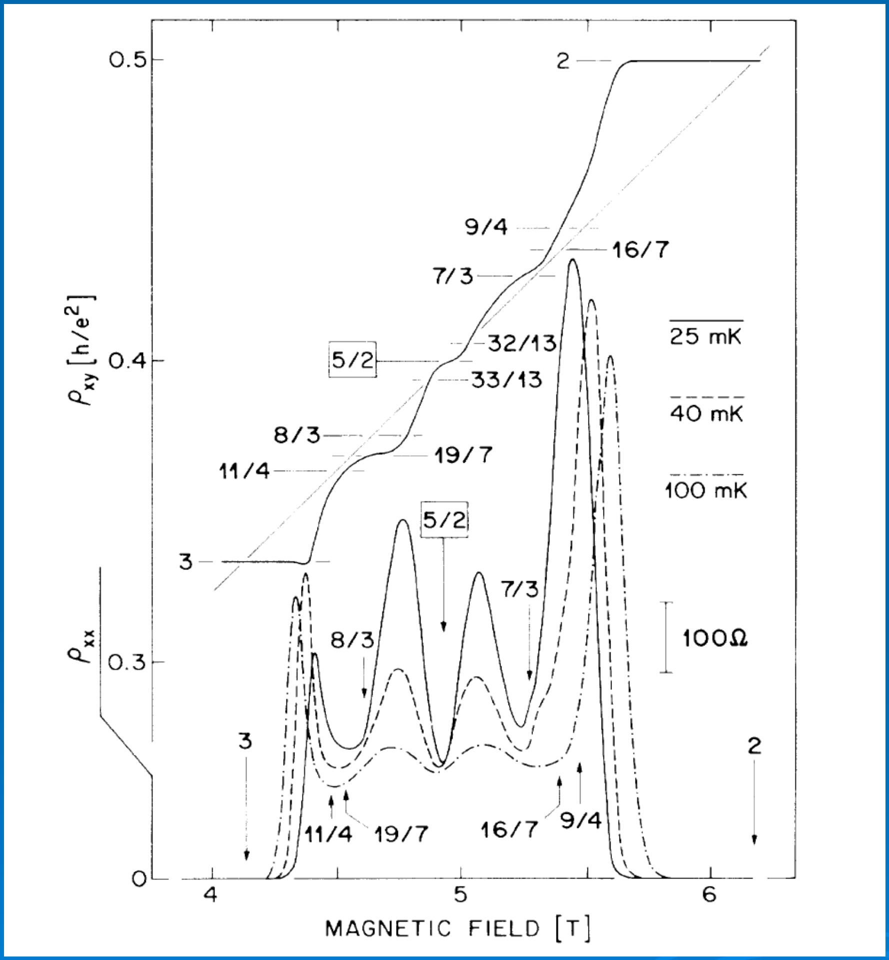
- Incompressible electron liquids
- Off-diagonal conductance:

values $\nu \dfrac{e^2}{h}$ Filling fraction $\nu = \dfrac{p}{q}$

- Vortices with fractional charge
- +AB-effect: fractional statistics

(Abelian) **ANYONS!**

An Unusual Hall Effect



Willett et al. PRL 59, 1776, 1987

Filling fraction 5/2: even denominator!

Now believed to have

- electrons paired in ground state (exotic p-wave 'superconductor')
- halved flux quantum
- charge e/4 quasiholes (vortices) which are **Non-Abelian Anyons**

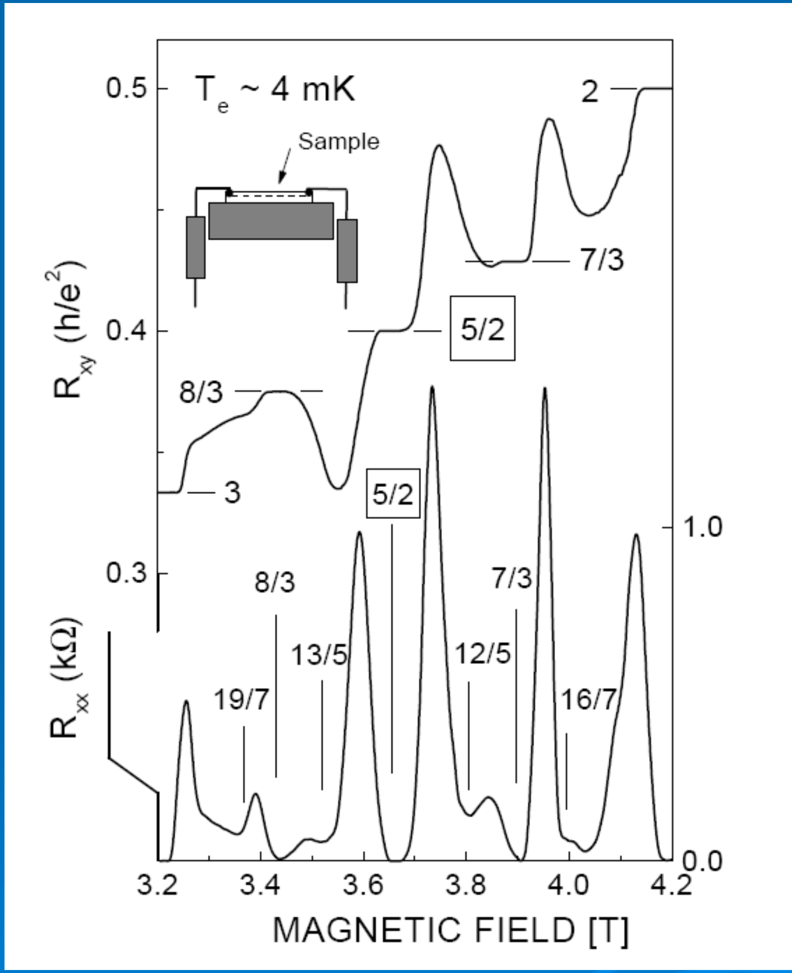
(exchanges implement non-commuting unitaries)

Moore, Read, Nucl. Phys. B360, 362, 1991

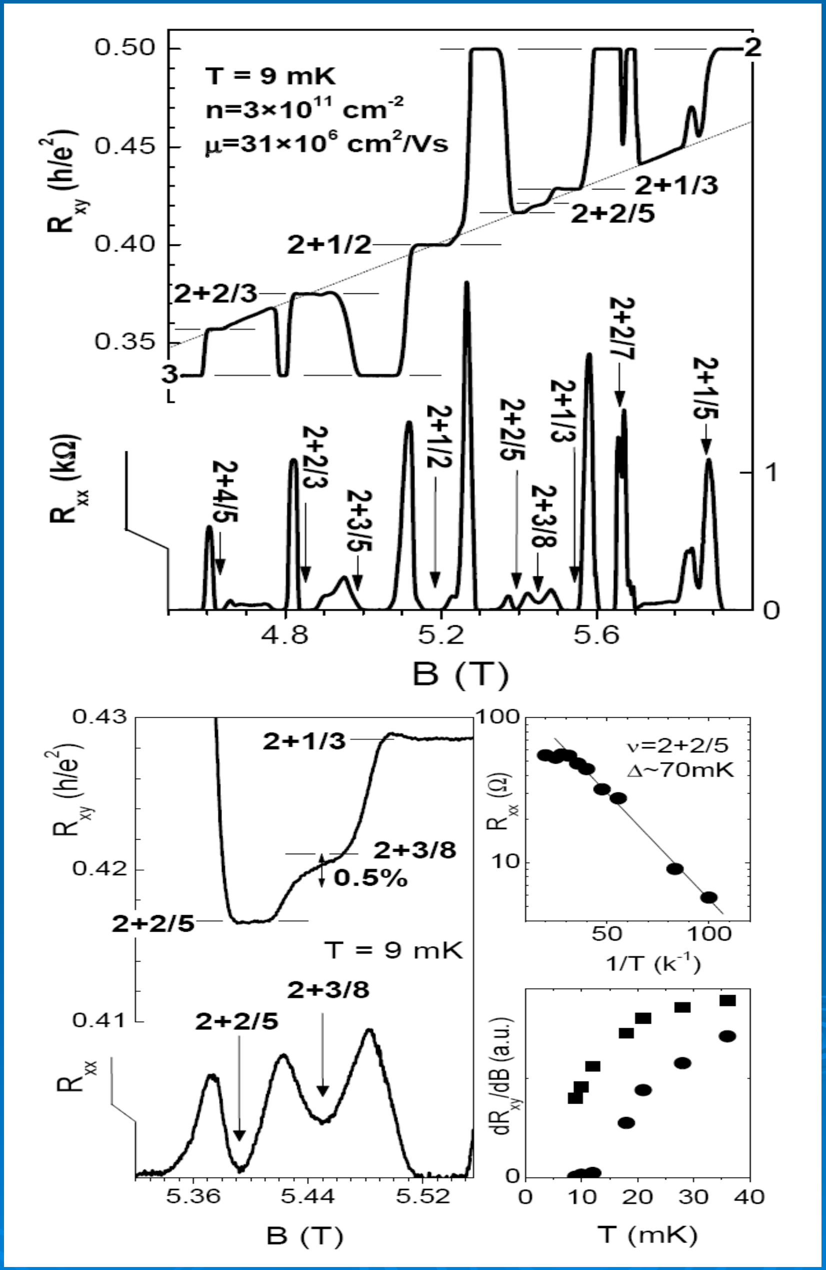
Can use braiding interaction for Topological Quantum Computation

(not universal for 5/2 state, but see later)

Experimental Progress

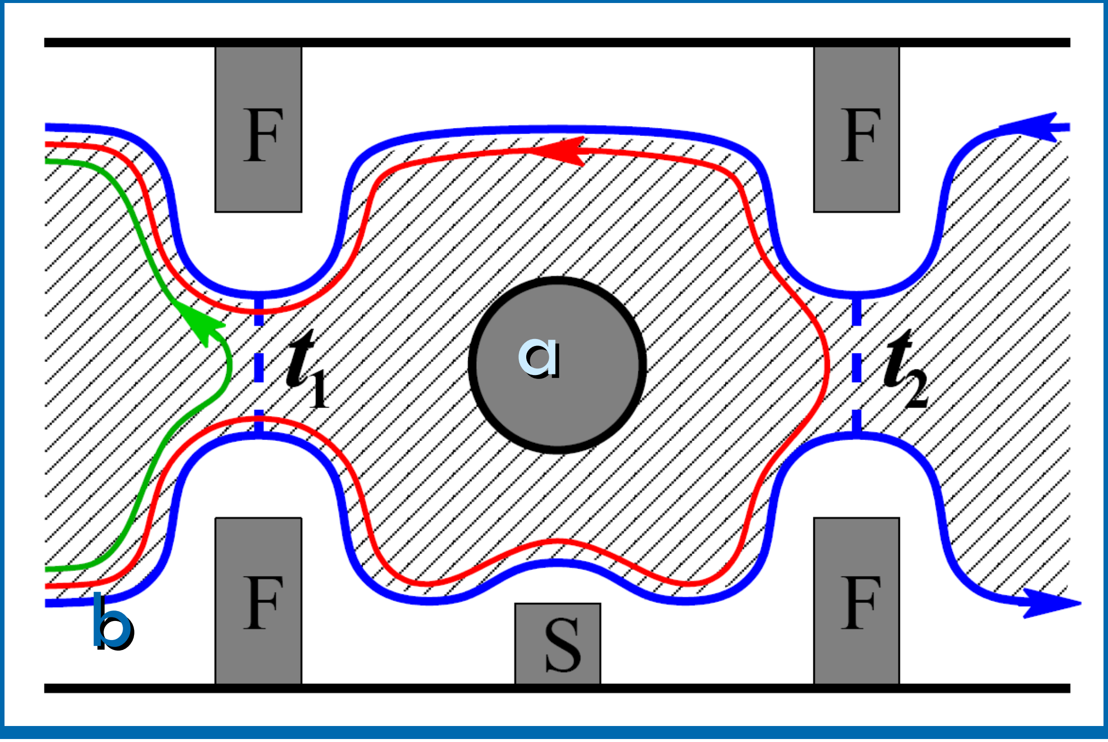


Pan et al. PRL 83, 1999
Gap at 5/2 is 0.11 K



Xia et al. PRL 93, 2004,
Gap at 5/2 is 0.5 K, at 12/5: 0.07 K

Quantum Hall Interferometry

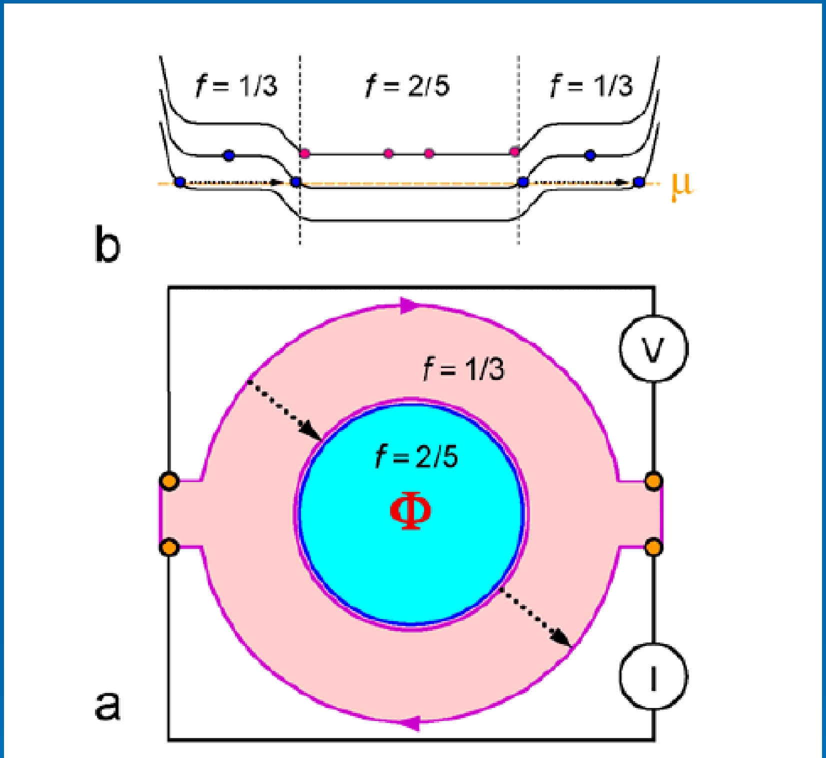


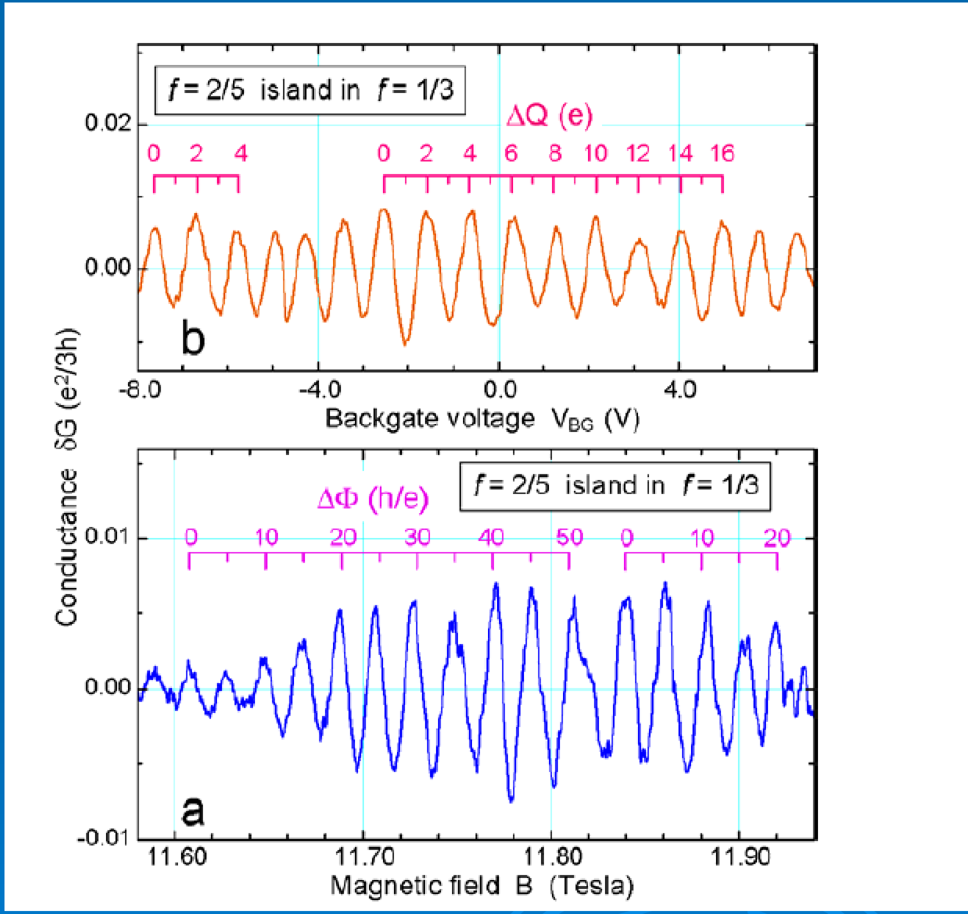
$$\sigma_{xx} \propto |t_1|^2 + |t_2|^2 + 2\mathrm{Re}\left\{ t_1^* t_2 \langle \Psi_{ab} | U_1^{-1} U_2 | \Psi_{ab} \rangle \right\}$$
$$= |t_1|^2 + |t_2|^2 + 2|t_1 t_2| \, |M_{ab}| \cos(\beta + \theta_{ab}) \; .$$

Interference suppressed by |M|: effect from non-Abelian braiding!

(This should actually be easier to observe than the phase shift from Abelian braiding…)

Actual experiments (abelian Anyons)





Camino, Zhou, Goldman, Phys. Rev. B72 075342, 2005

Part III
Topological Symmetry Breaking and Bose Condensation

- Can describe topological order by extended "symmetry" concepts: TQFTs, Tensor Categories, Hopf Algebras, Quantum Groups

Particle types	$\longleftrightarrow$	Irreducible representations
Fusion	$\longleftrightarrow$	Tensor Product
Braiding	$\longleftrightarrow$	R-matrix
Twist	$\longleftrightarrow$	Ribbon Element

- IDEA:

  Relate topological phases by "Symmetry Breaking"

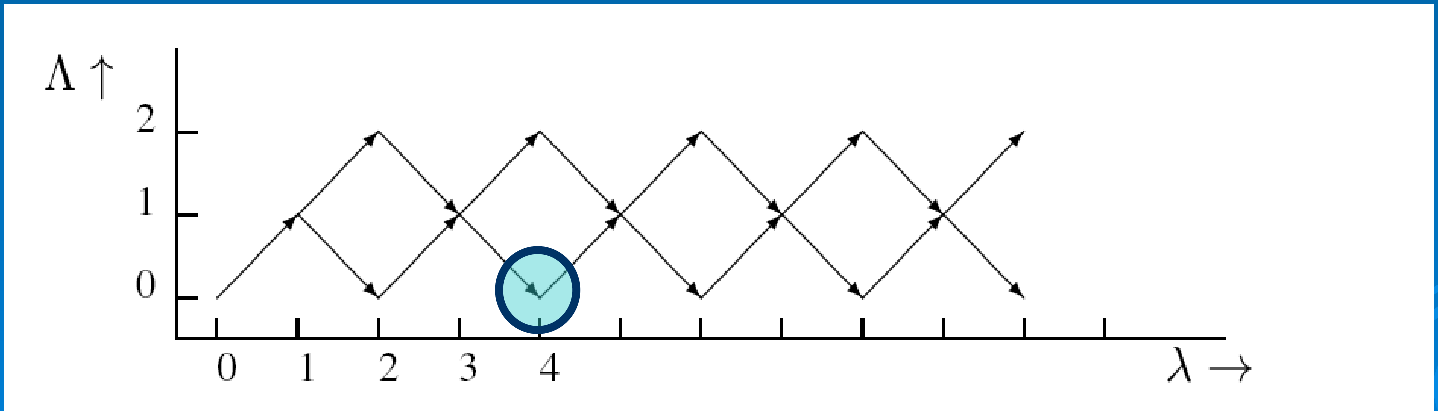
- Mechanism? Bose Condensation!

  Break the Quantum Group to the "Stabiliser" of the condensate's order parameter

On Bosons

- What is a boson? A particle with
  - trivial twist factor/ integer conformal weight
  - trivial self braiding in at least on fusion channel,
    i.e. at least one of the fusion products also has trivial twist/integer weight

- Have a boson in the Pfaffian state (below)
  and lots of bosons in the higher RR-states (k=4 upwards)

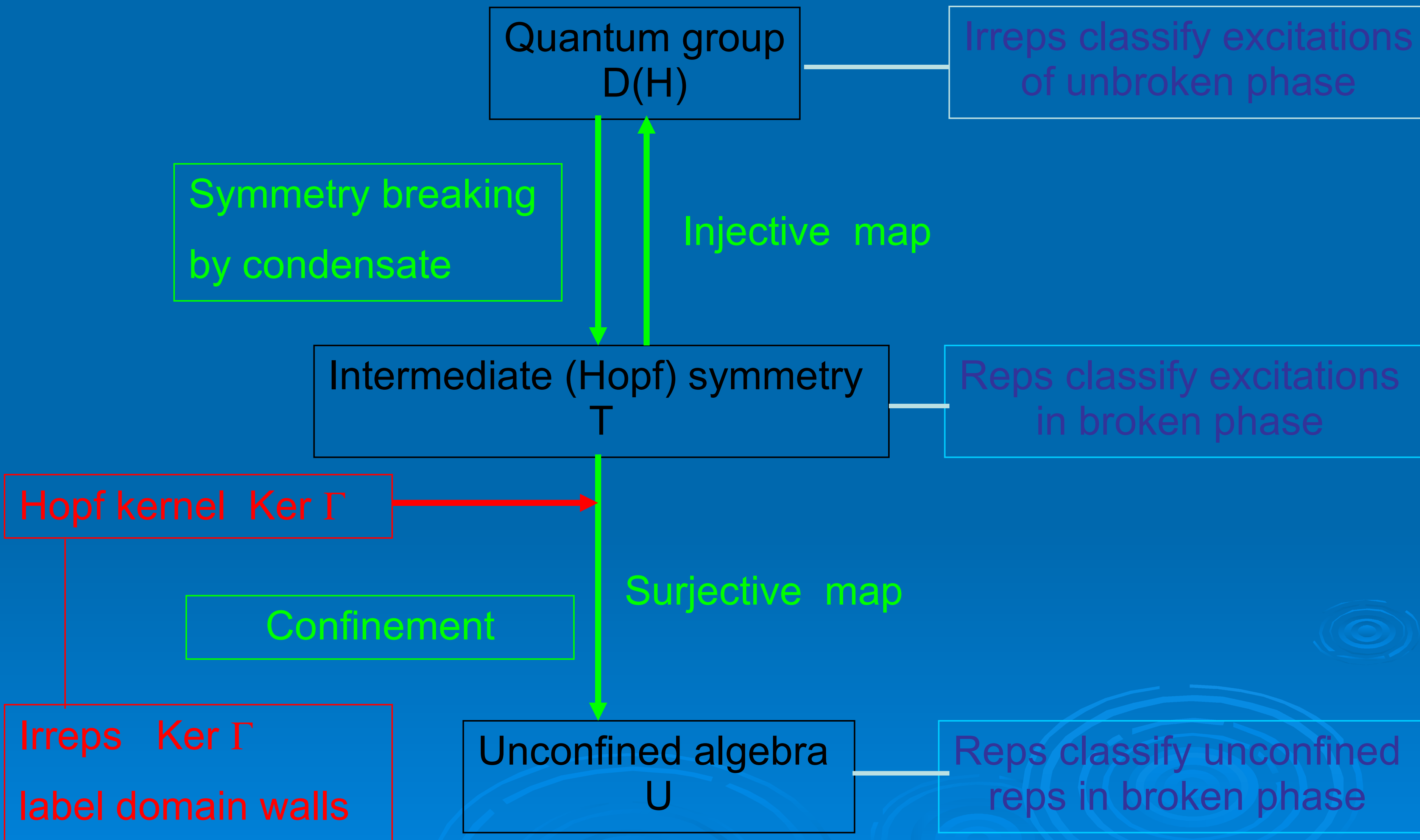


Contains

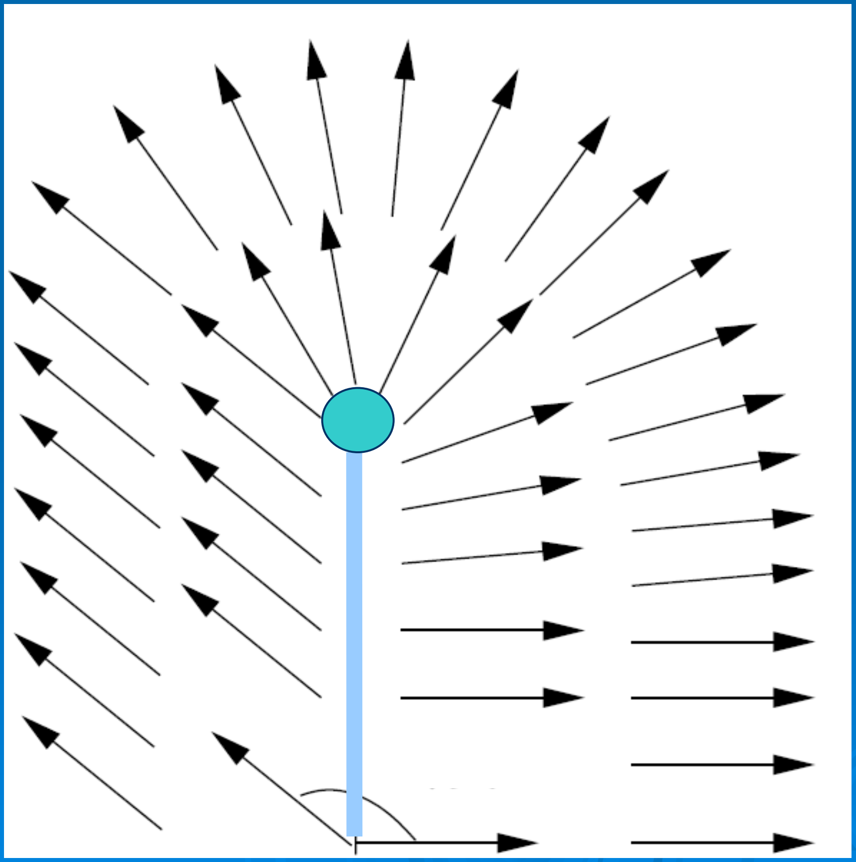
$$e^{i\sqrt{2}\varphi}$$

U(1) conformal primary with h_b=1,
Trivial self-braiding

Symmetry breaking scheme

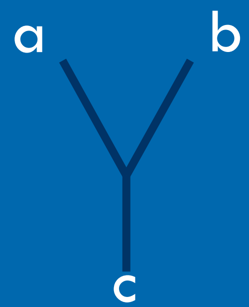


Confined Excitation



Quantum group symmetry breaking: What we will use here

Fusion



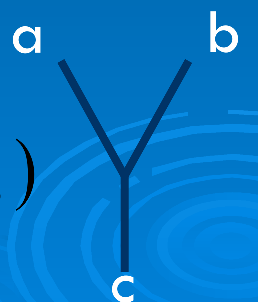
$$a \times b = \sum_c N_c^{ab} c$$

Twist

$$= e^{2\pi i h_a}$$

Monodromy

$$= e^{2\pi i (h_c - h_a - h_b)}$$



"Symmetry breaking" from the dual side

Inspired by usual algebra symmetry breaking,
introduce branchings for topological sectors:

$$a \to \sum_i n_{a,i} a_i$$

Note: condensate must branch to vacuum (+ possibly more)

Requirements

1. The new labels themselves form a fusion model
   (need associativity, vacuum and charge conjugation)
2. Branching and fusion commute,

$$a \otimes b \to (\sum_i n_{a,i} a_i) \otimes (\sum_i n_{b,i} b_i)$$

This implies preservation of quantum dimensions (useful in calculations)

Breaking $SU(2)_4$

$SU(2)_4$

0	$d_0 = 1$	$h_0 = 0$
1	$d_1 = \sqrt{3}$	$h_1 = 1/8$
2	$d_2 = 2$	$h_2 = 1/3$
3	$d_3 = \sqrt{3}$	$h_3 = 5/8$
4	$d_4 = 1$	$h_4 = 1$

$$1 \times 1 = 0 + 2$$
$$1 \times 2 = 1 + 3 \qquad 2 \times 2 = 0 + 2 + 4$$
$$1 \times 3 = 2 + 4 \qquad 2 \times 3 = 1 + 3 \qquad 3 \times 3 = 0$$
$$1 \times 4 = 3 \qquad 2 \times 4 = 2 \qquad 3 \times 4 = 1 \qquad 4 \times 4 = 0$$

Condensate, splitting and identification

Assume a bosonic condensate forms in the 4 rep of $SU(2)_4$:

$$2\times 2 = 0+2+4 = 0+2+0$$

$\Rightarrow\ 2 := 2_1 + 2_2$ possible because $d_2=2$

$$2_1\times 2_1 + 2_1\times 2_2 + 2_2\times 2_1 + 2_2\times 2_2 = 0+2_1+2_2+0$$

$$\Rightarrow\ 2_1\times 2_2 = 0$$

$$\text{if } 2_1\times 2_1 = 2_1$$

$$\text{then } 2_2\times(2_1\times 2_1) = 2_2\times 2_1 = 0$$

$$(2_2\times 2_1)\times 2_1 = 0\times 2_1 = 2_1$$

$$\Rightarrow 2_1\times 2_1 = 2_2 \text{ and } 2_2\times 2_2 = 2_1$$

$$1\times 1 = 0+2_1+2_2$$

$$1\times 3 = 0+2_1+2_2$$

$$\Rightarrow 1 \Leftrightarrow 3$$

$$0$$

$$1$$

$$2 := 2_1 + 2_2$$

$$3 \Leftrightarrow 1$$

$$4 \Leftrightarrow 0$$

$$1\times 1 = 0+1$$

$$1\times 2_1 = 1 \qquad 2_1\times 2_1 = 2_2$$

$$1\times 2_2 = 1 \qquad 2_1\times 2_2 = 0 \qquad 2_2\times 2_2 = 2_1$$

Confinement and Braiding

To see which of the particles in the broken theory are confined, look at braiding with the condensed particle.
How? For particle a_i, look in all channels of the old theory that cover $a_i x 1 = a_i$

Now notice: Fields that cover 1 have trivial twist factor (condensate is bosonic). Hence braiding with the vacuum is trivial and a_i is not confined precisely when all the fields that branch to a_i have equal twist factors (or conformal dimensions that differ by integers).

The non-confined particles all have well defined monodromies with each other, given by their twist factors (which are unambiguously defined from the branching).

Confinement for $SU(2)_4$

From branching rules and conformal weights one finds that the 1 and 3 are confined.

The unconfined algebra becomes $SU(3)_1$:

$$\begin{array}{lcl} 2_1 \times 2_1 = 2_2 & & 3 \times 3 = \bar{3} \\ 2_1 \times 2_2 = 0 & \longleftrightarrow & 3 \times \bar{3} = 1 \\ 2_2 \times 2_2 = 2_1 & & \bar{3} \times \bar{3} = 3 \end{array}$$

Relation to Conformal Embedding

Central charges satisfy c(G) = c(H) ==> c(G/H) = 0
Coset algebra is trivial.
==> Finite branching of inf. Dim. KM representations

Example: $SU(2)_4$ ==> $SU(3)_1$ (c=2)

$SU(3)_1$ Irreps:

1	$d_1 = 1$	$h_1 = 0$
3	$d_3 = 1$	$h_3 = 1/3$
$\bar{3}$	$d_{\bar{3}} = 1$	$h_{\bar{3}} = 1/3$

$$\boxed{\begin{array}{l} 3 \times 3 = \bar{3} \\ 3 \times \bar{3} = 1 \\ \bar{3} \times \bar{3} = 3 \end{array}} \longrightarrow \boxed{\begin{array}{l} 1 \to 0 + 4 \\ 3 \to 2 \\ \bar{3} \to 2 \end{array}}$$

branching

Summary and Outlook

Results

- Extended Topological symmetry breaking to TQFTs with non-integer quantum dimensions
- Found connection to conformal embeddings
- Had a first go at application to nonabelian FQH states

Questions/Future Work

- Found Fusion and twist factors.
  How to determine the rest of the TQFT (half-braidings, F-symbols…) ?
  Note: often fixed by consistency (always?)
- Work suggests conformal embeddings of coset chiral algebras.
  Interesting CFT problem…
- Further Physical applications….