Mini-Symposium on Topological Quantum Computation

National University of Ireland at Maynooth

D-Colexes & Topological Color Codes

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References

2-Colexes

H.Bombin, M.A. Martin-Delgado, “Topological Quantum Distillation”, Phys. Rev. Lett. **97** 180501 (2006)

3-Colexes

H.Bombin, M.A. Martin-Delgado,
“Topological Computation without Braiding”, quant-ph/0605138

Topological order

H.Bombin, M.A. Martin-Delgado, “Exact Topological Quantum Order in D=3 and Beyond: Branyons and Brane-Net Condensates”,
Phys. Rev. B accepted

Outline

- **Stabilizer codes**
  - Transversal gates. Reed-Muller codes and universality.
- **Topological stabilizer codes**
  - Surface codes.
- **2-Colexes**
  - 2D-lattice. Stabilizer.
  - Strings and string-nets.
  - Implementation of the Clifford group.
- **3-Colexes**
  - Universal quantum computation.

Stabilizer Codes

- A **stabilizer code**[1] C of length n is a subspace of the Hilbert space of a set of n qubits. It is defined by a stabilizer group S of Pauli operators, i.e., tensor products of Pauli matrices.

$$|\psi\rangle \in \mathcal{C} \qquad \Longleftrightarrow \qquad \forall\, s \in \mathcal{S} \quad s|\psi\rangle = |\psi\rangle$$

- It is enough to give the **generators** of S. For example:

$$\{ZXXZI, IZXXZ, ZIZXX, XZIZX\}$$

- Operators O that belong to the **normalizer** of S

$$O \in \mathbf{N}(\mathcal{S}) \qquad \Longleftrightarrow \qquad O\mathcal{S} = \mathcal{S}O$$

leave invariant the code space C. If they do not belong to the stabilizer, then they act non-trivially in the code subspace.

[1] D. Gottesman 95

Stabilizer Codes

- A encoded state can be subject to **errors**.
- To correct them, we measure a set of generators of *S*. The results of the measurement compose the **syndrome** of the error. Errors can be corrected as long as the syndrome lets us distinguish among the possible errors.
- Since correctable errors always form a vector space, it is enough to consider Pauli operators, which form a basis.
- We say that a Pauli error *e* is **undetectable** if it belongs to **N**(*S*)-*S*. In such a case, the syndrome says nothing:

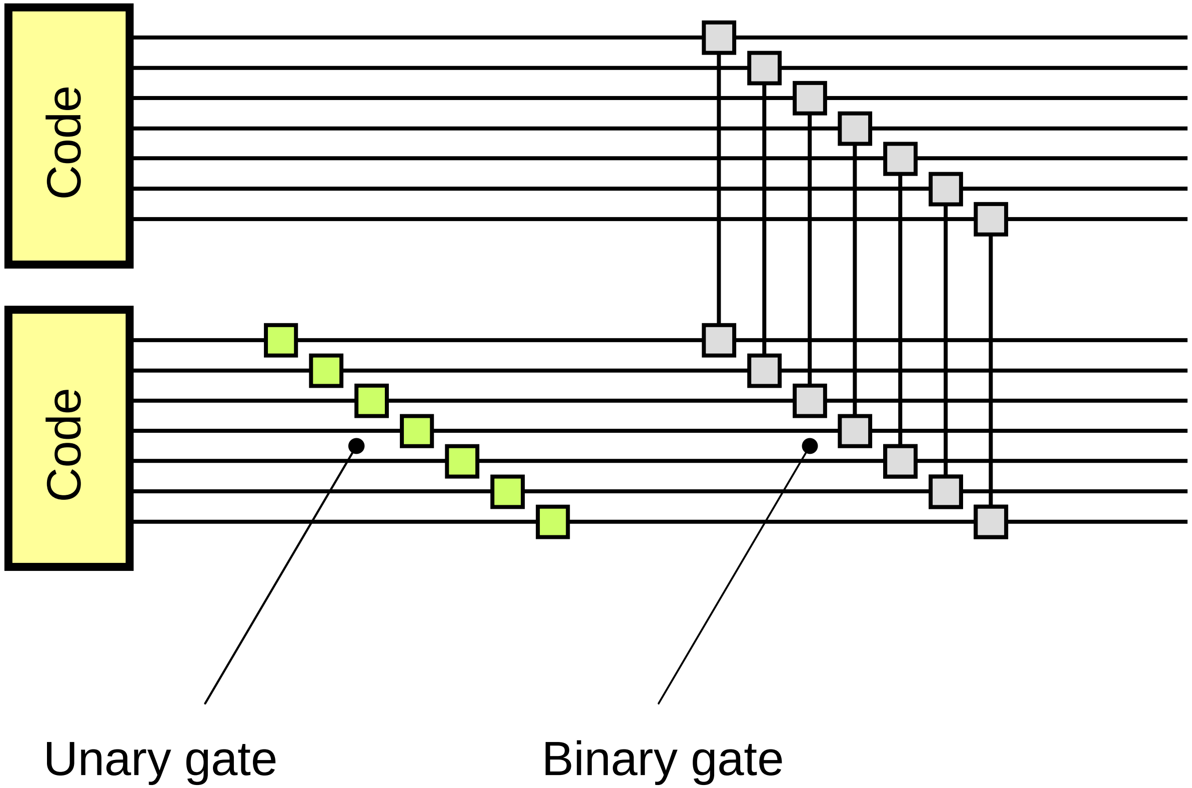
$$\forall\, s \in \mathcal{S} \qquad s\, e|\psi\rangle = e\, s'|\psi\rangle = e|\psi\rangle$$

- A set of Pauli errors *E* is correctable iff:

$$E^\dagger E \cap \mathbf{N}(\mathcal{S}) \subset \mathcal{S}$$

Stabilizer Codes

- Some stabilizer codes are specialy suitable for quantum computation. They allow to perform operations in a **transversal** and **uniform** way:



Stabilizer Codes

- Several codes allow the transversal implementation of

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \qquad K = \begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix} \qquad \Lambda = \begin{pmatrix}I_2 & 0\\ 0 & X\end{pmatrix}$$

  which generate the **Clifford group**. This is useful for quantum information tasks such as teleportation or **entanglement distillation**.
- Quantum **Reed-Muller** codes[1] are very special. They allow **universal computation** through transversal gates

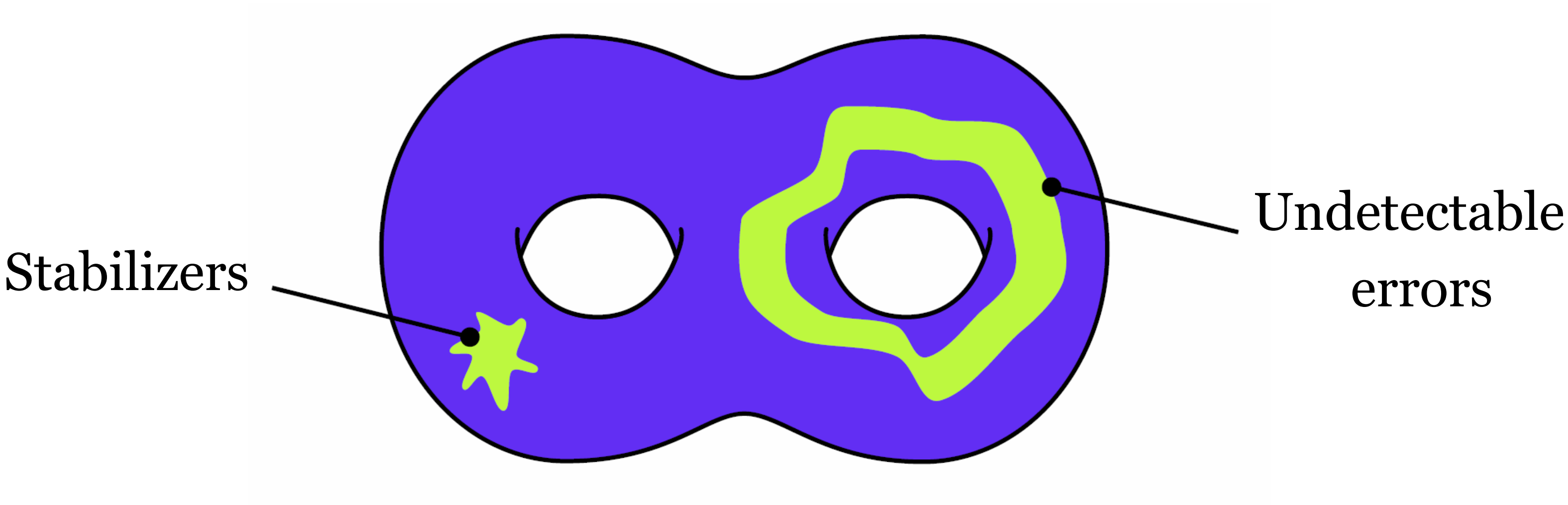
$$K^{1/2} = \begin{pmatrix}1 & 0\\ 0 & i^{1/2}\end{pmatrix} \qquad \Lambda = \begin{pmatrix}I_2 & 0\\ 0 & X\end{pmatrix}$$

  and transversal measurements of X and Z.
- We will see how both sets of operations can be transversally implemented in 2D and 3D topological color codes.

[1] E. Knill *et al.*

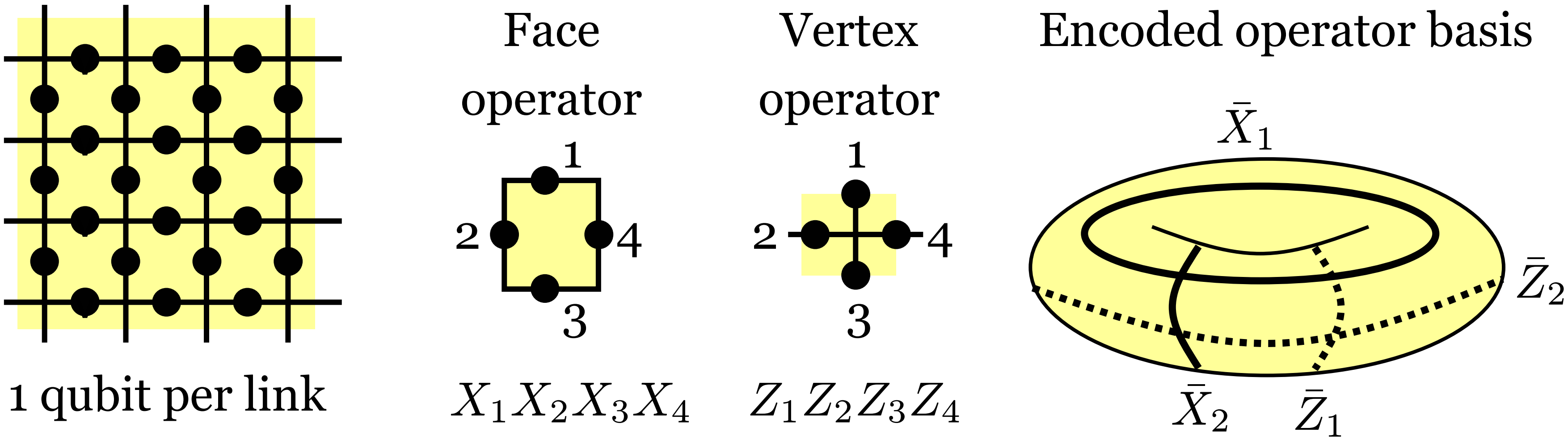
Topological Stabilizer Codes

- In order to introduce the idea of a topological stabilizer code (TSC), we must consider a topological space in which our physical qubits are to be placed, for example a surface.
- A TSC is a stabilizer code in which the generators of the stabilizer are **local** and undetectable errors (or encoded operators) are **topologically nontrivial**.

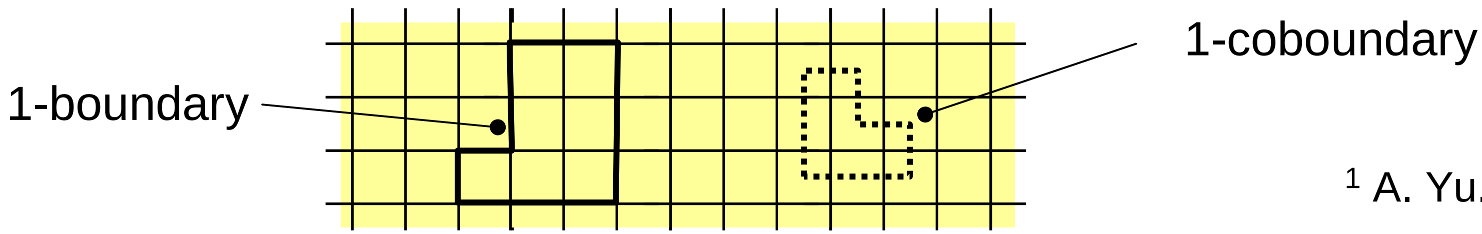


Topological Stabilizer Codes

- The first example of TSC were **surface codes**[1], which are based on $\boldsymbol{Z_2}$ homology and cohomology.



- S gets identified with 1-boundaries and 1-coboundaries, and $\mathbf{N}(S)$ with 1-cycles and 1-cocycles.



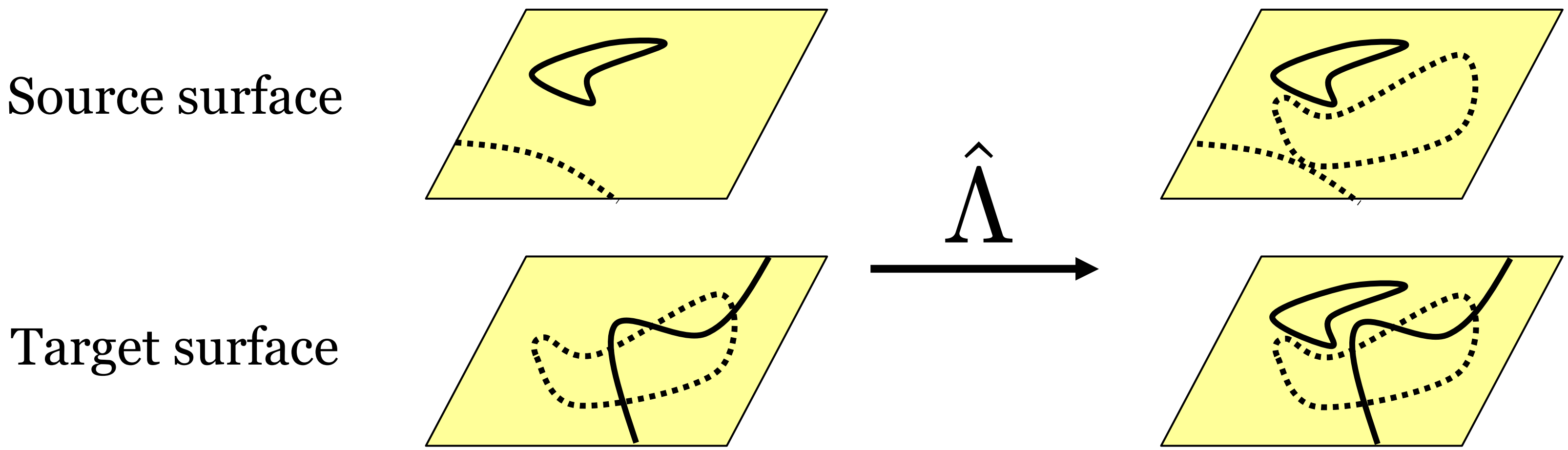
[1] A. Yu. Kitaev 97

Topological Stabilizer Codes

- The CNot gate can be implemented transversally on surface codes. First, its action under conjugation on operators is:

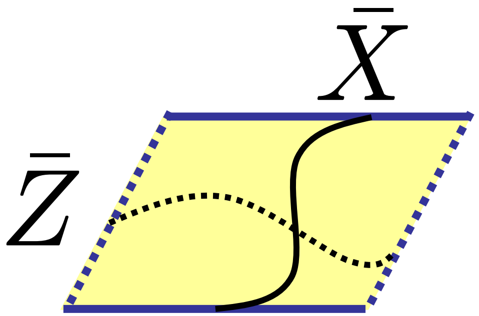
$$\Lambda : \quad \begin{array}{ll} IX \longrightarrow IX & IZ \longrightarrow ZZ \\ XI \longrightarrow XX & ZI \longrightarrow ZI \end{array}$$

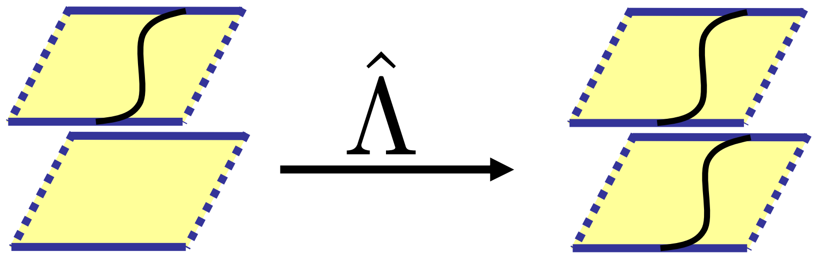
- Thus the transversal action of the CNot on a surface code, at the level of operators, is simply to copy chains forward and cochains backwards.



Topological Stabilizer Codes

- Finally, to see the action of the tranversal CNOT on the code, we have to choose a Pauli basis for the encoded qubits. In the simplest example we have a single qubit in a square surface with suitable borders:



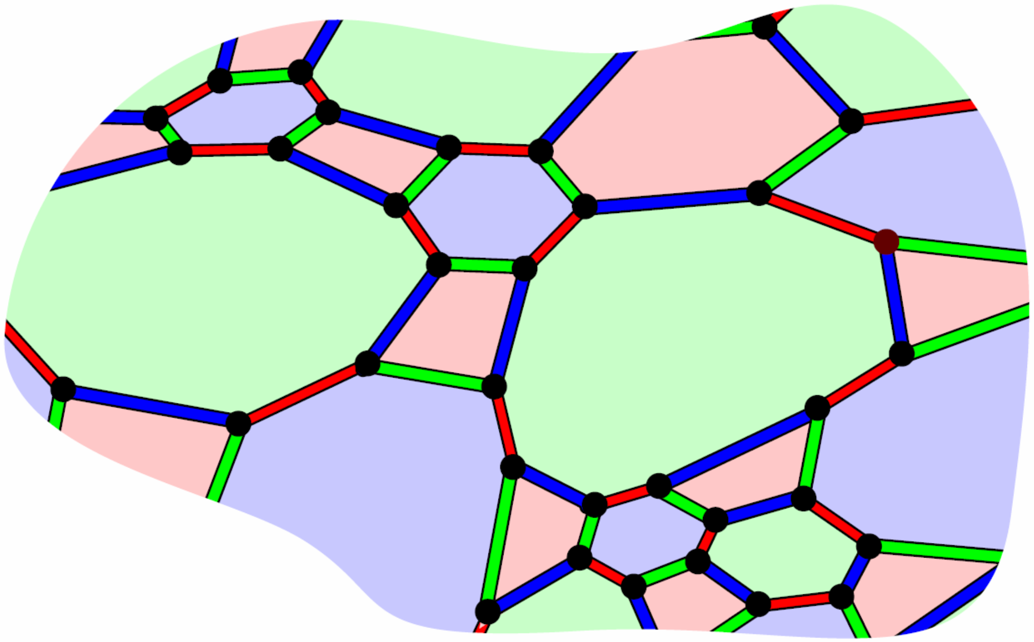


$$\hat{\Lambda}\,\bar{X}\hat{I}\,\hat{\Lambda}^{\dagger} = \bar{X}\bar{X}$$

- Clearly the action of a transversal CNot is itself a CNot gate on the encoded qubits. However, this is the only gate we can get with surface codes. If we want to get further, we have to go beyond homology.

2-Colexes

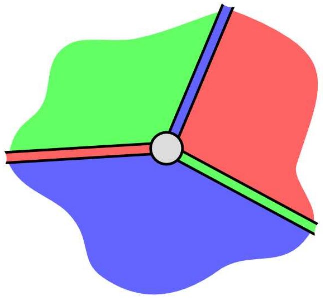
- A 2-colex is a **trivalent** 2-D lattice with **3-colored faces**.

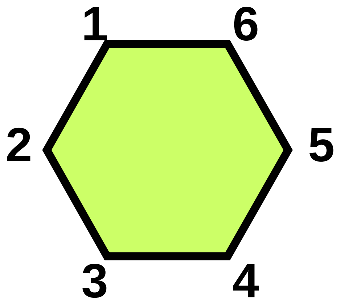


- Edges can be 3-colored accordingly. Blue edges connect blue faces, and so on.
- The name 'colex' is for 'color complex'. *D*-colexes of arbitrary dimension can be defined. Their key feature is that the whole structure of the complex is contained in the 1-skeleton and the coloring of the edges.

2-Colexes

- To construct a **color code** from a 2-colex, we place 1 qubit at each **vertex** of the lattice. The generators of S are **face operators**:





$$B_f^X = X_1X_2X_3X_4X_5X_6$$
$$B_f^Z = Z_1Z_2Z_3Z_4Z_5Z_6$$

- Transversal Clifford gates should belong to $\mathbf{N}(S)$. We have:

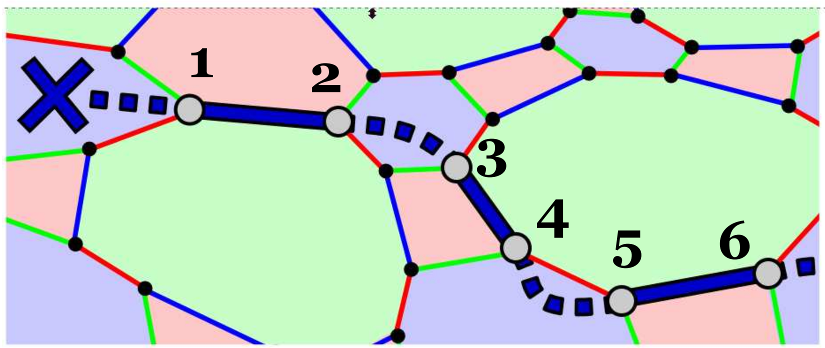
$$\hat{H}B_f^X\hat{H}^\dagger = B_f^Z \qquad \hat{K}B_f^X\hat{K}^\dagger = (-)^{\frac{v}{2}}B_f^XB_f^Z$$
$$\hat{H}B_f^Z\hat{H}^\dagger = B_f^X \qquad \hat{K}B_f^Z\hat{K}^\dagger = B_f^Z$$

- Here v is the number of vertices in the face. If it is a multiple of 4 for every face, then K is in $\mathbf{N}(S)$. H always is.
- As for the CNot gate, it is clearly in $\mathbf{N}(S)$ (it is a CSS code).

2-Colexes

- In order to understand 2-D color codes, we have to introduce string operators in the picture. As in surface codes, we play with $\boldsymbol{Z_2}$ homology. However, there is a new ingredient, color.
- A blue string is a collection of blue links:

Endpoint



String operators

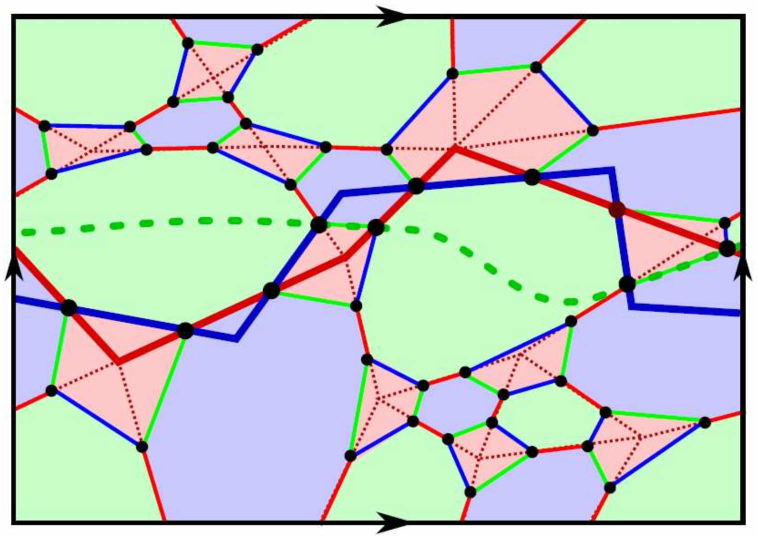
$$S^X = X_1 X_2 X_3 X_4 X_5 X_6 \cdots$$

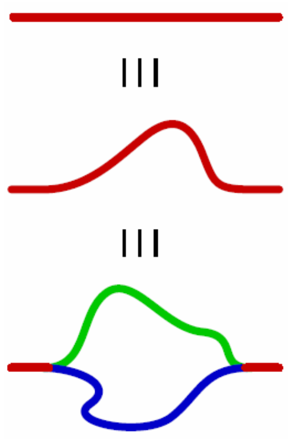
$$S^Z = Z_1 Z_2 Z_3 Z_4 Z_5 Z_6 \cdots$$

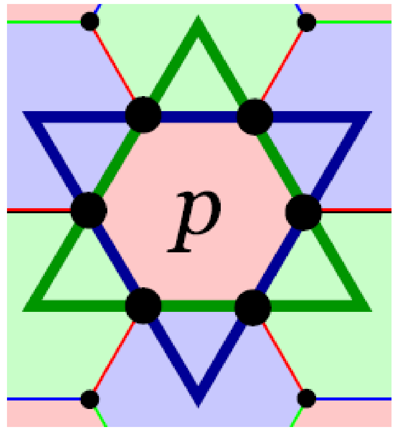
- Strings can have endpoints, located at faces of the same color. However, in that case the corresponding string and face operators will not commute. Therefore, a string operator belongs to $\mathbf{N}(S)$ iff the string has no endpoints.

2-Colexes

- For each color we can form a **shrunk graph**. The red one is:







Red faces $\longrightarrow$ vertices
Red edges $\longrightarrow$ edges
Blue and green faces $\longrightarrow$ faces

A red face is also blue or green string

- Thus for each color homology works as in surface codes. The new feature is the possibility to **combine** homologous blue and red string operators of the same kind to get a green one.

2-Colexes

- Since there are two independent colors, the number of encoded qubits should double that of a surface code. Lets check this for a surface **without boundary** using the Euler characteristic $\chi = V + F - E$ for any *shrunk* lattice.
- Face operators are subject to the **conditions**

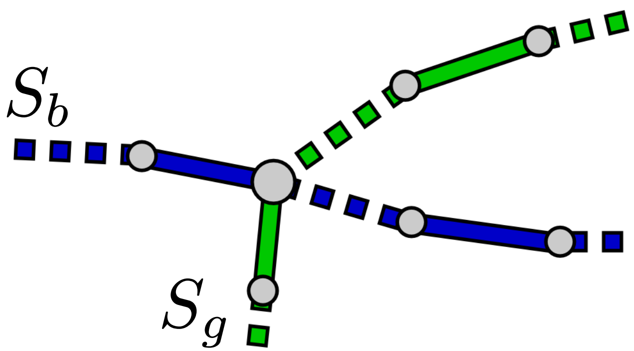
$$\prod_{f\in \textcolor{red}{\bullet}} B_f^\sigma = \prod_{f\in \textcolor{blue}{\bullet}} B_f^\sigma = \prod_{f\in \textcolor{green}{\bullet}} B_f^\sigma$$

  so that the total number of generators is $g = 2(F + V - 2)$.
- The number of physical qubits is $n = 2E$. Therefore the number of encoded qubits q is twice the first Betti number of the manifold:

$$q = n - g = 4 - 2\chi = 2h_1$$

2-Colexes

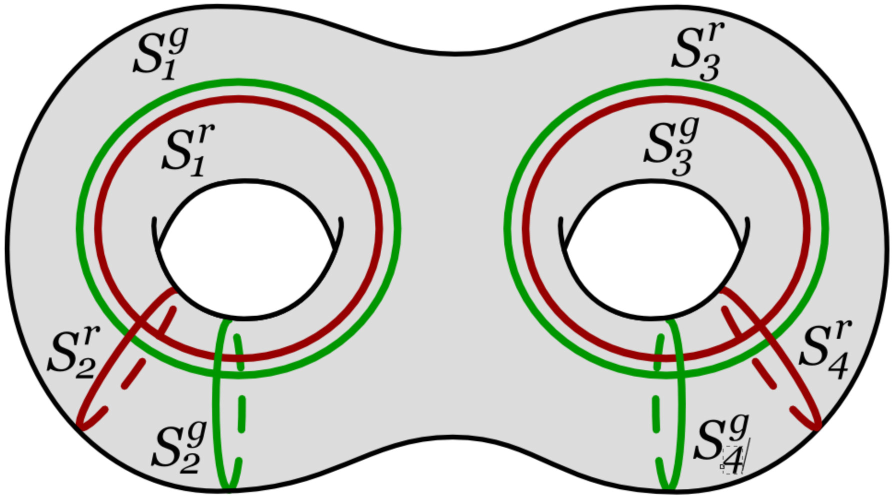
- In order to form a Pauli basis for the operators acting on encoded qubits, we can use as in surface codes those string operators (SO) that are not homologous to zero.
- To this end, we need the commutation rules for SO.
- Clearly SO of the same type (X or Z) always commute.
- A string is made up of edges with two vertices each. Therefore, two SO of the same color have an even number of qubits in common an they commute.
- SO of **different colors** can **anticommute**, but only if they **cross** an odd number of times:



$$\{S_b^X, S_g^Z\} = 0$$

2-Colexes

- Now we can construct the desired operator basis for the encoded qubits. In a 2-torus a possible choice is:



$$\begin{array}{ll} S_1^{gX} \leftrightarrow X_1 & S_2^{rZ} \leftrightarrow Z_1 \\ S_2^{rX} \leftrightarrow X_2 & S_1^{gZ} \leftrightarrow Z_2 \\ S_2^{gX} \leftrightarrow X_3 & S_1^{rZ} \leftrightarrow Z_3 \\ \vdots & \vdots \end{array}$$

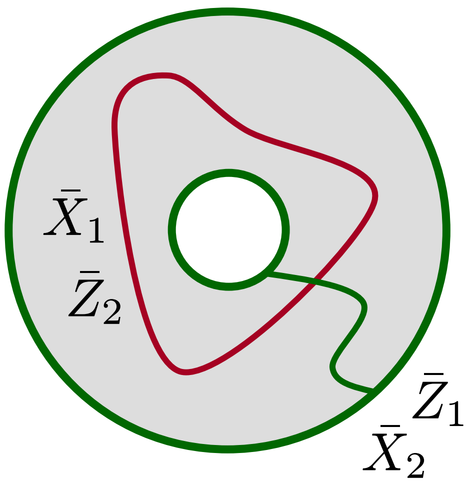
$$X_i Z_j = (-1)^{\delta_{i,j}} Z_j X_i$$

- However, if we apply the transversal H gate to such a code the resulting encoded gate is not H. The underlying reason is that for a string S we **never** have

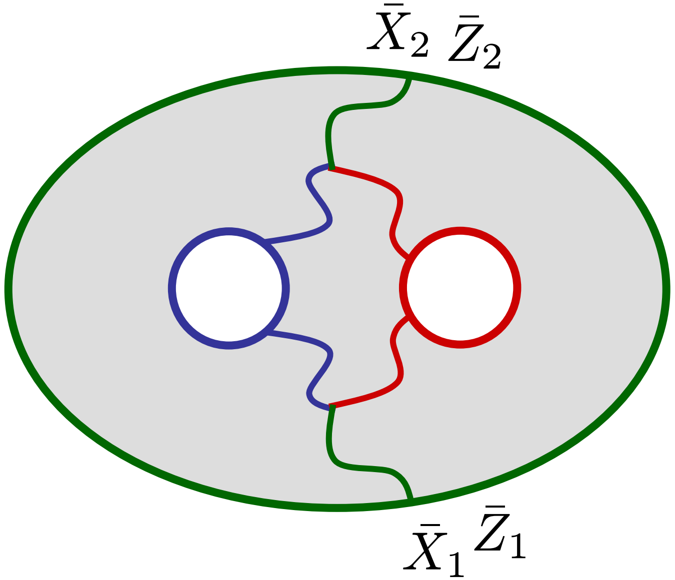
$$\{S^X, S^Z\} = 0$$

2-Colexes

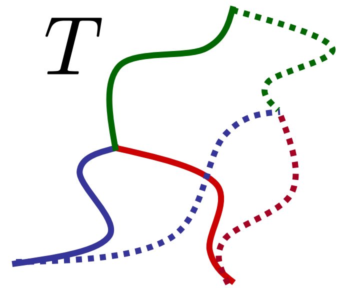
- But we can consider surfaces with **boundary**. To this end, we take a sphere, which encodes no qubit, and **remove faces**.
- When a face is removed, the resulting boundary must have its color, and only strings of that color can end at the boundary.



2 qubits



2 qubits

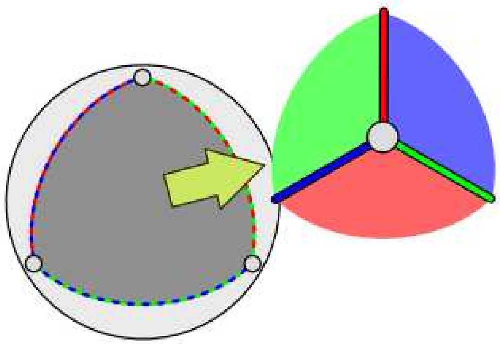


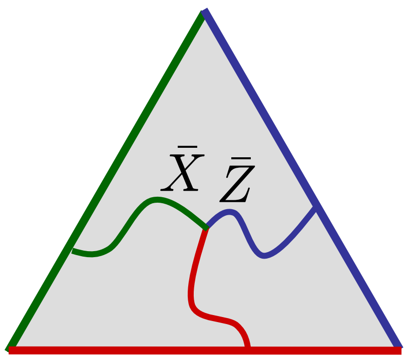
$$\{T^X, T^Z\} = 0$$

As desired!

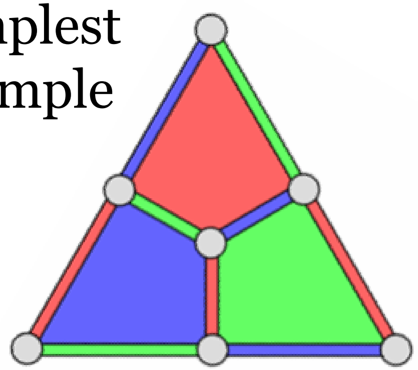
2-Colexes

- We can even encode a single qubit an remove the need for holes. If we remove a site and neighboring links and faces from a 2-colex in a sphere, we get a **triangular** code:

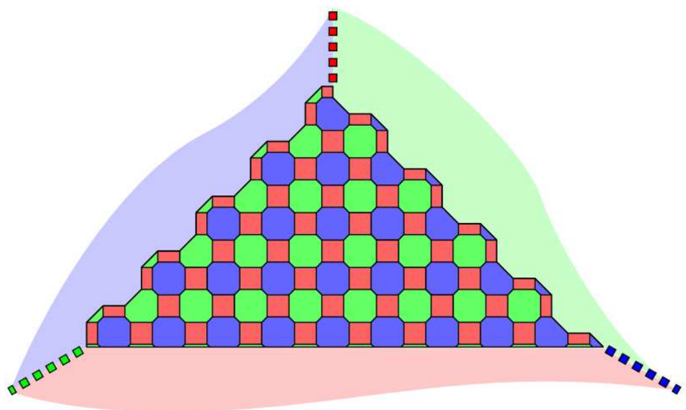




Simplest example



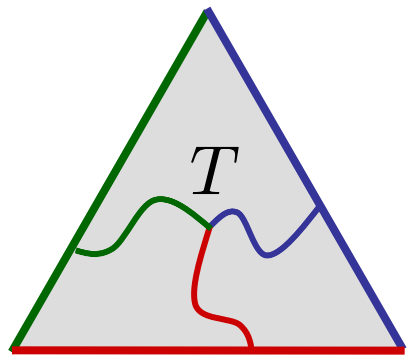
- We can construct triangular codes of arbitrary sizes. The vertices per face can be 4 and 8 so that K is in $\mathbf{N}(S)$.



2-Colexes

- The transversal H clearly amounts to an encoded H:

$$H : \begin{array}{l} X \longrightarrow Z \\ Z \longrightarrow X \end{array} \qquad \hat{H} : \begin{array}{l} T^X \longrightarrow T^Z \\ T^Z \longrightarrow T^X \end{array}$$



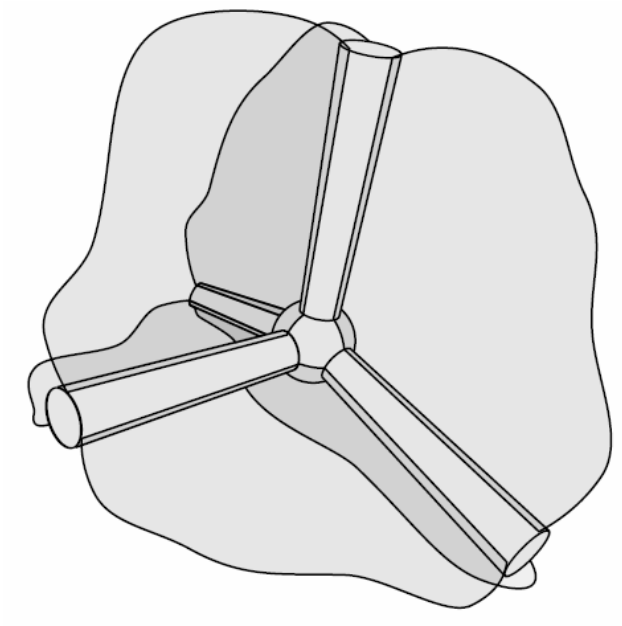
- This is also true for K. The anticommutation properties of T imply that its support consists of an odd number of qubits:

$$K : \begin{array}{l} X \longrightarrow iXZ \\ Z \longrightarrow Z \end{array} \qquad \hat{K} : \begin{array}{l} T^X \longrightarrow \pm i T^X T^Z \\ T^Z \longrightarrow T^Z \end{array}$$

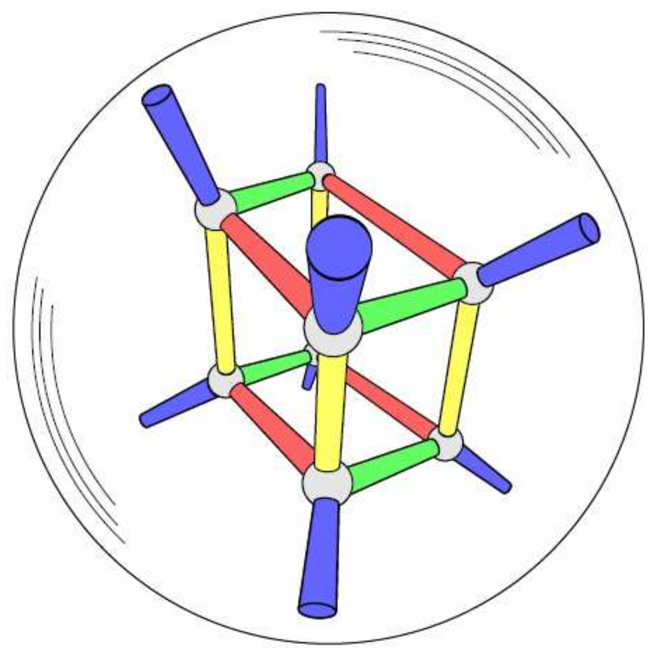
- Therefore, the **Clifford group** can be implemented transversally in triangular codes.

3-Colexes

- **3-colexes** are tetravalent lattices with a particular local appearance such that their 3-cells can be 4-colored. They can be built in any compact 3-manifold without boundary.
- Edges can be colored accordingly, as in the 2-D case.



The neigborhood of a vertex.

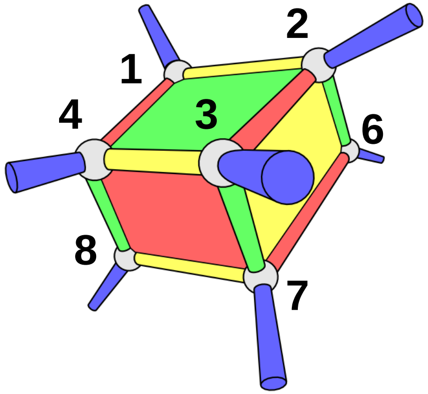


The simplest 3-colex in the projective space.

3-Colexes

- This time the generators of S are face and (3-) cell operators.

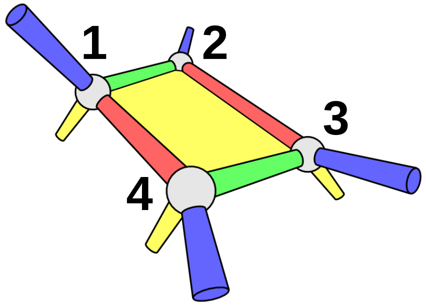
A b-cell



Cell operators

$$B_c^X = \bigotimes_{i=1}^{8} X_i$$

A by-face separates b- and y-cells.



Face operators

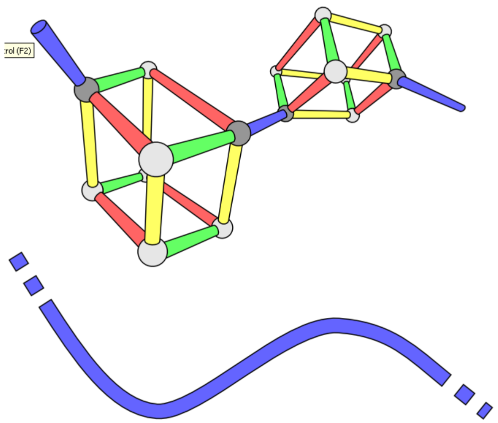
$$B_f^Z = \bigotimes_{i=1}^{8} Z_i$$

- Therefore there are two different homology groups in the picture, those for 1-chains and for 2-chains. But in fact, due to Poincaré's duality they are the same.

3-Colexes

- Strings are constructed as in 2-D, but now come in four colors. Branching is again possible.
- The new feature are membranes. They come in 6 color combinations and also have branching properties.

String operators

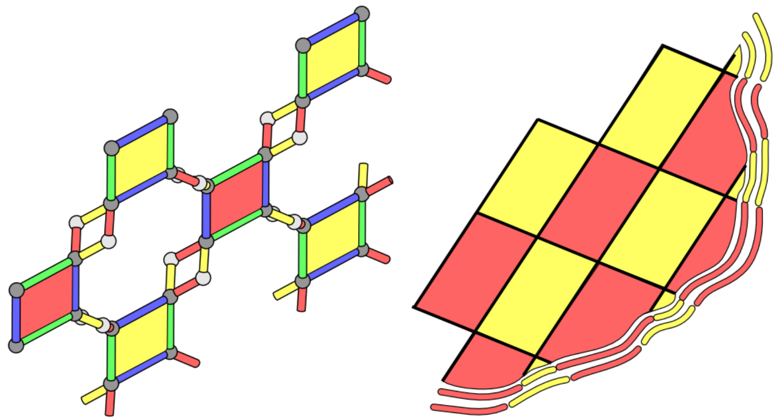


b-string

$$S^Z = \bigotimes_{\text{string}} Z_i$$

Membrane operators

$$M^X = \bigotimes_{\text{membrane}} X_i$$

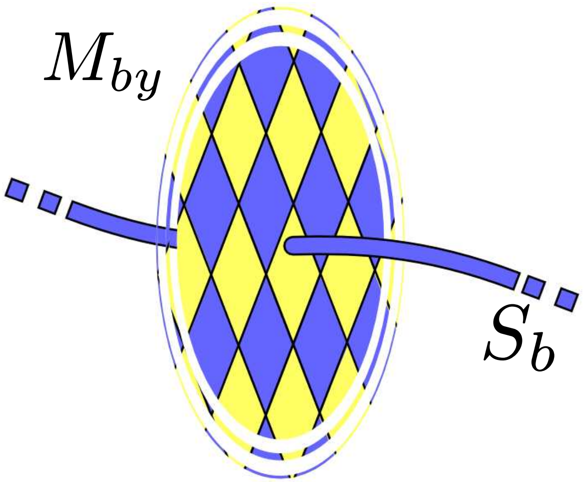


ry-membrane

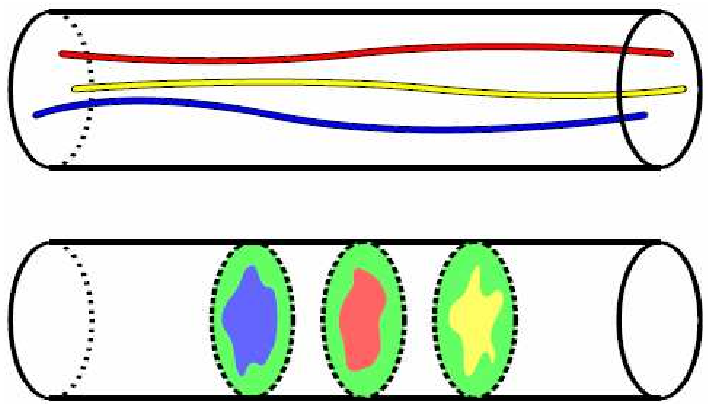
- There exist appropiate shunk complexes both for strings and for membranes.

3-Colexes

- Now there are 3 indendent colors for strings (and similarly 3 color combinations for membranes). Therefore, we expect that the number of encoded qubits will be $3h_1 = 3h_2$.
- String and membrane operators always commute, unless they share a color and the string crosses an odd number of times the membrane.



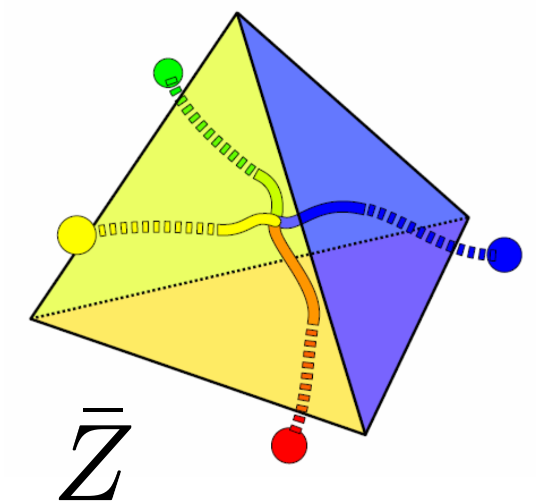
$$\{S_b^Z, M_{by}^X\} = 0$$

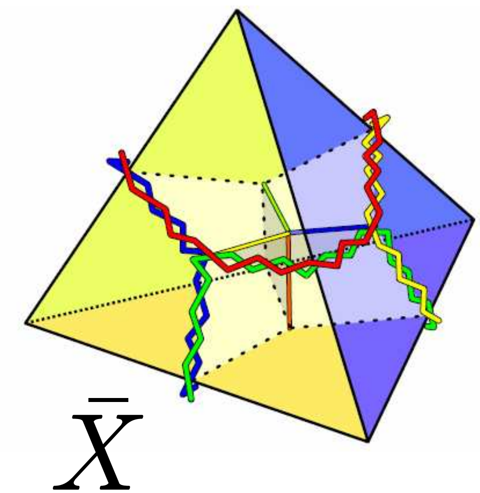


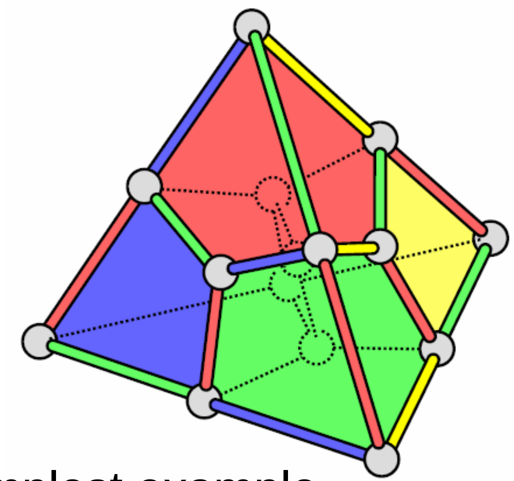
A pauli basis for the operators on the 3 qubits encoded in $S^2 x S^1$.

3-Colexes

- 3-Colexes cannot have a practical interest unless we allow boundaries. But this is just a matter of erasing cells. As in two dimensions, boundaries have the color of the erased cell.
- The analogue of triangular codes are tetrahedral codes, obtained by erasing a vertex from a 3-sphere.







Simplest example

- The desired transversal $K^{1/2}$ gate can be implemented as long as faces have 4x vertices and cells 8x vertices.

Conclusions

- *D*-colexes are *D*-valent complexes with certain coloring properties.
- Topological color codes are obtained from colexes. They have a richer structure than surface codes.
- 2-colexes allow the transversal implementation of Clifford operations.
- 3-colexes allow the transversal implementation of the same gates as Reed-Muller codes.