Quantum State Engineering In Low-Dimensional Quantum Gases

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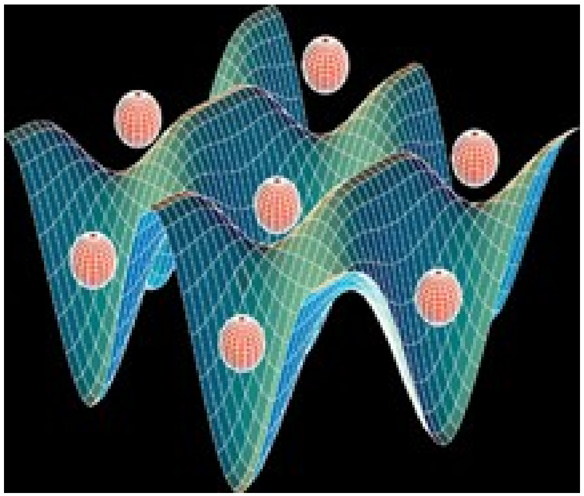
Outline

- Introduction to some cold atom concepts
- Hard-Core Bosons and the Tonks-Girardeau gas
- The Fermi-Bose Mapping theorem
- Tonks-Girardeau gas in a delta-split trap
- Analytic eigenstates and many-body properties :
  - Single particle densitys and pair-distribution functions
  - Reduced single particle density matrices
  - Momentum space distibutions
- Varying interaction between two particles in the trap
- 2-particle Von-Neumann entropy and entanglement
- Further Work
- Summary

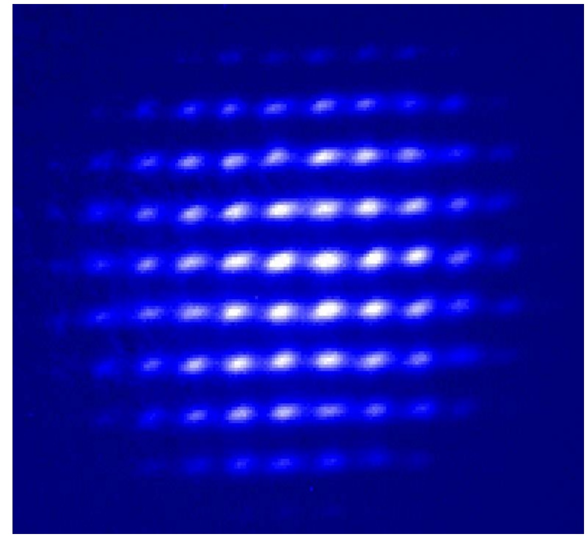
Cold Atoms

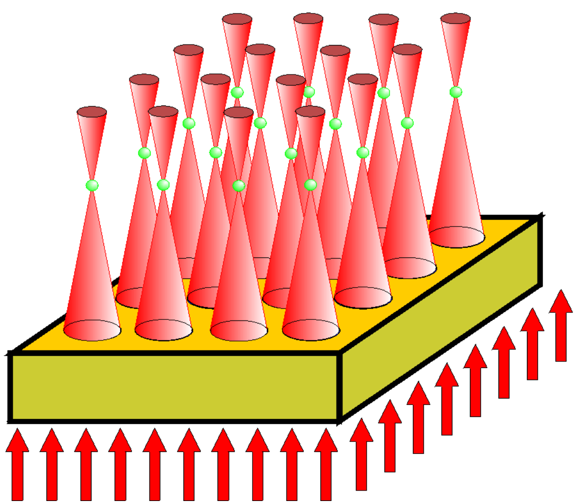
- In the last two decades remarkable progress has been made in the field of atom trapping and cooling
- 1995 – First B.E.C. created in lab by E. Cornell and C. Wiemann
- Combined laser cooling and evaporative cooling to reach the nano-kelvin range
- Optical-Lattice potentials and micro-trap arrays
- Ideal arena for implementation of quantum information processing protocols (clean, highly controllable)

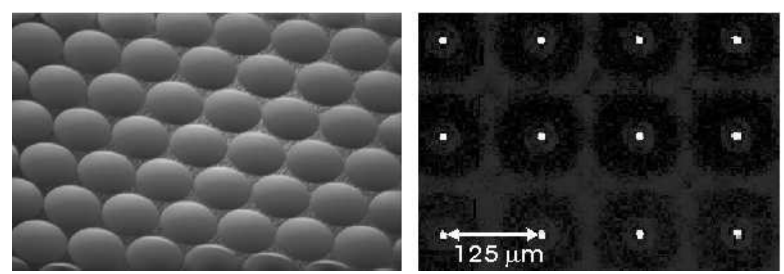
Cold Atoms



Optical Lattice



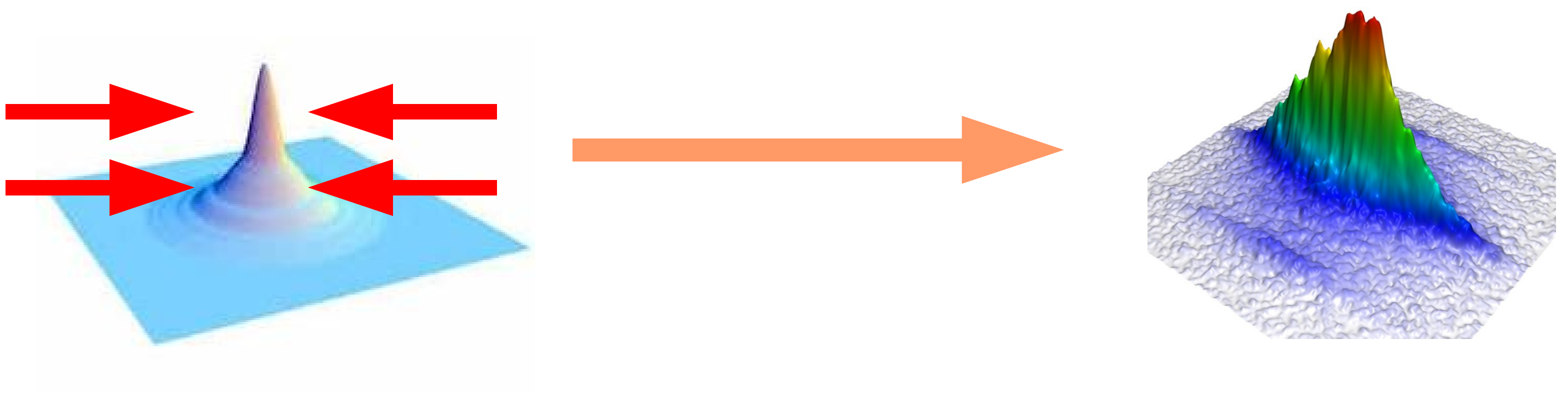




Microtraps in University of Hannover/ Darmstadt (G. Birkl)

Cold Atoms

- Experimentalists can vary the interaction between trapped atoms using 'Feshbach Resonances'
- One can also modify the shape of trapping potentials
  - By stiffening the transverse trapping frequencies can prepare quasi 1-d gases



- Atoms motion gets 'stuck' in one dimension
- When atoms are strongly repulsive and interact with a 'hard-core' potential we are in the *Tonks-Girardeau* regime

'Hard-Core' Bosons

What is a gas of hard core bosons?

- Particles behave like impenetrable hard-spheres
- Interact via repulsive hard-core potential
- L. Tonks gave first statistical treatment in 1936
  - Restricted to the classical high temperature limit
  - No light shed on the extreme quantum limit $T \rightarrow 0$
- Here the De-Broglie wavelength >> interparticle distance

$$\lambda_T = \left(\frac{2\pi\hbar^2}{mk_B T} \right)^{\frac{1}{2}}$$

Many-Body Hamiltonian

- At T~0 interaction can be approximated by point-like potential
- Hamiltonian for a gas of hard core atoms in an arbitary 1-d trapping potential V(x)

$$\widehat{H} = \sum_{i=1}^{N} \left[-\frac{\hbar}{2m} \frac{\partial^2}{\partial x_i^2} + V(x_i) \right] + g_1 \sum_{i<j}^{N} \delta(|x_i - x_j|)$$

$$g_1 \propto \frac{1}{a_{1d}}$$

- Tonks-Girardeau regime: $g_1 \longrightarrow \infty$
- In this limit of strong repulsion the particles are sitting like 'beads on a string'

The Fermi-Bose mapping

- First theoretical treatment of a one dimensional quantum gas was given by M. Girardeau
- Fermi-Bose Mapping theorem first appeared in his 1960 paper J.Math. Phys. 1,516
- Discovered clever way to treat the interaction part of the Hamiltonian
  - Replace it by a constraint on the allowed wave-functions:

$$\psi = 0 \; if \; |x_i - x_j| < a$$

Implications!

- The constraint creates a traffic jam of bosons
- Look carefully at the constraint:
  - It is equivalent to the Pauli principle for a gas of spinless fermions!
- The Fermi-Bose mapping states that in the 1D TG regime we may calculate the many body wavefunction from

$$\Psi_B(x_1, \ldots, x_N) = |\Psi_F(x_1, \ldots, x_N)|$$

→ Bosons should acquire fermionic signatures e.g.

$$|\Psi_B|^2 = |\Psi_F|^2$$

Implications!

- The FB mapping theorem maps a strongly interacting many-boson problem to a non interacting many-fermion problem
- This is nice! Why?

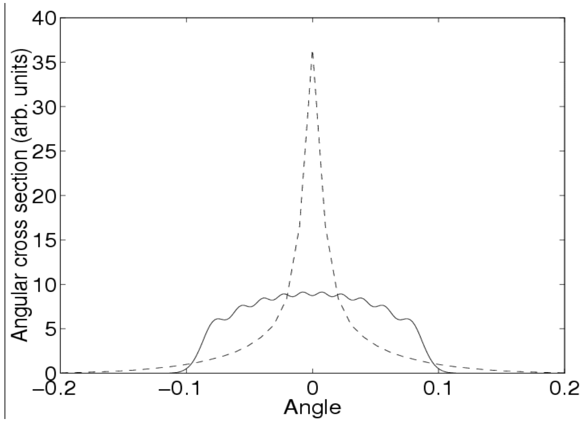
  -The many-body wavefunction can be calculated via the single particle eigenstates !

→ Slater determinant

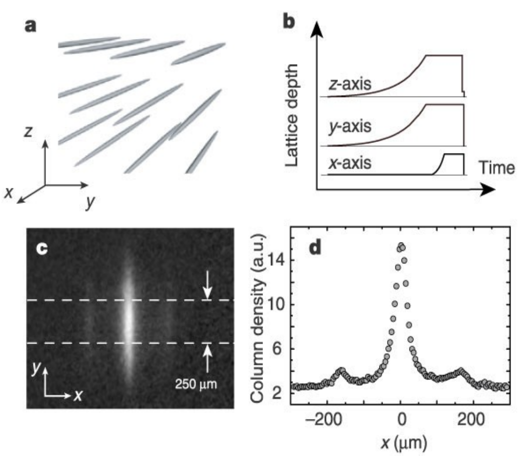
→
$$\Psi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N) = \left| \frac{1}{\sqrt{N!}} \begin{vmatrix} \chi_1(\mathbf{x}_1) & \chi_1(\mathbf{x}_2) & \cdots & \chi_1(\mathbf{x}_N) \\ \chi_2(\mathbf{x}_1) & \chi_2(\mathbf{x}_2) & \cdots & \chi_2(\mathbf{x}_N) \\ \vdots & \vdots & & \vdots \\ \chi_N(\mathbf{x}_1) & \chi_N(\mathbf{x}_2) & \cdots & \chi_N(\mathbf{x}_N) \end{vmatrix} \right|$$

Trapped TG gas

- 40 years after discovering the Fermi Bose mapping
- Girardeau investigated the many-body properties of the TG gas in the harmonic potential
- Able to calculate the momentum distribution, interference patterns, etc



Momentum State: Girardeau (2001)

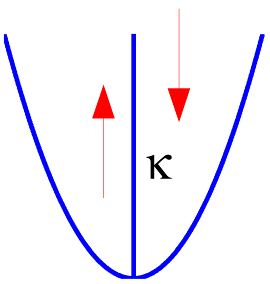


Immanuel Bloch(2004)

TG gas in the δ-split trap

Consider the Hamiltonian

$$\hat{H}_0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2 + \kappa\delta(x)$$

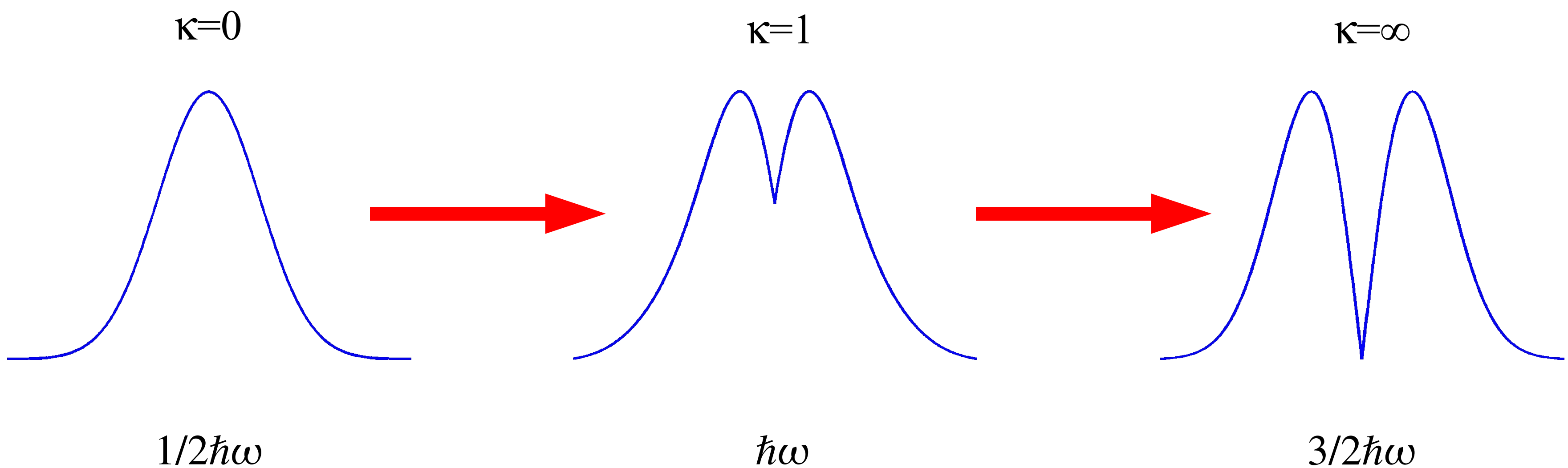


Have a trap split by central barrier,

- Splitting strength parameterised by κ
- Can we solve Schrödinger's equation?

Single Particle Eigenstates

Ground state wavefunction



As you increase κ the cusp in the centre gets deeper and the ground state gets lifted in energy space

Single Particle Eigenstates

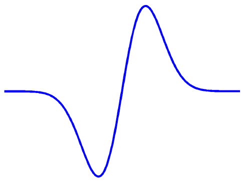
- Solutions are Whittaker functions

$$U(\epsilon_n, x) = cos(\frac{\pi}{4} + \frac{\pi\epsilon_n}{2})Y_1 - \sin\left(\frac{\pi}{4} + \frac{\pi\epsilon_n}{2}\right) Y_2$$

$$Y_1 = \frac{\Gamma\left(\frac{1}{4} - \frac{1}{2}\epsilon_n\right)}{\sqrt{\pi}2^{\left(\frac{1}{4}+\frac{1}{2}\epsilon_n\right)}} e^{-\frac{1}{4}x^2} M\left(\frac{1}{4} + \frac{1}{2}\epsilon_n, \frac{1}{2}, \frac{1}{2}x^2\right)$$

$$Y_2 = \frac{\Gamma\left(\frac{3}{4} - \frac{1}{2}\epsilon_n\right)}{\sqrt{\pi}2^{\left(-\frac{1}{4}+\frac{1}{2}\epsilon_n\right)}} e^{-\frac{1}{4}x^2} xM\left(\frac{3}{4} + \frac{1}{2}\epsilon_n, \frac{3}{2}, \frac{1}{2}x^2\right)$$

- Symmetric states $\phi_n(x) = CU(\epsilon_n, |x|)$
- Anti-symmetric states are the usual harmonic oscillator states



  - In the limit κ=∞ we get a doubly degenerate spectrum

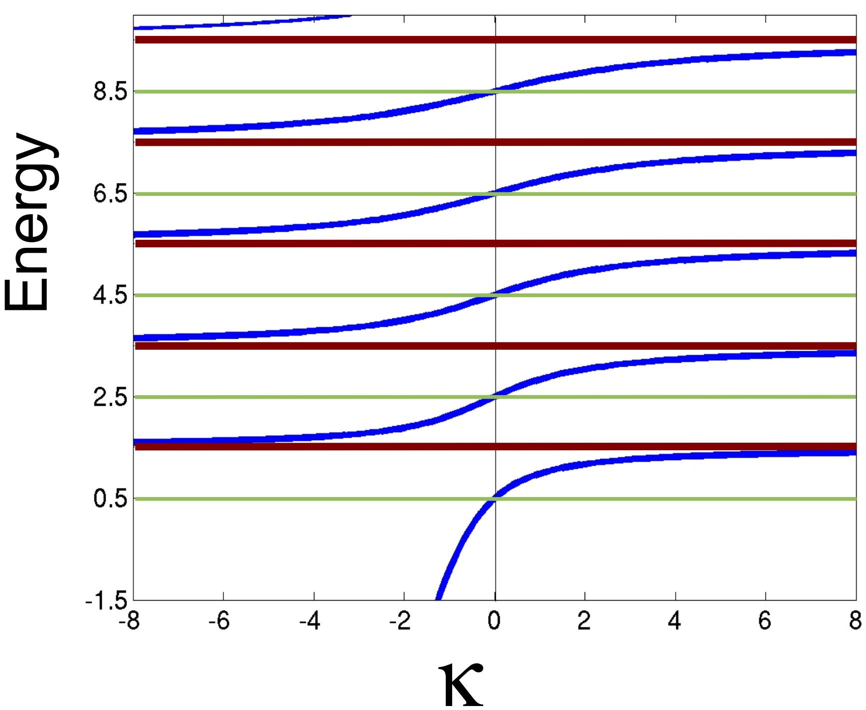
Energy Eigenvalues

Evaluate the continuity condition at x=0

$$\frac{d}{dx}\phi_n(0^+) - \frac{d}{dx}\phi_n(0^-) = \tilde{\kappa}\phi_n(0)$$

We obtain an implicit relationship for the energy eigenvalues

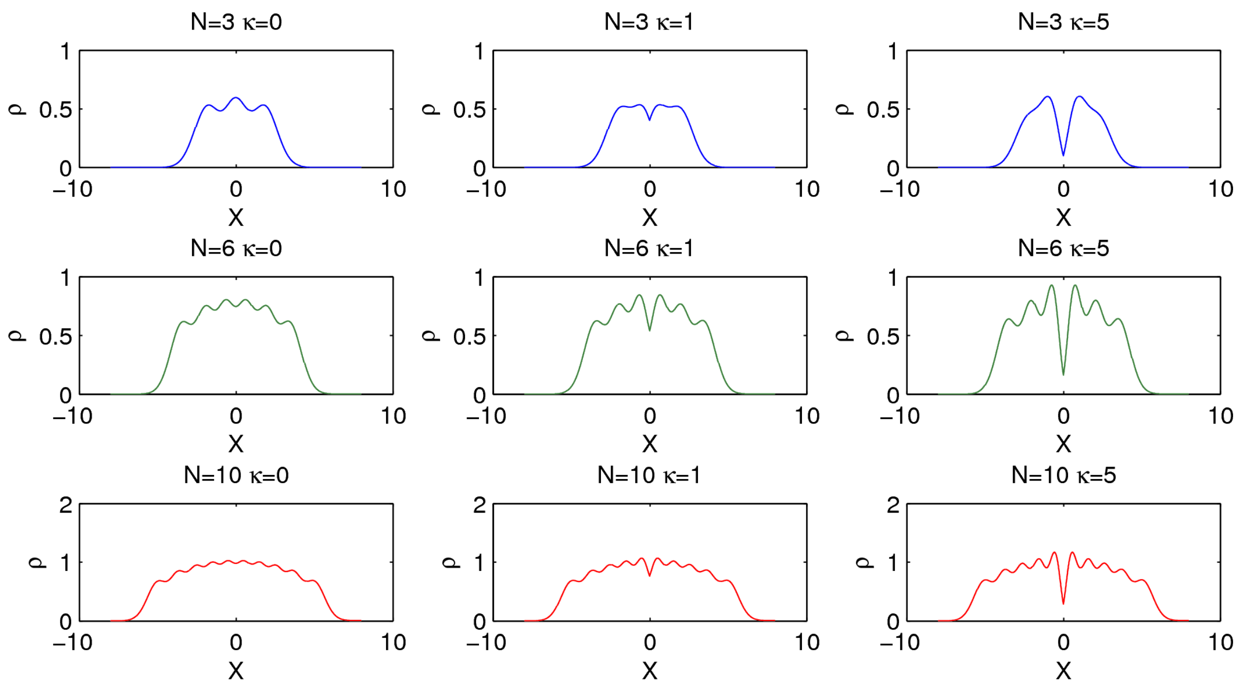
$$\frac{\Gamma\left(\frac{3}{4}+\frac{1}{2}\epsilon_n\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}\epsilon_n\right)} = -\tilde{\kappa}$$



Many-Body Properties

- We are now in a position to explore many body properties of the TG gas in the split trap
  - Single particle densitys

$$\rho(x) = N\int_{-\infty}^{+\infty} |\Psi_B(x, x_2 \ldots, x_N)| dx_1 \ldots dx_N \quad = \sum_{n=0}^{N-1} |\psi_n(x)|^2$$

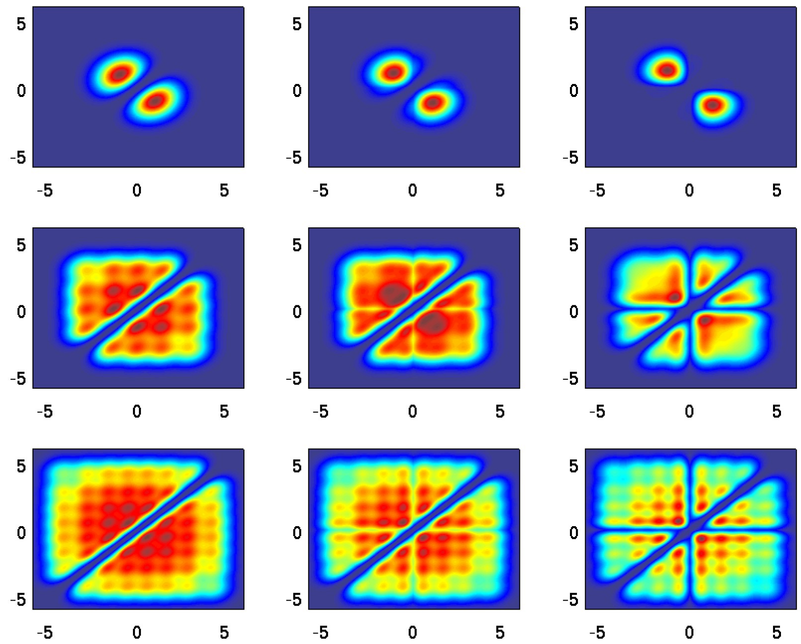


Many-Body Properties

- Pair Distribution Functions
  - These are joint measurement probability densities

$$D(x_1, x_2) = N(N-1) \int_{-\infty}^{+\infty} |\Psi_B(x_1, x_2 \ldots, x_N)| dx_1 \ldots dx_N$$

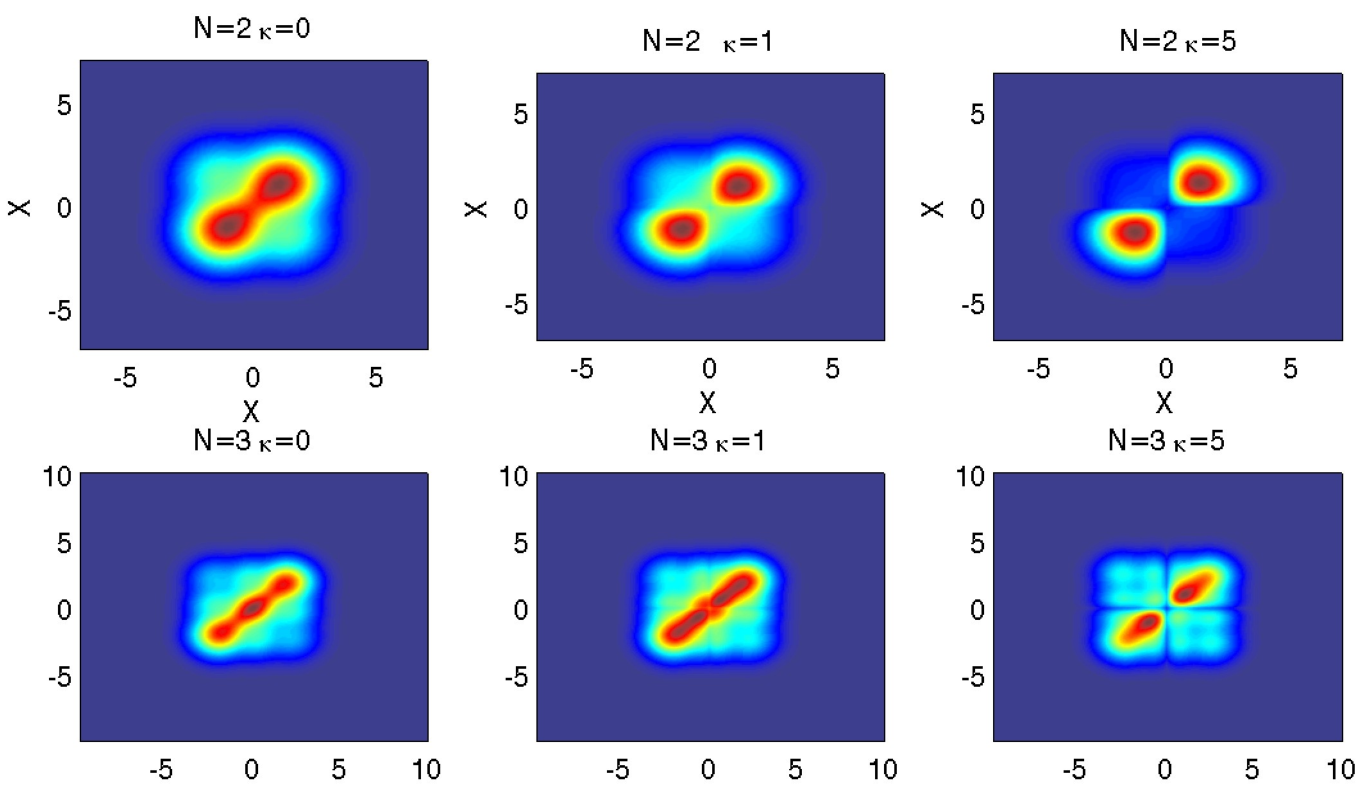
$$= \rho(x_1))\rho(x_2) - |\Delta(x_1, x_2)|^2$$



Many-Body Properties

- Reduced single particle density matrices

$$\rho_1(x,x') = N\int_{-\infty}^{\infty} \Psi_B(x,x_2,\ldots,x_N) * \Psi_B(x',x_2,\ldots,x_N)dx_2\ldots dx_N$$

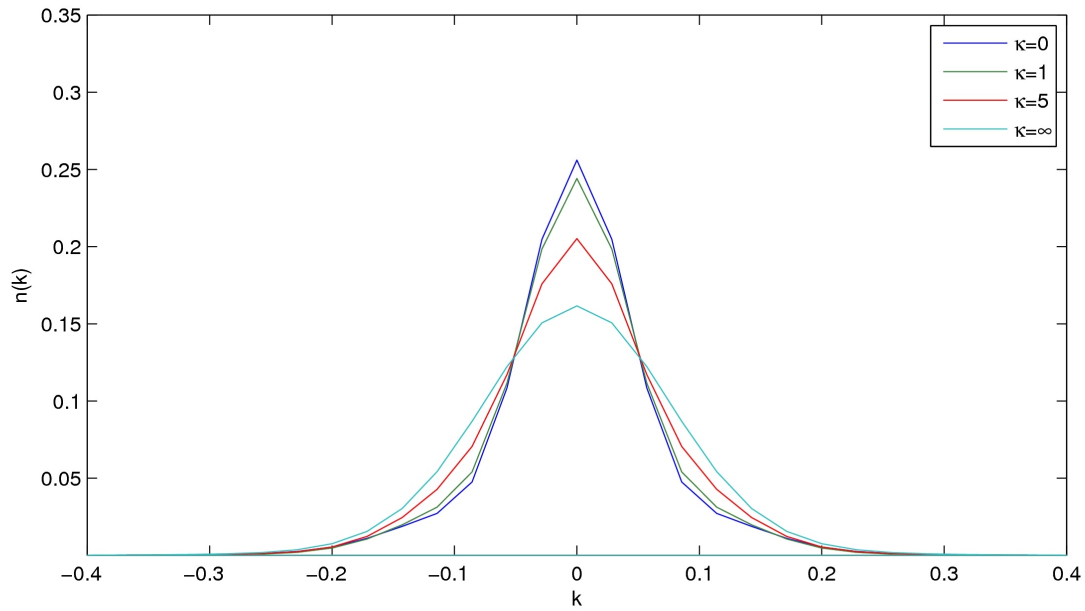


Momentum Distributions

We can obtain momentum distributions from the reduced single particle density matrices

$$n(k) = (2\pi)^{-1} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' \rho_1(x, x') e^{-ik(x-x')}$$

$$n(k) = \sum_j \lambda_j |\mu_j(k)|^2$$



2 Particle Case

- 2 particles in split trap is special
- Can solve it for all interaction strengths!!!
- Eigenstates are confluent hypergeometric functions

→ Investigate entanglement between particles as a function of interaction and splitting strength !

→ Good measure of entanglement for 2 particles?

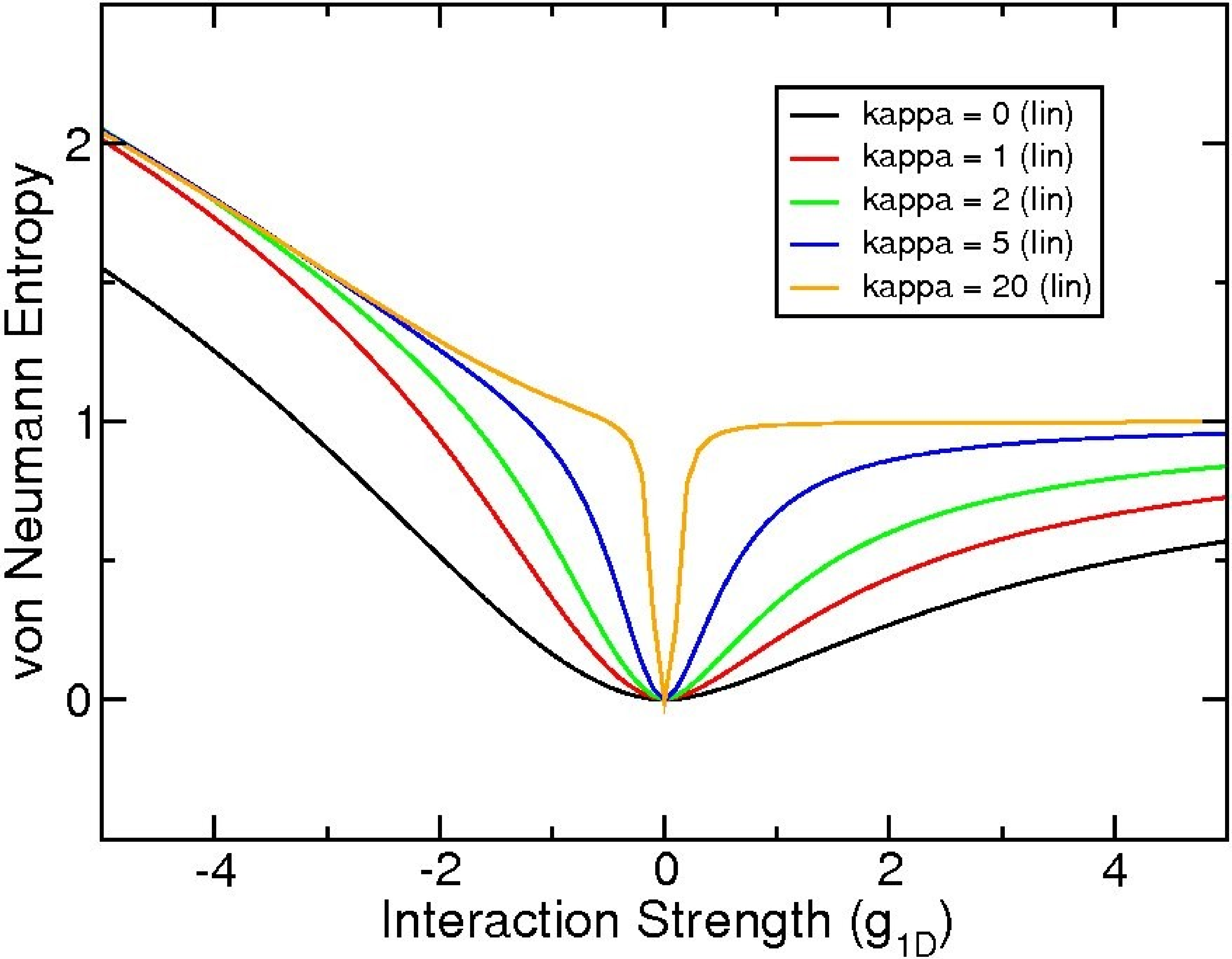
2 Particle Entanglement

- The Von Neumann entropy is a good entanglement measure for two particles*
- Caution must be taken! Why?
- There will be entropy attributed to the indistinguishability criterion
- Can calculate Von Neumann from RSPDM

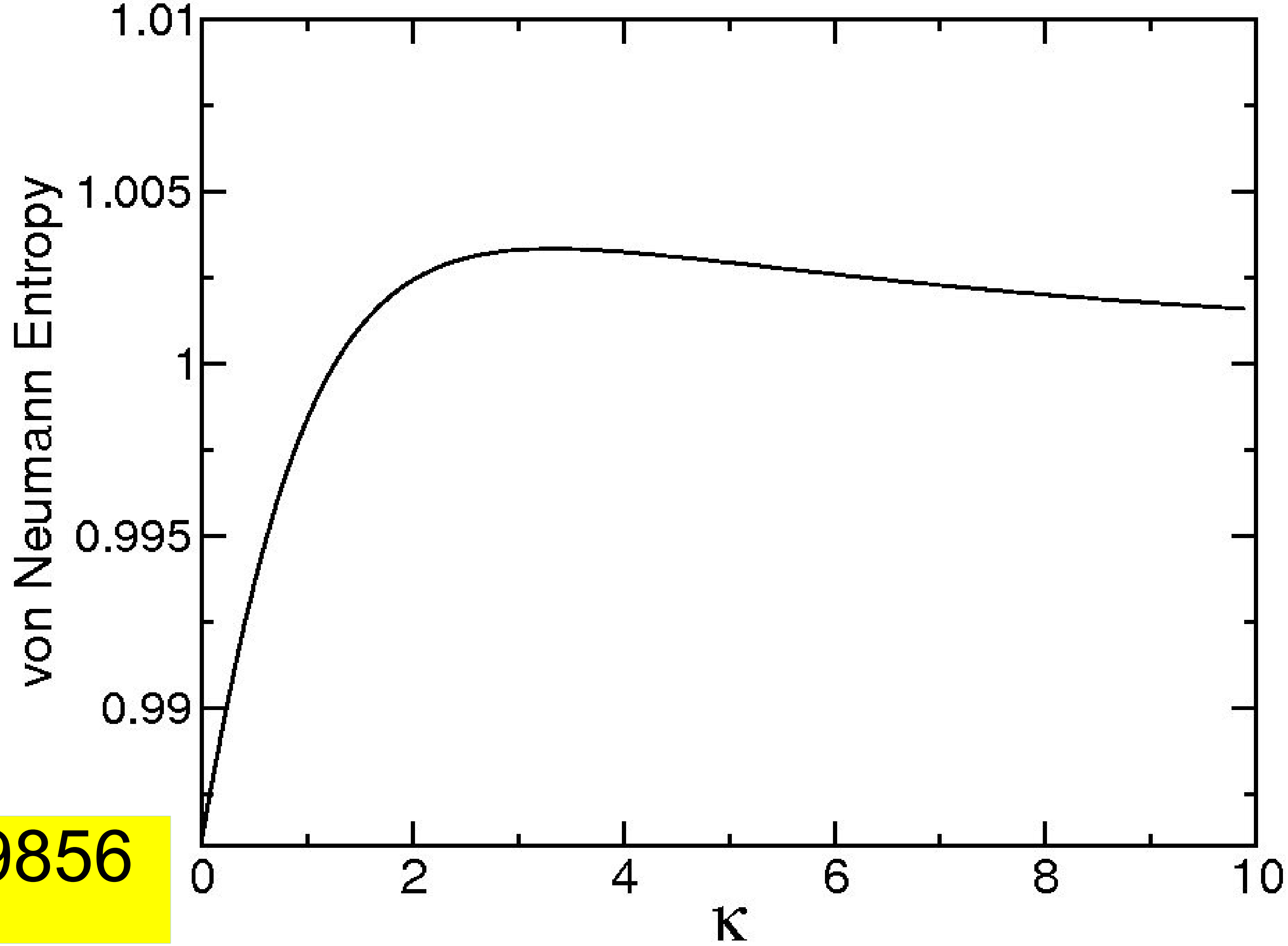
$$\rho = -\sum_j \lambda_j \log_2 \lambda_j$$

*L.You Phys Rev. A 64,042310 (2001)

Entanglement



Entanglement in TG Regime ?



~0.9856

Further Work

- Explore entanglement criteria for larger particle numbers
- Ideas for controlled creation and engineering of multi-particle entanglement using projective measurments
- Tonks gas can be used as a quantum processor along the lines of the linear ion trap
- Scalable system

Conlusions

- What experimentalists can do with cold atoms
- Ultra-cold one dimensional gases
- Fermi-Bose mapping theorem and the Tonks-Girardeau Gas
- The split-trap model
- Quantum many-body results
- Entanglement between two trapped bosons
- Further ideas for quantum information processing

Thank You For Listening