Opgaven Quantumfysica II, Week 6, 9 februari 2000

Inleveren voor maandag 14 februari 2000, 12.00 uur, bij de portier van de Diamantslijperij. Met vragen over stof en opgaven kun je terecht bij de werkcollegedocenten:

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Vermeld alstjeblieft de naam van je werkcollegedocent op het in te leveren werk. Succes !

Exercise 6.1: The Hydrogen Atom

In this problem we make a leap from the two dimensional quantum mechanics to the three dimensional case, which is far more relevant to our physical world, studying the behaviour of an electron placed in the electromagnetic field of a proton, namely the hydrogen atom. At t=0, the electron is in the following state:

$$\Phi(r,\theta,\phi,t=0) = N\left\{\exp(-r/a)\frac{1}{\sqrt{4\pi}} + \frac{1}{12\sqrt{6}}(\frac{r}{a})\exp(-r/2a)(\sqrt{7}Y_{1,0} - iY_{1,-1} + Y_{1,1})\right\}. \tag{1}$$

Where a is the Bohr-radius, 0.053 nm.

To save calculational effort we give the normalized eigenfunctions Ψ for the hamiltonian of an electron in the electromagnetic field of the proton. Because of the symmetry of the system it seems to be a good idea to work with spherical coordinates r, θ, φ. With these coordinates the eigenfunctions can be written as a product of a radial part $R_{nl}(r)$ which only depends on r and an angular part which is written in terms of the *spherical harmonics* $Y_{lm}(\theta,\varphi)$ which only depend on θ and φ:

$$\Psi_{nlm}(r,\theta,\varphi) = R_{nl}(r)Y_{lm}(\theta,\varphi). \tag{2}$$

These eigenfunctions that we give for this problem are eigenfunctions for both L^2 and L_z operators simultaneously. These quantum mechanical operators commute $[L^2, L_z] = 0$ and are very tightly related to the angular momentum of the system and they will play an important role in the study of the hydrogen atom as it will be discussed in the coming lectures. The spherical harmonics $Y_{lm}(\theta,\varphi)$ are the eigenfunctions of the operators L^2 and L_z, having as eigenvalues:

$$\begin{aligned} L^2 Y_{lm}(\theta,\varphi) &= \hbar^2 l(l+1) Y_{lm}(\theta,\varphi) \\ L_z Y_{lm}(\theta,\varphi) &= \hbar m Y_{lm}(\theta,\varphi) \end{aligned} \tag{3}$$

and they are normalized in the sphere:

$$\int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\theta\ \sin\theta\ Y^*_{lm}(\theta,\varphi)Y_{l'm'}(\theta,\varphi) = \delta_{l,l'}\delta_{m,m'} \tag{4}$$

The explicit form of the spherical harmonics $Y_{lm}(\theta,\varphi)$ and for the radial part $R_{nl}(r)$ for different combinations of nlm can be found in the book of *Griffiths* in pages 128 and 141 respectively. The subscripts nlm are the quantum numbers of the system which label the different eigenfunctions. The energy levels corresponding to those quantum numbers nlm are:

$$E_{nlm} = \frac{E_1}{n^2}. \tag{5}$$

The value for E_1 is -13.6 eV.

(Hint: Despite its appearance, this problem can be done without using a lot of mathematics! Try to recognize the eigenfunctions $R_{nl}(r)$ and Y_{lm} in Φ.)

a. Calculate the value of the normalization constant N.

b. Write the wave function Φ in Dirac bra-ket notation. (help: You can label the states by their quantum numbers nlm as $|\, n, l, m\,\rangle$)

c. Which are the possible outcomes if we measure L^2? And which are the corresponding probabilities?

d. What is the expectation value of L^2? (Hint: use the results of c.)

e. What is the possibility for finding the electron between a distance r and $r + \mathrm{d}r$ of the proton?

f. Which outcomes are possible if you measure L_z?

g. What is the expectation value of L_z?

h. What is the time dependence of $\Phi(r, \theta, \varphi, t)$? (Hint: Solve the time-dependent Schrödinger equation.)

i. What is $\Phi(r, \theta, \varphi, t)$ if at $t = 0$ a measurement of L_z leads to the value $+\hbar$?

j. What is $\Phi(r, \theta, \varphi, t)$ if at $t = 0$ a measurement of L_z gives 0?

k. What is the expectation value of the energy?

l. Which is the lowest possible value for the energy that you could find at $t = 0$?