Quantum Hall trial wave functions and CFT

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1 The quantum Hall effect

When an electric field is applied to a conductor, a current parallel to this field will usually arise. However, when a magnetic field orthogonal to this current is switched on, the charged particles that carry of the current are deflected by the Lorentz force and as a result the current acquires a component orthogonal to the applied electric field. This effect is called the Hall effect, after Edwin Hall, who discovered it in 1879. Let us assume that the particles move in a plane and that the magnetic field is orthogonal to this plane. We may then write the following formula for the dependence between the current density $\mathbf{J}$ and the electric field $\mathbf{E}$ in the medium

$$\left(\begin{array}{c} J_x \\ J_y \end{array} \right) = \left(\begin{array}{cc} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{xx} \end{array} \right) \left(\begin{array}{c} E_x \\ E_y \end{array} \right) \tag{1}$$

Here $\sigma_H := \sigma_{xy}$ is the so called Hall conductivity. The inverse of the conductivity tensor σ is the resistivity tensor ρ and again, we call $\rho_H := \rho_{xy}$ the Hall resistivity. Classical theory predicts that the Hall resistivity is proportional to the applied magnetic field.

The quantum Hall effect is an extraordinary version of the Hall effect that occurs when electrons caught at the interface between two semiconducting materials (for example GaAs and AlGaAs) are cooled to very low temperatures (~ 10mK) and subjected to a very strong magnetic field (~ 20T). Under these conditions, the Hall resistance does not rise linearly with the applied B-field, but instead exhibits plateaus. At these plateaus, the Hall conductance takes values which are integer [1] and fractional [2] multiples of the fundamental unit $\frac{e^2}{h}$ (see figure 1). At each plateau, the diagonal elements of the conductance tensor vanish; the current is perpendicular to the applied voltage. The electrons form a fluid state which is incompressible, meaning that there is an energy gap against excitations (for example, it would be impopssible to make an arbitrarily soft sound in these fluids). Nevertheless these electron liquids have interesting localized excitations. These are called quasiparticles if they correspond to a local peak in the electron density and quasiholes if they correspond to a local dip. The quantum Hall quasiparticles and quasiholes at the fractional plateaus exhibit many exotic properties. For example, their charge is typically a fraction of the charge of an electron. Also they are believed to be neither bosons nor fermions. Their exchanges are governed by the braid group and are even predicted to be non-Abelian in some cases.
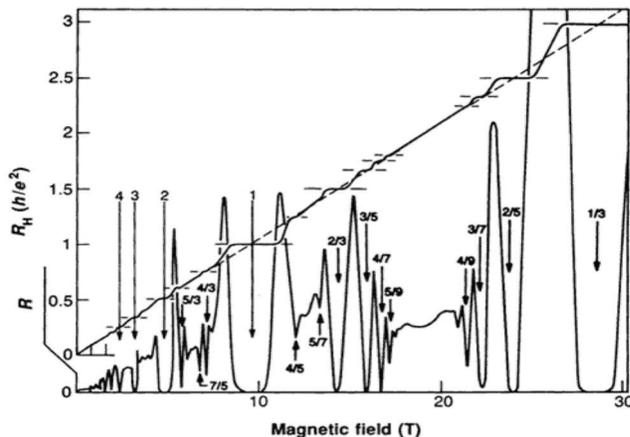
Figure 1: Experimental results for the Hall resistance, taken from [3]

For a full introduction to the quantum Hall effect, one may for instance consult the books [4, 5, 6]. These notes will give a short and necessarily biased introduction, because I will focus on what is needed to understand the conformal field theory description of the bulk properties of the non-Abelian states.

2 Theory of the integer effect

The most striking characteristic of the quantum Hall effect is the occurrence of plateaus in the conductance at values $\nu\frac{e^2}{h}$, where ν is an integer or a simple fraction. It turns out that there is a fundamental difference between these two cases: the fractional quantum Hall effect depends essentially on the repulsive interactions between the electrons, while the integer effect ($\nu \in \mathbb{N}$) may be understood in terms of a system of non-interacting electrons in a magnetic field, which scatter on impurities. In order to introduce some of the basic concepts in the quantum Hall literature, it is useful to have a brief look at the system without even the impurities. This is the problem of free particles of charge $-e$ and mass m in two dimensions, under the influence of a magnetic field $\mathbf{B} = (0, 0, B)$. It was solved by Landau in 1930 [7](see for instance [8] for a treatment in English).

The single particle Hamiltonian for the system is[1]

$$H = \frac{1}{2m}\left((p_x - \frac{e}{c}A_x)^2 + (p_y - \frac{e}{c}A_y)^2\right).$$

(2)

where $\mathbf{A}$ is a vector potential which gives rise to the required magnetic field. We will work in symmetric or central gauge, which means that

$$\mathbf{A} = (-\frac{B}{2}y, \frac{B}{2}x, 0).$$

(3)

In terms of a dimensionless complex coordinate $z = (x + iy)/\ell$, where $\ell = \sqrt{\hbar c/(eB)}$ is the magnetic length, the Hamiltonian H and the angular momentum $L = xp_y - yp_x$

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[1]For simplicity, we ignore the spin of the electrons. In many Hall systems, this is actually a good way to proceed, since only one spin direction occurs, due to the large Zeeman splitting.

become

$$H = \frac{1}{2}\hbar\omega_c(-4\partial_z\partial_{\bar{z}} - z\partial_z + \bar{z}\partial_{\bar{z}} + \frac{1}{4}z\bar{z})$$
$$L = \hbar(z\partial_z - \bar{z}\partial_{\bar{z}}) \tag{4}$$

Here $\omega_c = \frac{eB}{mc}$ is the cyclotron frequency.

The eigenvalues and eigenfunctions of this Hamiltonian can be found in a way similar to the usual solution of the Harmonic oscillator. Introducing operators

$$a = -\partial_{\bar{z}} - \frac{z}{4} \quad a^\dagger = \partial_z - \frac{\bar{z}}{4}$$
$$b = -\partial_z - \frac{\bar{z}}{4} \quad b^\dagger = \partial_{\bar{z}} - \frac{z}{4}, \tag{5}$$

we find that a and $a^\dagger$ commute with b and $b^\dagger$ and that

$$[H, a^\dagger] = \hbar\omega_c a^\dagger \quad [H, b^\dagger] = 0 \tag{6}$$

Hence $b^\dagger$, when applied to an eigenstate of H, creates a new eigenstate with the same eigenvalue, while $a^\dagger$ creates an eigenstate with the eigenvalue increased by $\hbar\omega_c$. Their conjugates a and b are the corresponding annihilation operators. We may find a "lowest weight" ground state $\psi_{0,0}$ by solving $a\psi_{0,0} = b\psi_{0,0} = 0$. This yields $\psi_{0,0}(z,\bar{z}) := e^{-z\bar{z}/4}$. Applying $a^\dagger$ and $b^\dagger$ to $\psi_{0,0}$, we obtain a basis of eigenstates of the Hamiltonian, given by

$$\psi_{m,n}(z) = (\partial_{\bar{z}} - \frac{z}{4})^m(\partial_z - \frac{\bar{z}}{4})^n e^{-z\bar{z}/4} = e^{z\bar{z}/4}\partial_{\bar{z}}^m\partial_z^n e^{-z\bar{z}/2}. \tag{7}$$

The corresponding energy levels are called Landau levels. They are independent of m and hence infinitely degenerate,

$$E_n = \hbar\omega_c(n + \frac{1}{2}). \tag{8}$$

The states in each Landau level can be distinguished by their angular momentum; we have $L\psi_{m,n} = \hbar(m-n)\psi_{m,n}$. This follows easily from $L\psi_{0,0} = 0$ and the commutation relations

$$[L, a^\dagger] = \hbar a^\dagger \quad [L, b^\dagger] = -\hbar b^\dagger. \tag{9}$$

The first Landau level is of particular importance to us, as it is the only level that plays a role in the physics at very high magnetic fields. From (7), we see that the wave functions in this Landau level are exactly all functions which are a product of the Gaussian factor $e^{-z\bar{z}/4}$ and a holomorphic function. The angular momentum eigenstates in the first Landau level are just the functions $z^m e^{-z\bar{z}/4}$, with L-eigenvalues $m\hbar$. The main effect of confining the particles to a finite region in the plane (the sample) is that the Landau levels are no longer infinitely degenerate. Wave functions of exceedingly high angular momentum would place their electron outside the sample. Effectively, each single-particle state takes a surface area $\frac{hc}{eB} = \ell^2$ so that the Landau levels now contain eBA/hc states each, where A is the surface area of the sample. The number of states in a Landau level thus equals the number of fundamental flux quanta $\frac{e}{hc}$ that pierce the sample. These results are really independent of the detailed geometry of the sample, but for convenience, we will always take the sample to be circular.

The quotient of the number of electrons in the sample by the number of states in a Landau level is called the filling factor or filling fraction. In a system of free electrons, it is just the number of filled Landau levels, hence the term. It is seen experimentally that the conductance plateau at conductance $\nu\frac{e^2}{h}$ occurs at filling fraction ν. Hence, one speaks of the plateau at filling fraction ν.[2]

Integer filling "fractions" are special, since a system at integer filling fraction has a gap of $\hbar\omega_c$ to the next unoccupied single electron state. This suggests that scattering of electrons should be inhibited at these filling fractions and hence provides an explanation for the dips in the longitudinal resistance of the system at these values of ν. To explain the fact that there is a plateau in the resistance around integer filling fractions, one has to go beyond free electrons and introduce impurities. These impurities localize some of the states in each Landau level and shift their energies away from the quantized values (8). The states which remain extended also don't have their energies shifted by much. Now the crucial idea is that, at low temperatures, only the extended states contribute to the transport of electric charge across the system. Thus, when the B-field is varied and the Fermi level of the system sweeps through the energy levels, the conductivity remains constant as long as the Fermi level is in a band of localized states and changes rapidly as it moves through a band of extended states. In other words, the plateaus correspond to bands of localized states between the Landau levels.

Of course, in the model with impurities there is no longer a real gap, but there is still a gap between the bands of extended states, a mobility gap. Another aspect of the addition of impurities seems more problematic at first. The number of extended states in the system depends on the number and nature of the impurities and therefore it would seem that the conductance would also depend on these. However an argument of Laughlin's [9] which was later refined by Halperin [10], shows that the contribution of each band of extended states to the conductance is actually independent of the number of states in that band.

3   Theory of the fractional effect

The explanation of the integer quantum Hall effect which was so sparsely sketched above does not provide an understanding of the fractional quantum Hall effect; there seems no reason why there should be a gap or a mobility gap at fractional ν. In order to understand the fractional effect, one has to take the interactions between the electrons into account.

3.1   Laughlin's wave functions

Crucial stepping stones in the theory of the fractional effect were Laughlin's variational wave functions for a system of N electrons on a disc [11]. In terms of the complex coordinates z_k for the electrons, the ground state wave functions he proposed are

$$\Psi_N^m(z_1,\ldots,z_N) = \prod_{i<j}(z_i - z_j)^{2m+1}\, e^{-\left(\frac{1}{4}\sum_i z_i \bar{z}_i\right)} \tag{10}$$

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[2]Note that the "location" of a plateau is much less accurately determined than the conductance at the plateau, so that when people speak of "the plateau at $\nu = \frac{1}{3}$", this is a reference to the value of the conductance, rather than to the filling fraction.

where m is an integer. One may arrive at these wave functions in the following way. First, one restricts to the space of functions of the Jastrow form:

$$\Psi(z_1, \ldots, z_N) = \prod_{i<j} f(z_i - z_j). \tag{11}$$

The choice of this form for the variational wave function is really where the repulsive interactions between the electrons are included; any f with $f(0) = 0$ will tend to keep the particles apart. After the assumption of the Jastrow form, the wave functions (10) are determined by three physical requirements.

1. The wave function must be totally antisymmetric, since the electrons are fermions. Hence f must be odd.

2. In order to minimize energy, the wave function must be built up from single electron wave functions in the lowest Landau level. That is, it must be holomorphic up to a factor of $e^{-z\bar{z}/4}$ for each electron. This requirement is reasonable if the energy scale for the interaction is small compared to $\hbar\omega_c$

3. The ground state must be an eigenstate of total angular momentum. This means that the holomorphic function multiplying the Gaussian factors must be a homogeneous polynomial in the z_k. Since angular momentum commutes with the Hamiltonian, this condition is certainly satisfied if the ground state is non-degenerate (i.e. if there is a gap).

The wave functions (10) are the only wave functions of the Jastrow form which satisfy these three requirements. Therefore, this argumentation predicts a discrete series of ground states, corresponding to different filling fractions.

A simple way to find the filling fraction straight from the expression (10) is the following. Since the electrons fill the sample, the highest occupied single particle angular momentum state will always be the highest state in the first Landau level. On the other hand, we may read off the maximal angular momentum for a single particle from (10); it is just the maximal power of any single z_k, which is $(2m + 1)(N - 1)$. This means that the first Landau level contains $\sim (2m + 1)N$ states, while there are only N electrons and hence the filling fraction is $\nu = \frac{1}{2m+1}$. For small numbers of electrons (~ 10), one may, by numerical methods, check that Ψ_N^m has very good overlap with the exact ground state of the system at $\nu = \frac{1}{2m+1}$.

References

[1] K. von Klitzing, G. Dorda, and M. Pepper. New method for high-accuracy determination of the fine- structure constant based on quantized Hall resistance. *Phys. Rev. Lett.*, 45:494–497, 1980.

[2] D. C. Tsui, H. L. Stormer, and A. C. Gossard. Two-dimensional magnetotransport in the extreme quantum limit. *Phys. Rev. Lett.*, 48:1559–1562, 1982.

[3] J.P. Eisenstein and H.L. Stormer. The fractional quantum Hall effect. *Science*, 248(4962):1510–1516, 1990.

[4] R.E. Prange and S.M. Girvin, editors. *The quantum Hall effect*. Graduate texts in contemporary physics. Springer-Verlag, New York, 1987.

[5] S. Das Sarma and A. Piczuk, editors. *Perspectives in quantum Hall effects, novel quantum liquids in low-dimensional semiconductor structures*. John Wiley and Sons, New York, 1997.

[6] Z.F. Ezawa, editor. *Quantum Hall effects, field theoretical approach and related topics*. World Scientific, Singapore, 2000.

[7] L. Landau. Diamagnetismus der Metalle. *Z. Phys.*, 64:629–637, 1930.

[8] L.D. Landau and E.M. Lifshitz. *Quantum mechanics, non-relativistic theory*, volume 3 of *Course of theoretical physics*. Pergamon Press, London-Paris, 1958.

[9] R. B. Laughlin. Quantized Hall conductivity in two dimensions. *Phys. Rev.*, B23:5632–5733, 1981.

[10] B. I. Halperin. Quantized Hall conductance, current carrying edge states, and the existence of extended states in a two-dimensional disordered potential. *Phys. Rev.*, B25:2185–2190, 1982.

[11] R. B. Laughlin. Anomalous quantum Hall effect: An incompressible quantum fluid with fractionally charged excitations. *Phys. Rev. Lett.*, 50:1395, 1983.