MP466: Particle Physics — Problem Sheet

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(1) An α-particle, a ^{4}He nucleus consisting of two protons and two neutrons, has mass $6.644656 \times 10^{-27}\ kg$. An α-particle is less massive than the combined mass of two protons and two neutrons and the energy equivalent of the difference is the binding energy of the α-particle. Calculate the binding energy of an α-particle, given $m_p = 1.672622 \times 10^{-27}\ kg$ and $m_n = 1.674927 \times 10^{-27}\ kg$.

(2) A particle of mass M decays to two daughter particles of masses m_1 and m_2. Use conservation of relativistic 3-momentum to show that the speeds v_1 and v_2 of the emerging particles, in the rest frame of M, are related by

$$m_1^2\left(\gamma^2(v_1) - 1\right) = m_2^2\left(\gamma^2(v_2) - 1\right),$$

where $\gamma(v_1)$ and $\gamma(v_2)$ are the Lorentz γ-factors of the emerging particles.

Combine this expression with conservation of energy to show that

$$\gamma(v_1) = \frac{M^2 + m_1^2 - m_2^2}{2Mm_1}.$$

Use this to show that the kinetic energy of m_1 in the rest frame of M is

$$\left(\frac{(M - m_1)^2 - m_2^2}{2M}\right)c^2$$

and hence that the maximum kinetic energy, $T = E - m_e c^2$, of the electron in β-decay is

$$\begin{aligned} T = m_e c^2(\gamma(v_e) - 1) &= \frac{\{(m_n - m_e)^2 - m_p^2\}c^2}{2m_n} \\ &= (m_n - m_p - m_e)c^2 - \frac{\{(m_n - m_p)^2 - m_e^2\}c^2}{2m_n}\,. \end{aligned}$$

Thus T is slightly less than the figure $(m_n - m_p - m_e)c^2$ quoted in the lectures. How much less?

(3) In a two body collision two incoming particles a and b are anihilated and produce two outgoing particles c and d, with masses m_c and m_d, $a + b \to c + d$. In the centre of mass frame the total incoming energy of a and b is E. Use relativistic kinematics to show that c and d emerge with energies

$$\begin{aligned} E_c &= \frac{1}{2E}\left\{E^2 + (m_c^2 - m_d^2)c^4\right\}, \\ E_d &= \frac{1}{2E}\left\{E^2 + (m_d^2 - m_c^2)c^4\right\}, \end{aligned}$$

and that each has relativistic 3-momentum $\underline{P}$ with magnitude squared, $P^2 = \underline{P}.\underline{P}$, given by

$$P^2 = \frac{1}{4E^2c^2}\left\{E^2 - (m_c + m_d)^2c^4\right\}\left\{E^2 - (m_c - m_d)^2c^4\right\}.$$

Show that the velocity of particle c is given by

$$v_c^2 = \frac{\left(E^2 - (m_c + m_d)^2c^4\right)\left(E^2 - (m_c - m_d)^2c^4\right)c^2}{\left(E^2 + (m_c^2 - m_d^2)c^4\right)^2}.$$

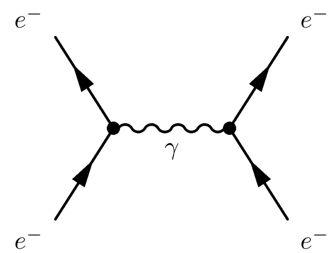
(4) Show, using conservation of momentum, that an isolated electron moving through space cannot spontaneously emit a photon with non-zero energy. (Use the fact that the 4-momentum $\underset{\sim}{\mathrm{P}}$ of a particle of mass m satisfies $\underset{\sim}{\mathrm{P}}.\underset{\sim}{\mathrm{P}} = -m^2c^2$ — a particle who's 4-momentum satisfies this condition is said to be on its *mass-shell*.

When there is another particle present the situation is different. Due to the uncertainty principle a particle can be off its mass shell and live for a short time as a 'virtual' particle, provided the time τ is small enough that

$$\tau\sqrt{|\underset{\sim}{\mathrm{Q}}.\underset{\sim}{\mathrm{Q}}|} \leq \frac{\hbar}{2c}$$

where $\underset{\sim}{\mathrm{Q}}$ is the virtual particle's 4-momentum.

Two electrons scatter off each other by exchanging a single photon



If the 4-momenta of the incoming electrons are $\underset{\sim}{\mathrm{P}}_1$ and $\underset{\sim}{\mathrm{P}}_2$ and the outgoing momenta are $\underset{\sim}{\mathrm{P}}'_1$ and $\underset{\sim}{\mathrm{P}}'_2$, the 4-momentum transfer in the collision is defined as

$$\underset{\sim}{\mathrm{Q}} := \underset{\sim}{\mathrm{P}}'_1 - \underset{\sim}{\mathrm{P}}_1 = -(\underset{\sim}{\mathrm{P}}'_2 - \underset{\sim}{\mathrm{P}}_2).$$

Assuming that the electrons are all on mass-shell show that the photon cannot have a light-like 4-momentum. *i.e.* the photon is off its mass-shell. Using the uncertainty principle calculate the maximum distance that this 'virtual' photon can travel in terms of $\underset{\sim}{\mathrm{Q}}.\underset{\sim}{\mathrm{Q}}$.

(5) Show that

$$\frac{dp}{dE} = \frac{1}{v}$$

for a particle with relativistic 3-momentum p and energy E.

(6) Two particles c and d emerge from a 2-body collision process

$$a + b \rightarrow c + d$$

where the total energy is $E = E_a + E_b = E_c + E_d$ in the centre of mass frame, with E_a and E_b the individual energies of the incident particles and E_c and E_d the energies of the emerging particles. If the final velocities are non-relativistic show that

$$\frac{dE}{dp_f} = v_c + v_d$$

where v_c and v_d are the final velocities in the centre of mass frame.

Show that the answer is the same when the velocities are relativistic.

(7) There is a resonance in the cross-section for pion-nucleon scattering at $E_0 = 776\ MeV$ with a width of $\Gamma = 0.149\ MeV$. Calculate the mass and life-time of the associated particle (this particle is called the ρ-meson, which decays predominantly to two pions).

(8) Two particle of mass m_1 and m_2 collide elastically, with speeds v_1 and v_2 respectively, in the centre of mass frame. Show that the speeds of the two particles are unchanged in the collision and that the differential cross-section given in the lectures can be written in terms of the particles' energies, E_1 and E_2, as

$$\frac{d\sigma}{d\Omega} = \frac{1}{4\pi^2\hbar^4 c^4} \frac{E_1^2 E_2^2}{(E_1 + E_2)^2} |M_{if}|^2,$$

(9) Determine which of the following reactions are allowed by conservation laws and which are forbidden:

$$\pi^0 \rightarrow e^+ + e^-$$
$$\pi^- \rightarrow \mu^- + \nu_\mu$$
$$e^- + p \rightarrow n + \nu_e$$
$$\mu^+ \rightarrow e^+ + e^- + e^+$$
$$\mu^- \rightarrow e^+ + e^- + \nu_\mu$$
$$K^0 + n \rightarrow \Lambda + \pi^0$$
$$K^- + \pi^0 \rightarrow \Lambda + \pi^-$$
$$\Xi^0 \rightarrow \Lambda + \pi^0$$

(in each case explain your reasoning clearly and in detail).

(10) A 2×2 complex matrix has, in general, 4 complex (*i.e.* 8 real) components. If the matrix is unitary so $U^{-1} = U^\dagger$ with † denoting Hermitian conjugation, *i.e.* complex conjugation followed by transpose, how many free components does U have? What constraint does this put on $\det(U)$?

(11) In terms of the 2×2 Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

a rotation through an angle θ about an axis pointing in the direction $\mathbf{n}$, with $\mathbf{n}.\mathbf{n} = 1$, can be represented by

$$U(\theta, \mathbf{n}) = e^{-i\theta(\sigma.\mathbf{n})/2}.$$

Derive the following forms for rotations through an angle θ about the directions indicated below:

$$U(\theta) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}, \qquad \mathbf{n} = (0,0,1);$$

$$U(\theta) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \qquad \mathbf{n} = (0,1,0);$$

$$U(\theta) = \begin{pmatrix} \cos(\theta/2) & -i\sin(\theta/2) \\ -i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \qquad \mathbf{n} = (1,0,0).$$

In each case show that $U(\theta)U(\theta') = U(\theta + \theta')$, $U^\dagger U = 1$ and $U(2\pi) = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(12) Draw quark flow diagrams for the following processes and decide whether or not they are suppressed by the OZI rules:

$$\begin{aligned}
\rho^+ &\to \pi^+ + \pi^0 \\
K^{*+} &\to K^0 + \pi^+ \\
K^{*+} &\to K^0 + \pi^+ + \pi^0 \\
\phi &\to K^+ + K^- \\
\phi &\to \pi^+ + \pi^0 + \pi^- \\
J/\psi &\to D^0 + \overline{D}^0 \\
J/\psi &\to \pi^+ + \pi^0 + \pi^-.
\end{aligned}$$

In the last two decays the J/ψ (affectionately called the 'gypsy') is a $c\bar{c}$ meson with mass 3.1 Gev/c^2. Given that the charmed D-mesons have mass 1.9 GeV/c^2 do you expect the sixth decay above to happen?

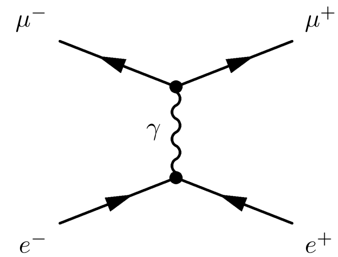
(13) Two particles $X^0(1193)$ and $Y^-(1321)$ are produced in the strong interaction processes

$$K^- + p \to X^0 + \pi^0, \qquad K^- + p \to K^+ + Y^-$$

respectively. Examine the baryon number and strangeness quantum numbers in each case and, using these, determine the quark content of the X and Y.

(14) A $B^{\pm}$-meson has a lifetime of $1.7 \times 10^{-12} s$ and a mass of $5279 MeV/c^2$. How far can a very energetic meson with $E = 500 GeV$) travel before it decays?

(15) Show that the virtual photon



in the electron-positron annihilation process, $e^+ + e^- \rightarrow \mu^+ + \mu^-$, is time-like when the leptons are all on-shell.

(16) If an electron and a positron collide with total centre of mass energy $> m_\pi c^2$, is the process e$^+ + e^- \rightarrow \pi^0$ possible? If the electron-positron pair have enough energy, could they produce a neutral vector meson in a process such as e$^+ + e^- \rightarrow \omega$ or $e^+ + e^- \rightarrow \phi$?

Are the reverse processes $\pi^0 \rightarrow e^+ + e^-$, $\phi \rightarrow e^+ + e^-$ and $\omega \rightarrow e^+ + e^-$ allowed? By thinking in terms of virtual photon production can you decide which of these three processes has the smallest amplitude?

What about $\pi^0 \rightarrow \mu^+ + \mu^-$, $\phi \rightarrow \mu^+ + \mu^-$ and $\omega \rightarrow \mu^+ + \mu^-$ or $\pi^0 \rightarrow \tau^+ + \tau^-$, $\phi \rightarrow \tau^+ + \tau^-$ and $\omega \rightarrow \tau^+ + \tau^-$?

(17) The neutral kaon system is in some ways analogous to neutrinos in that the particles that are produced in interactions are the K^0 and the $\bar{K}^0$ whereas the mass eigenstates are K_L and K_S. Important differences however are that, unlike neutrinos, the kaons decay to lighter particles (pions) and the kaons themselves have masses $\approx 500\ MeV/c^2$ and so are not relativistic for energies significantly less than 500 MeV (the calculation of K^0-$\bar{K}^0$ mixing in the lectures was done in the rest frame of the kaons).

Given that

$$|K^0 >= \frac{1}{\sqrt{2}}(|K_S > +|K_L >)$$

and

$$|K^0 >= \frac{1}{\sqrt{2}}(|K_S > -|K_L >)$$

calculate the mixing angle. From the mass difference

$$\Delta m = 3.5 \times 10^{-6}\ eV$$

calculate $\Delta m^2 = |m^2_{K_L} - m^2_{K_S}|$.

Calculate the oscillation length of a non-relativistic kaon moving with speed $0.1c$.

(18) Show that the observation of the process $\bar{\nu}_\mu + e^- \rightarrow \bar{\nu}_\mu + e^-$ provides evidence for neutral currents but the observation of the process $\bar{\nu}_e + e^- \rightarrow \bar{\nu}_e + e^-$ does not.

One type of neutrino detector consists of a large tank of heavy water, that is water in which some of the hydrogen nuclei consist of an isotope called deuterium, denoted by $D = ^2H$, which is a bound state of a proton and a neutron with atomic mass 2. Electron neutrinos can be detected by virtue of the fact that they can change the neutron in the heavy hydrogen into a proton, via inverse β-decay,

$$\nu_e + n \rightarrow e^- + p \qquad \Leftrightarrow \qquad \nu_e + D^+ \rightarrow e^- + 2p, \tag{1}$$!

thus breaking a deuterium ion, D^+, into two protons. Sometimes though the neutrinos simply scatter off the heavy hydrogen without changing their identity,

$$\nu + n \rightarrow \nu + n \qquad \Leftrightarrow \qquad \nu + D \rightarrow \nu + D. \tag{2}$$

Show that only electron neutrinos can participate in the former process but all three flavours of neutrino can participate in the latter. These processes can be used to detect oscillations in neutrinos from the Sun. Nuclear reactions in the Sun produce only ν_e neutrinos and the number of neutrinos produced at any given energy can be calculated, knowing the physical conditions prevailing in the centre of the Sun. The flux of ν_e detected at the Earth, using (1), is about a half of the expected value. However the total flux detected using (2) is exactly the same as the calculated flux of ν_e. This is interpreted as evidence for the fact that some of the electron neutrinos produced in the centre of the Sun transform into other flavours on their way to the Earth.

(19) Show that the total centre of mass energy, E, in inverse β-decay,

$$\nu_e + n \quad \rightarrow \quad e^- + p,$$

is related to the energy of the incoming neutrino, E_ν, by

$$E = \sqrt{E_\nu^2 + m_n^2 c^4} + E_\nu,$$

where m_n is the mass of the neutron.

(20) Using the expression derived in the lectures for the differential cross-section for inverse β-decay when the incoming neutrino energy $E_\nu >> m_n c^2$,

$$\frac{d\sigma}{d\Omega} = \frac{g_W^4}{32\pi^2} \frac{E_\nu^2}{\{m_W^2 c^4 + 2E_\nu^2(1 - \cos\theta)\}^2}$$

calculate the total cross-section.

(21) Charged pions produced in the upper atmosphere by cosmic rays decay to muons

$$\pi^+ \rightarrow \mu^\pm + \nu_\mu$$

and the muons subsequently decay to electrons or positrons

$$\mu^\pm \rightarrow e^\pm + \nu_e + \nu_\mu$$

(the distinction between neutrinos and anti-neutrinos is ignored here).

If both neutrinos and anti-neutrinos are detected at ground level, the flux of neutrinos from pions produced by cosmic rays should then consist of twice as many muon neutrinos as electron neutrinos. Observationally the ratio is one, and this is interpreted as being due to muon neutrinos oscillating to another type of neutrino which is not ν_e, probably to ν_τ.

Given the mass difference of the oscillation $\Delta m^2 = 2 \times 10^{-3}\ eV^2/c^4$ calculate the oscillation length of a 500 MeV muon neutrino. What do you think the mixing angle might be?