Angular Momentum in quantum Mechanics

In classical physics angular momentum is a vector $\mathbf{J}$ with a magnitude J and a direction specified by three components (J_x, J_y, J_z). Of course these are not independent, since

$$J^2 = \mathbf{J}.\mathbf{J} = J_x^2 + J_y^2 + J_z^2,$$

and angular momentum is specified by three real numbers. Vectors are added using the usual triangle law from which

$$|J_1 - J_2| \le |\mathbf{J}_1 + \mathbf{J}_2| \le J_1 + J_2. \tag{1}$$

Quantum mechanically things are different in a number of ways:

- Angular momentum is *quantised.* J is not a continuous variable but can take only discrete values which are multiples of the fundamental quantum unit $\hbar$. In fact

  $$J^2 = j(j+1)\hbar^2$$

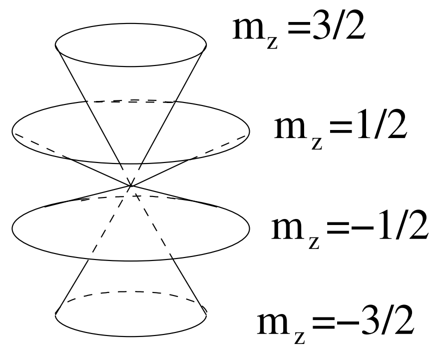
  where j is either a non-negative integer, $j = 0, 1, 2, \ldots$ or a positive half-integer $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$.
- The uncertainty principle says that J_x, J_y and J_z cannot be simultaneously specified — there are only two degrees of freedom in quantum angular momenta. If J is specified then we can measure only one linear combination of J_x, J_y or J_z, not all three independently. For example if J and J_z are known for a given quantum state then J_x and J_y are completely undetermined and have no physical value. Furthermore J_z is also quantised and has only $2j + 1$ possible values

  $$J_z = m_z\hbar$$

  where

  $$m_z = -j, -j+1, \ldots, j-1, j. \tag{2}$$

  For $j = \frac{3}{2}$, for example, there are 4 possible values of J_z, given by $m_z = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$, but J_x and J_y are completely undetermined. Pictorially the situation looks something like this:



  where $\mathbf{J}$ is represented by a cone whose height is determined by m_z.

- Because of the above two restrictions, addition of angular momenta in quantum mechanics is also different to the classical case. If two angular momenta $\mathbf{J}_1$ and $\mathbf{J}_2$, are added to give a third $\mathbf{J}$ then the triangle inequality (1) still holds, but J can only take the discrete values

  $$J = |J_1 - J_2|, |J_1 - J_2| + \hbar, \ldots, J_1 + J_2 - \hbar, J_1 + J_2,$$

  or equivalently

  $$j = |j_1 - j_2|, |j_1 - j_2| + 1, \ldots, j_1 + j_2 - 1, j_1 + j_2.$$

  If the z-components are known there are further constraints on the sum: if $\mathbf{J}_1$ has third component $J_{1,z}$ and $\mathbf{J}_2$ has third component $J_{2,z}$ then these do just add like the classical case and $\mathbf{J}$ will have third component

  $$J_z = J_{1,z} + J_{2,z}$$

  or $m = m_1 + m_2$. This requires $j \geq |m|$ because of (2).

In general a particle need not be in a definite state of angular momentum, it may be in a linear superposition of different j and m. A general state is a vector in a Hilbert space and we can use definite states of j and m as basis vectors, denoted by $|j; m>$. This basis can be chosen so that it is orthonormal with

$$< j; m | j'; m' >= \delta_{jj'} \delta_{mm'}.$$

A general state is then a linear sum of basis vectors

$$\Psi = \sum_j \sum_{m=-j}^{j} \psi_{jm} |j; m>$$

where ψ_{jm} are complex numbers. For example adding a state with definite angular momentum $|j_1; m_1>$ to a second state with definite angular momentum $|j_2; m_2>$ produces a linear superposition of states with angular momenta j between $|j_2 - j_1|$ and $j_2 + j_1$. A standard notation for this is

$$|j_1; m_1 > \otimes |j_2; m_2 >= \sum_{j=|j_2-j_1|}^{j_2+j_1} C^{j;m}_{j_1 j_2; m_1 m_2} |j; m> \tag{3}$$

where the numbers $C^{j;m}_{j_1 j_2; m_1 m_2}$ are called *Clebsch-Gordon coefficients*. Conservation of angular momentum requires that only states with $m = m_1 + m_2$ appear in the sum on the right-hand side, so $C^{j;m}_{j_1 j_2; m_1 m_2} = 0$ unless $m = m_1 + m_2$.

The reason for the $\otimes$ symbol is that, for fixed j_1 and j_2, there are $2j_1 + 1$ possible values of m_1 and $2j_2 + 1$ possible values of m_2. Thus the set

$$\{|j_1; m_1 >; m_1 = -j_1, -j_1 + 1, \ldots, j_1 - 1, j_1\}$$

is a basis for a $2j_1+1$ dimensional vector space and

$$\{|j_2;m_2>;m_2=-j_2,-j_2+1,\dots,j_2-1,j_2\}$$

is a basis for a $2j_2+1$ dimensional vector space. There are therefore $(2j_1+1)(2j_2+1)$ possible states on the left-hand side of equation (3) so vector addition in quantum mechanics is like multiplication in terms of Hilbert space dimensions.[1] A slightly more compact notation, which we shall often use, is

$$|j_1,j_2;m_1,m_2>=|j_1;m_1>\otimes|j_2;m_2>.$$

For each value of j on the right-hand side of (3) there are $2j+1$ possible values of m and j runs from $|j_2-j_1|$ to j_2+j_1 giving, assuming $j_2\geq j_1$ for example,

$$\sum_{j=j_2-j_1}^{j_2+j_1}(2j+1)=\big\{(j_2+j_1)+(j_2-j_1)\big\}(2j_1+1)+(2j_1+1)=(2j_1+1)(2j_2+1)$$

different $|j;m>$.

The quantum Hilbert space associated with these angular momentum states is a finite dimensional complex vector space with dimension $(2j_1+1)(2j_2+1)$ and either

$$\{|j;m>;j=|j_1-j_2|,|j_1-j_2|+1,\dots,j_1+j_2-1,j_1+j_2,m=-j,-j+1,\dots,j-1,j\}$$

or

$$\{|j_1,j_2;m_1,m_2>;m_1=-j_1,-j_1+1,\dots,j_1-1,j_1,m_2=-j_2,-j_2+1,\dots,j_2-1,j_2\}$$

can be used as an orthonormal basis for the space. The Clebsch-Gordon coefficients just give a linear transformation between two different orthonormal bases.

Consider for example combining two spin-1/2 electrons, so $j_1=j_2=1/2$. Each electron has a spin which is a vector in a two-dimensional Hilbert space of possible spins, its spin is some linear combination of two possible spin states, often called spin up ($|\uparrow>$) and spin down ($|\downarrow>$) relative to some fiducial direction such as the z-axis of a Cartesian co-ordinate system. In our notation

$$\Big|\frac{1}{2};\frac{1}{2}>=|\uparrow>,\qquad\text{and}\qquad\Big|\frac{1}{2};-\frac{1}{2}>=|\downarrow>.$$

There are only two possible results for the total angular momentum when the electron spins are combined, $j=\frac{1}{2}-\frac{1}{2}=0$ or $j=\frac{1}{2}+\frac{1}{2}=1$. If both electrons are spin up, $m_1=m_2=1/2$, the combined spin must have $m=1$ and only $j=1$ is allowed in the combined state:

$$\Big|\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2}>=|1;1>\qquad\Leftrightarrow\qquad|\uparrow>\otimes|\uparrow>=|\uparrow\uparrow>.$$

[1] For historical reasons the $\otimes$ operation is called a *tensor product* — a term that comes originally from the theory of elasticity in solids.

Similarly if both electrons are spin down, $m_1 = m_2 = -1/2$, the combined spin must have $m = -1$ and again only $j = 1$ is allowed in the combined state:

$$\Big|\frac{1}{2},\frac{1}{2};-\frac{1}{2},-\frac{1}{2}>=|1;-1> \qquad \Leftrightarrow \qquad |\downarrow>\otimes|\downarrow>=|\downarrow\downarrow>.$$

Obviously these two states are symmetric under interchange of the two electrons.[2]

If however one electron has $m_1 = 1/2$ and the other $m_2 = -1/2$, or *vice versa*, then the combination has $m = 0$ and this could be either $j = 1$ or $j = 0$: $j = 1$ is the symmetric combination

$$|1;0>=\frac{1}{\sqrt{2}}\big(|\uparrow\downarrow>+|\downarrow\uparrow>\big)$$

and $j = 0$ the orthogonal combination

$$|0;0>=\frac{1}{\sqrt{2}}\big(|\uparrow\downarrow>-|\downarrow\uparrow>\big),$$

which is necessarily anti-symmetric.

These can be inverted to give

$$\begin{aligned} |\uparrow\downarrow> &= \Big|\frac{1}{2},\frac{1}{2};\frac{1}{2},-\frac{1}{2}>=\frac{1}{\sqrt{2}}\big(|1;0>+|0;0>\big|\big),\\ |\downarrow\uparrow> &= \Big|\frac{1}{2},\frac{1}{2};-\frac{1}{2},\frac{1}{2}>=\frac{1}{\sqrt{2}}\big(|1;0>-|0;0>\big), \end{aligned}$$

from which the Clebsch-Gordon coefficients can be read off

$$C^{1;0}_{\frac{1}{2},\frac{1}{2};\frac{1}{2},-\frac{1}{2}}=\frac{1}{\sqrt{2}}, \qquad C^{0;0}_{\frac{1}{2},\frac{1}{2};\frac{1}{2},-\frac{1}{2}}=\frac{1}{\sqrt{2}}, \qquad C^{1;0}_{\frac{1}{2},\frac{1}{2};-\frac{1}{2},\frac{1}{2}}=\frac{1}{\sqrt{2}}, \qquad C^{0;0}_{\frac{1}{2},\frac{1}{2};-\frac{1}{2},\frac{1}{2}}=-\frac{1}{\sqrt{2}}.$$

The total Hilbert space for the electrons' spin is 4-dimensional and one can use either

$$\left\{\Big(\Big|\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2}>\Big),\ \Big(\Big|\frac{1}{2},\frac{1}{2};\frac{1}{2},-\frac{1}{2}>\Big),\ \Big(\Big|\frac{1}{2},\frac{1}{2};-\frac{1}{2},\frac{1}{2}>\Big),\ \Big(\Big|\frac{1}{2},\frac{1}{2};-\frac{1}{2},-\frac{1}{2}>\Big)\right\}$$

or

$$\big\{|1;1>,|1;0>,|1;-1>,|0;0>\big\}$$

as a set of basis vectors. The decomposition into $j = 1$ (triplet) and $j = 0$ (singlet) sectors is

$$\begin{aligned} |1;1> &= |\uparrow\uparrow> \\ |1;0> &= \frac{1}{\sqrt{2}}\big(|\uparrow\downarrow>+|\downarrow\uparrow>\big) \\ |1;-1> &= |\downarrow\downarrow> \end{aligned} \tag{4}$$

[2] The total quantum state corresponding to two Fermions should of course be anti-symmetric under interchange of the two particles. Spin is only part of the story, a complete quantum description should also include position. For example electrons with j=1 could have a relative orbital angular momentum of l=1 making the total quantum state antisymmetric under interchange of the two electrons.

and

$$|0;0>= \frac{1}{\sqrt{2}}(|\uparrow\downarrow> - |\downarrow\uparrow>). \tag{5}$$

It is often useful to exhibit the linear transformation that the Clebsch-Gordon coefficients represent in the form of a table.

		$j=1$			$j=0$
m_1	m_2	$m=1$	$m=0$	$m=-1$	$m=0$
$1/2$	$1/2$	1	0	0	0
$1/2$	$-1/2$	0	$1/\sqrt{2}$	0	$1/\sqrt{2}$
$-1/2$	$1/2$	0	$1/\sqrt{2}$	0	$-1/\sqrt{2}$
$-1/2$	$-1/2$	0	0	1	0