Final Honours Mathematical Physics

Particle Physics (MP466)

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1. Introduction to Forces and Particles

History credits the idea that ordinary matter is not infinitely divisible, but is made up of atoms, to the ancient Greek philosopher Democritus (460-370 BC). But even at the turn of the 20th century atoms were not universally accepted, though the concept was gaining support. Boltzmann (1844-1906) was perhaps one of the first physicists to take atoms really seriously but it was not until a very influential paper by Einstein in 1905, on Brownian motion, that the general consensus swung in favour of atoms. Nowadays the concept is not questioned, we can even see individual atoms in electron microscopes. Atoms are very small — typically about $10^{-10}\,m$ in size.

The modern view is that atoms themselves have substructure, consisting of a small atomic "nucleus" surrounded by a cloud of electrons. A naive, but instructive, way of visualising atoms is to think of them as being like a small solar system, with the nucleus replacing the Sun and the electrons replacing the planets. Consider for example an atom of the monatomic gas Helium, ^{4}He, which consists of a nucleus, made up of two protons and two neutrons, with two electrons buzzing around it so that the total electric charge is zero. The nucleus is tiny, $3.4 \times 10^{-15}\,m$ across, and in fact, just like the solar system, an atom is mostly empty space. For helium (consisting of a nucleus with two electrons orbiting around it) the ratio of the size of the nucleus to the size of the atom ($3.1 \times 10^{-11}\,m$) is the dimensionless number $3.4 \times 10^{-15}/3.1 \times 10^{-11} = 1.1 \times 10^{-4}$. Compare these numbers to the size of the Sun, $7.0 \times 10^{8}\,m$, and the distance between the Sun and Venus (the second planet) which is $1.1 \times 10^{11}\,m$, giving a ratio $7.0 \times 10^{8}/1.1 \times 10^{11} = 6.4 \times 10^{-3}$. There is more empty space in an atom than there is in empty space!

From the above picture you might suspect, and indeed it is true, that atoms themselves are not indivisible, they can be broken up into smaller pieces. But the smaller pieces would not be called "atoms" according to the modern usage, they are the electrons and the nucleus. It takes about $6.5 \times 10^{-21}\,J$ of energy to pull the two electrons completely out of a Helium atom (this is called the *ionisation energy* of Helium). This is such a small number that a Joule is not really a useful unit in which to measure energies at these small scales. Instead we use **electron-Volts**, denoted by eV, where $1\,eV$ is the energy acquired by one electron falling through a potential difference of $1\,Volt$. The conversion factor is $1\,eV = 1.6021765 \times 10^{-19}\,J$. [1] The ionisation energy of Helium is $24.6\,eV$ in these units.

[1] Particle physicists also use multiples of electron-Volts: $1\,keV{=}10^3\,eV$ is one thousand electron-Volts or a **kilo electron-Volt**; $1\,MeV{=}10^6\,eV$ is one million electron-Volts or a **mega electron-Volt**; $1\,GeV{=}10^9\,eV$ is one billion electron-Volts or a **giga electron-Volt**.

But even that is not the end of the story. In 1932 two physicists working in Cambridge, John Cockroft and Ernest Walton, showed that the nucleus itself could be sub-divided. They broke up not a Helium nucleus but a nucleus of the element Lithium, specifically ^{7}Li containing 3 protons and 4 neutrons. They achieved this by bombarding the nucleus with high energy ($0.2 \sim 0.5\, MeV$) protons.[2] Occasionally one of the protons is absorbed by the nucleus which then splits into two nuclei of ^{4}He, releasing 17 MeV of energy which is equally shared between the two emerging ^{4}He-particles. Symbolically this nuclear reaction is written as

$$^7\mathrm{Li} + p \longrightarrow 2\ ^4\mathrm{He},$$

where p denotes the proton.

The production of 17 MeV of energy in this reaction can be understood through Einstein's equation

$$E = mc^2. \tag{1}$$

(Because of this equation particle physicists often quote masses in units of MeV/c^2 rather than in kilogrammes.) Protons, neutrons and ^{4}He nuclei have masses

$$\begin{aligned} m_p &= 1.6726 \times 10^{-27}\, kg = 938.27\, MeV/c^2 \\ m_n &= 1.6749 \times 10^{-27}\, kg = 939.57\, MeV/c^2 \\ m_{\,^4\mathrm{He}} &= 6.6465 \times 10^{-27}\, kg = 3728.41\, MeV/c^2. \end{aligned}$$

The mass of a ^{7}Li nucleus is $11.6501 \times 10^{-27}\ kg = 6535.21\ MeV/c^2$. Add to this the mass of the incoming proton and the result is more than twice the mass of a ^{4}He nucleus

$$m_{\,^7\mathrm{Li}} + m_p - 2m_{\,^4\mathrm{He}} = 6535.21 + 938.27 - 2(3728.41) = 16.7\ MeV/c^2. \tag{2}$$

The Cockroft-Walton experiment can be understood in the following way. A ^{7}Li nucleus consists of 3 protons and 4 neutrons. While an incoming proton with 0.2 MeV of energy does not have enough energy to overcome the Coulomb repulsive barrier classically and enter the nucleus, it can nevertheless get close enough to tunnel through the potential barrier and enter the nucleus by virtue of quantum mechanical tunnelling. Once the proton is inside the Lithium nucleus has 4 protons and 4 neutrons and is unstable, decaying to two ^{4}He nuclei. The mass difference is expressed as energy via Einstein's equation (1). Indeed Cockroft and Walton's experiment was a milestone, not only because they were the first to split the nucleus, but also because it was the first direct experimental verification of Einstein's formula.

It takes a lot of energy to break a Helium nucleus into its constituents (*i.e.* 2 protons and 2 neutrons) $28.3\, MeV$ in fact. A Helium nucleus $^4\mathrm{He}^{++}$ (the plus signs denote the electric charge — a fully ionised Helium nucleus has two units of positive electric charge) is very stable and difficult to break up. For this reason it is a common product in nuclear reactions and is even given its own symbol, α — in nuclear physics a $^4\mathrm{He}^{++}$ nucleus is usually called an α-particle. An ionised Hydrogen nucleus, $^1H^+$, is, of course, a proton.

[2] In 1951 Cockroft and Walton received the Nobel prize for physics for their work. Ernest Walton was born in Dungarvan, County Waterford, and taught in Trinity College, Dublin for many years.

The value of $28.3\,MeV$ is called the *binding energy* of the protons and neutrons in an α particle, and it is related to the masses of its 4 constituent particles again through Einstein's equation. There is an energy difference of

$$E = \Big(m_\alpha - 2(m_p + m_n)\Big)c^2 = 3728.4\,MeV - 2(939.6 + 938.3)\,MeV = -27.4\,MeV.$$

This is very close to the binding energy. It is negative because the α-particle is more stable than a situation with 2 free protons and 2 free neutrons. Binding energies of the order of MeV are so large that they manifest themselves as mass differences through Einstein's equation (1).

But it doesn't end there. There are further indications that protons and neutrons themselves are not fundamental, but are made up of smaller constituents. The first hint of this is the fact that neutrons in free space are unstable and are subject to radioactive decay to a proton and an electron on a time scale of about $15\,mins$. If n denotes a neutron and e an electron, this process is shown by the following formula:[3]

$$n \longrightarrow p + e.$$

This is termed *β-decay* in nuclear physics, since the electrons that are produced were originally called β-rays. If there is a given number of neutrons in a sample at time $t = 0$, N_0 say, then the number that is left at time t will be

$$N(t) = N_0 \mathrm{e}^{-t/\tau_n}$$

where $\tau_n = 886\,s \approx 15\,mins$ is called the *lifetime* of the neutron.[4]

The mass of an electron is $m_e = 9.109 \times 10^{-31}\,kg = 0.511\,MeV/c^2$, so

$$(m_n - (m_p + m_e))c^2 = 939.6\,MeV - (938.3 + 0.5)\,MeV = 0.8\,MeV.$$

[3] Since the mass of a neutron is greater than the mass of a proton and an electron combined this process is allowed by conservation of energy.

[4] There are two important points to note here. Firstly, as we will see later, it is *not* correct to think of a neutron as a bound state of an electron and a proton. A neutron is really made up of three smaller particles called *quarks*. What happens in β-decay is that one of the quarks in a neutron changes its identity and turns into a different kind of quark, thus turning the neutron into a proton, and at the same time emits an electron. Secondly many atomic nuclei, such as ^{4}He for example, contain neutrons and yet are stable. Although neutrons are unstable in free space they are often stable inside atomic nuclei. This is because nuclei contain protons and protons are *fermions*, obeying the Pauli *exclusion principle* in quantum mechanics. The exclusion Principle states that no two fermions can be in the same quantum state at the same time. Now the two protons inside the Helium nucleus occupy quantum states with a definite energy and there is no room left in these states for a third proton. If one of the neutrons tried to β-decay into a proton that proton would have to occupy the next energy level above the two that are occupied and there is not enough energy available to do this — the neutron is destined to remain a neutron. The Pauli exclusion principle ensures that the Helium nucleus is stable, but there is no such restriction on a neutron in free space. For more complicated nuclei the stability under β-decay depends on the specific energy levels for the protons in that nucleus. Some nuclei are stable and others are not.

and this excess energy might be expected to manifest itself as kinetic energy of the daughter particles, the proton and electron. Since the proton is much more massive than the electron conservation of momentum tells us that, in the neutron's rest frame, the proton will be produced almost at rest, with only a small recoil velocity, while the electron will shoot off with an excess energy of nearly $0.8\,MeV$ over and above its rest mass energy, $m_e c^2 = 0.5\,MeV$. If this is the case the total energy of the electron will be

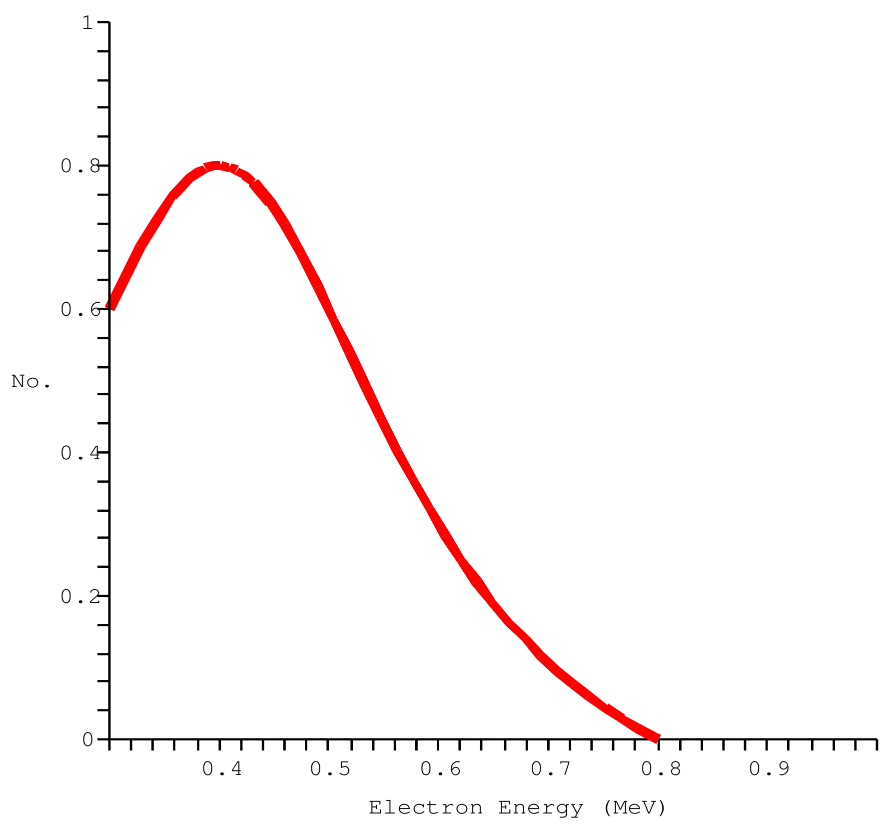
$$E = \gamma(v) m_e c^2 = 0.5\,\gamma(v)\,MeV = (0.5 + 0.8)\,MeV = 1.3\,MeV,$$

where $\gamma(v) = 1/\sqrt{1 - v^2/c^2}$ is the Lorentz γ-factor for the electron. Thus

$$\gamma(v) = 1.3/0.5 = 2.6 \qquad \Rightarrow \qquad v = 0.92\,c,$$

and the electron would be moving off at 92% of the speed of light.

However this is *not* what is observed. If we measure the energies of the electrons emerging from a large number of neutron decays we find a spread of energies well below $0.8\,MeV$ with $0.8\,MeV$ being the maximum energy, but certainly not the only energy. The experiments show that the relative number of electrons with a given energy looks something like this:[5]



When this type of curve was first measured people were rather perplexed as to what

[5] In practice it is difficult to do these measurements for free neutrons, they are more easily done for neutrons bound in unstable nuclei such as Tritium, an isotope of Hydrogen with two neutrons, ^{3}H. This changes the energy scale along the horizontal axis and the cut-off is no longer at $0.8\,MeV$, but the principle is the same.

was happening and a number of interpretations was put forward, including the possibility that energy might not be conserved in the, then newly developed, theory of quantum mechanics. The correct explanation of the continuous β-spectrum was given by Pauli in 1930 when he suggested that there might be missing energy that was not being observed because it was being carried away by a ghostly neutral particle, subsequently christened a *neutrino* and given the symbol ν.[6] The total energy is always $0.8\,MeV$ but we are only seeing that part of it carried by the electron. Actually with modern definitions it is the anti-particle of the neutrino, denoted by $\bar{\nu}$, that is produced in β-decay and the decay of a neutron is written as

$$n \quad \longrightarrow \quad p^+ + e^- + \bar{\nu}.$$

(It was predicted in 1931, by Paul Dirac, that all particles should come with Doppelgängers called *anti-particles* which have exactly the same mass as the particle but opposite electric charge, a prediction that nowadays has ample experimental confirmation.) The electric charge is indicated here as the superscript on p^+ and e^-, in units in which the charge on the proton is $+1$. Since the charge on the neutron is zero, as its name implies, conservation of charge demands that $\bar{\nu}$ also carry zero charge. Neutrinos were first observed directly, rather than being inferred because of missing energy, in 1956 by Reines and Cowan.

As already mentioned, the neutron is not fundamental. Like the proton it is made up of three constituents called "quarks".[7] Protons and neutrons are made up of two different kinds of quark, rather prosaically called an *up* quark and a *down* quark, and denoted by u and d respectively. A u-quark has electric charge $+2/3$ and a d-quark has charge $-1/3$, in units in which the charge on an electron is -1. A proton is made of two u-quarks and one d-quark, giving a configuration uud with a total charge of $+1$, and a neutron is made of one u-quark and two d-quarks, giving a configuration udd with a total charge of 0. To date nobody has ever observed a quark in isolation, the observed charges on free particles are always integer multiples of the charge on a proton, and quarks appear to be permanently locked up inside other particles — a phenomenon termed *confinement*. The reason for this is not fully understood even today.

In the quark model of protons and neutrons, β-decay of a neutron is due to a d-quark in the neutron decaying to a u-quark (d-quarks are heavier than u-quarks, so this process is compatible with conservation of energy since $m_d c^2 > m_u c^2$), thus turning the neutron into a proton. The fundamental reaction is

$$d \quad \longrightarrow \quad u + e^- + \bar{\nu}.$$

There is no evidence to date that quarks themselves are composite particles, all current experiments are compatible with quarks, and electrons, being fundamental particles with no observable substructure.

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[6] The unobserved particle must be electrically neutral because of conservation of charge, the neutron is electrically neutral so the sum of the electric charges of all the decay products must be zero. Because the neutrino is electrically neutral and almost massless it is not easy to detect.

[7] The name comes from the book *Finnegans Wake* by James Joyce. Murray Gell-Mann, who came up with the idea of quarks at the same time as another physicist George Zweig in 1964, happened to be reading *Finnegans Wake* at the time and came across the phrase "Three quarks for muster Mark", which Joyce used in his book supposedly to imitate the cry of a seagull.

In 1932 yet another particle was identified in cosmic ray experiments, a *positron*, e^+, the anti-particle of the electron with the same mass as an electron but with opposite charge, $+1$.

But there was more to come. To see that there must be more, think a little about a ^{4}He nucleus. Two protons and two neutrons bound together in space only $10^{-15}\,m$ across in an extremely stable configuration that takes more than $28\,MeV$ to break it up. Protons have the same electric charge and the repulsive electrostatic energy between two protons a distance r apart is, in a vacuum,

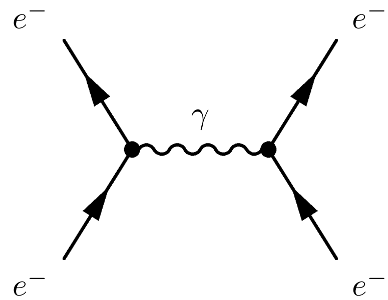
$$V(r) = \frac{e^2}{4\pi\epsilon_0 r}.$$

Using $e = 1.6 \times 10^{-19}\,C$ and $r = 10^{-15}\,m$ this works out as $2.3 \times 10^{-13}\,J = 1.4\,MeV$. So electrostatic repulsion should make ^{4}He unstable, with an excess of $1.4\,MeV$, instead of which it is stable with a deficit of $28\,MeV$, which must be supplied to break it up. The answer to this apparent riddle is that there must be a new force, a nuclear force, that is stronger than electrostatic repulsion in the nucleus and which causes attraction between the constituent particles inside a ^{4}He nucleus, thus holding it together. This force is called the *strong nuclear force.* There are actually two different kinds of nuclear force — the strong force holds stable atomic nuclei together and another force, the *weak nuclear force*, tries to make them decay. It is the weak nuclear force that causes β-decay. The strong force can also be repulsive, like electromagnetism it is sometimes repulsive and sometimes attractive. Heavy atomic atomic nuclei sometimes decay by breaking up and emitting α-particles, a characteristic of the strong force.

In 1935 the Japanese physicist Hideki Yukawa postulated a new particle, associated with the strong nuclear force. He reasoned in analogy with electromagnetism: just as there is a particle associated with the electromagnetic force, the photon, there ought to be a particle associated with the strong nuclear force. However, unlike electromagnetism for which the photon is massless, the particle associated with the strong force should be massive with a mass somewhere between that of an electron and a nucleon.[8] For this reason this new particle was christened a *Yukawa meson*, the modern name for it is the π*-meson*, or pion for short, and it is denoted by the Greek symbol π.

This way of looking at forces, as being mediated by particles, is now well established in particle physics. The electrostatic force, between two electrons say, is pictured as arising from photon exchange like this:

[8] *Nucleon* is a collective name use for protons and neutrons, the particles that constitute atomic nuclei, when it is not important to distinguish between them.



Note however that if an electron emits a photon in isolation there is an apparent inconsistency. If the electron has initial 4-momentum $\underset{\sim}{P} = (E/c, \mathbf{P})$ and final 4-momentum $\underset{\sim}{P}'$ then the photon has 4-momentum $\underset{\sim}{Q} = \underset{\sim}{P} - \underset{\sim}{P}'$ and

$$\begin{aligned} \underset{\sim}{Q}.\underset{\sim}{Q} &= (\underset{\sim}{P} - \underset{\sim}{P}').(\underset{\sim}{P} - \underset{\sim}{P}') = \underset{\sim}{P}.\underset{\sim}{P} + \underset{\sim}{P}'.\underset{\sim}{P}' - 2\underset{\sim}{P}'.\underset{\sim}{P} \\ &= -2m^2c^2 - 2m^2\gamma(v)\gamma(v')\big(-c^2 + \mathbf{v}.\mathbf{v}'\big), \end{aligned}$$

where $E = \gamma(v)mc^2$ and $\mathbf{P} = \gamma(v)m\mathbf{v}$ with $\gamma(v) = \frac{1}{\sqrt{1-(v^2/c^2)}}$ the Lorentz γ-factor for a particle of mass m moving with velocity $\mathbf{v}$.[9] Now $\underset{\sim}{Q}.\underset{\sim}{Q}$ is Lorentz invariant and evaluates to the same thing in any reference frame, so let us use a reference frame in which one of the electrons is initially at rest, with $\mathbf{v} = 0$ and $\gamma = 1$ say, giving

$$\underset{\sim}{Q}.\underset{\sim}{Q} = 2mc^2(\gamma' - 1).$$

If the final electron is a real physical particle it must have $\gamma' \geq 1$, with equality only if it is at rest. If the final state electron is at rest then $\underset{\sim}{Q} = \underset{\sim}{0}$ and nothing has happened (a photon with $\underset{\sim}{Q} = 0$ is no photon at all!) but, if $\underset{\sim}{Q} \neq 0$, then

$$\underset{\sim}{Q}.\underset{\sim}{Q} > 0,$$

so $\underset{\sim}{Q}$ is space-like and this is no ordinary photon!

Nevertheless the photon exchange depicted above can occur as a quantum mechanical process, by virtue of the uncertainty principle. The time component of $\underset{\sim}{Q}$, $Q^0 = \Delta E/c$, is an energy and Heisenberg's uncertainty relation between time and energy,

$$\Delta E \Delta t \approx \hbar,$$

says that a particle with energy ΔE can be created out of nothing provided that it only exists for a time less than

$$\Delta t \approx \hbar/\Delta E.$$

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[9] In this course we use the convention that a particle with mass m, energy E and relativistic 3-momentum $\mathbf{P}$ in an inertial reference frame S has 4-momentum, denoted by $\underset{\sim}{P}$, which decomposes into time-like and space-like parts as $(E/c,\mathbf{P})$ in S, and the square of the 4-momentum is $\underset{\sim}{P}.\underset{\sim}{P}{=}{-}E^2/c^2{+}\mathbf{P}.\mathbf{P}{=}{-}m^2c^2$.

In particle physics such a particle is called a *virtual particle*, because it is not a real physical particle, it only has a fleeting existence by virtue of the uncertainty principle and relies on borrowed energy that must be paid back before a time Δt.

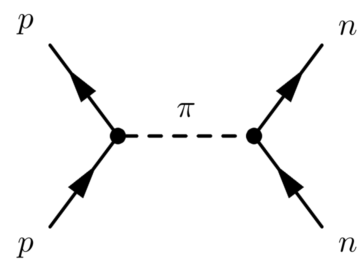
An analogy that may help here is with the banking system. If money is transferred from one bank account to another it is often the case that it does not appear in the second account until a few days after it has disappeared from the first account, even for electronic transactions that take a microsecond. Where is the money in the interim? The answer is that the banks can invest it and make a profit. They don't make a lot on one transaction, which might only make a small amount of money available for a few days, but if they have thousands of customers then at any one time there may be many transactions pending and the total amount of money available to invest could be quite considerable. Ultimately the books have to balance, any individual customer must get his or her money back after a few days, or they will start complaining. Nevertheless the banks have a permanent slush fund available to invest because, if they have a large enough number of customers, then statistically there will always be a significant number of transactions pending. This money is 'virtual', any one transaction only releases money for a very short period of time before it has to re-appear in the second account, but with enough virtual money around the banks have can have significant capital for long term investment.[10]

Although virtual particles are not themselves real physical entities, their presence can nevertheless have real physical effects! The photon exchanged above is a virtual photon and it has a space-like 4-momentum $\underset{\sim}{Q}.\underset{\sim}{Q} > 0$, but this does not violate any of the principles of special relativity since it is never observed as a real physical photon. The relations $\underset{\sim}{P}.\underset{\sim}{P} = -m^2c^2$ for a massive particle and $\underset{\sim}{Q}.\underset{\sim}{Q} = 0$ for a massless particle are classical relations that only hold for particles that live indefinitely, they are not always true for quantum phenomena. Quantum particles that do not satisfy the classical constraints on their momentum are said to be *off mass shell*.

Just as virtual photons mediate the electromagnetic force between charged particles Yukawa's pions mediate the strong force between, for example, a proton and a neutron binding them inside atomic nuclei:[11]

[10] In principle banks can carry on doing this indefinitely, as long as they are sensible about it. But when they start borrowing virtual money from each other, in order to increase their investment, the process become unsustainable, the virtual money runs out and the banks collapse!

[11] The force binding protons and neutrons in atomic nuclei must of course be an attractive force. The diagram here, showing the exchange of a pion, looks as though the proton and neutron are being repelled, but this is just a conventional way of drawing such diagrams. These diagrams are not really physical representations in ordinary space — in the relativistic theory of quantum fields they are actually graphical representations of terms in a mathematical series whose evaluation gives a quantum mechanical amplitude for the processes involved in proton-neutron interactions.



Yukawa managed to guess the mass of the pion by realising that it was very significant that the strong force is short range, it does not appear to have any effect outside of an atomic nucleus, that is beyond a distance of about the size of a nucleus, $\sim 10^{-15}\,m$. This therefore marks an important length scale for the strong force. Contrast this with the electromagnetic force, where the Coulomb potential[12]

$$V(r) = \frac{e^2}{4\pi r} \tag{3}$$

has no intrinsic length scale in it.

Yukawa argued that any particle associated with a short range force, like the strong force, must be massive because of the uncertainty principle. If $\Delta E \approx mc^2$ then in a time Δt a massive virtual particle will travel a distance

$$\Delta l = c\Delta t = \hbar/mc,$$

Using the strong force length scale $\Delta l \approx 10^{-15}\,m$ gives

$$m = \hbar/c\Delta l \approx 1.5 \times 10^{-28}\,kg \approx 100\,MeV/c^2,$$

intermediate between the mass of a proton $m_p = 938\,Mev/c^2$ and an electron $m_e = 0.5\,MeV/c^2$ — this is the expected mass of Yukawa's mesons.

Note that this argument, when applied to electromagnetism, implies that $\Delta l = \infty$, since the photon is massless, $m_\gamma = 0$. The smaller the mass the longer the range and electromagnetism is an infinite range force.[13]

When the pion was eventually observed for the first time, in cosmic rays in 1947 some 12 years after Yukawa's initial proposal, it was found to have mass $m_\pi = 140\,MeV/c^2$, pretty close to Yukawa's original estimate.

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[12] Particle physicists like to use units in which ϵ_0=1, because it tidies up many of the formulae. There is no physical significance to this, it merely reflects our choice of the units in which we decide to quote electric charges. This system of units is called the *Lorentz-Heaviside* system.

[13] The only reason we do not tend to feel the electromagnetic force over extremely large distances is that positive charges tend to cancel negative charges. The other inverse square force, gravity, in which there are no negative charges and therefore no possible cancellations, indeed has a huge range — the force of gravity can be felt across astronomical, and even cosmological, distances.

But Yukawa went even further and proposed a precise mathematical form for the strong force. His argument went as follows. Classically the relativistic 4-momentum for a massive particle, $\underset{\sim}{P} = (E/c, \underline{P})$ where E is the energy and $\underline{P}$ the relativistic 3-momentum in a given inertial reference frame, satisfies

$$\underset{\sim}{P}.\underset{\sim}{P} = -\frac{E^2}{c^2} + \underline{P}.\underline{P} = -m^2c^2. \tag{4}$$

Making the quantum mechanical substitutions

$$E \to i\hbar\frac{\partial}{\partial t} \qquad \text{and} \qquad P_i \to -i\hbar\frac{\partial}{\partial x^i}, \qquad (i = 1, 2, 3)$$

on wave-functions ψ translates this to[14]

$$\frac{\hbar^2}{c^2}\frac{\partial^2\psi}{\partial t^2} - \hbar^2\nabla^2\psi = -m^2c^2\psi.$$

A time independent solution of this equation must satisfy

$$\nabla^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) = \frac{m^2c^2}{\hbar^2}\psi. \tag{5}$$

Notice the appearance of the intrinsic length scale $\frac{\lambda_c}{2\pi} = \hbar/mc$ in this equation — λ_c is called the *Compton wavelength* of the particle with mass m, in analogy with the Compton wavelength of an electron which enters the analysis of Compton scattering of a photon hitting an electron.

A solution of (5) that vanishes as $r \to \infty$ is

$$\psi = -\frac{\mathrm{e}^{-\kappa r}}{r}, \tag{6}$$

where $\kappa = mc/\hbar$ is the Compton wavenumber, defined as 2π times the inverse of the Compton wavelength, $\lambda_c = h/mc$. Dividing λ_c by the speed of light gives a time-scale, $t_c = \frac{\lambda_c}{c} \approx 10^{-23}\, s$ which is characteristic of the strong nuclear force.

Notice the similarity of (6) with the electrostatic potential (3), in which $\kappa = 0$ since the photon is massless. Pursuing this analogy Yukawa suggested the *Yukawa potential*

$$U_{Yukawa}(r) = -g_\pi^2\frac{\mathrm{e}^{-\kappa r}}{4\pi r} \tag{7}$$

as a mathematical description of the strong force.[15] The "charge" g_π in this equation is a strong force charge for the pion, analogous to the electric charge e in (3), called the

[14] This equation is known as the Klein-Gordon equation, though it was first considered by Schrödinger. It is a relativistic version of the usual Schrödinger equation of non-relativistic quantum mechanics.

[15] Of course a quantum mechanical wave-function is not the same thing as a physical potential energy, nevertheless this procedure gives the correct answer because the potential is a Green function for the appropriate differential operator, $-\nabla^2$ for electromagnetism and $-\nabla^2+\kappa^2$ for the strong force, and the Green function satisfies the same differential equation as the wave-function.

pion-nucleon coupling constant. Compared to the fine structure constant (using units with $\epsilon_0 = 1$) $\alpha = \frac{e^2}{4\pi\hbar c} \approx 1/137$, $\frac{g_\pi^2}{4\pi\hbar c} \approx 1.14$. Yukawa's strong force is, as its name implies, about $1.14 \times 137 \approx 160$ times stronger than the electromagnetic force.

As mentioned above, the pion was first observed in 1947 but before this confusion reigned because *another* new particle was found in cosmic rays in 1937, with a mass very similar to Yukawa's estimate of the pion mass. Originally these particles were also called mesons, because they were intermediate in mass between electrons and protons, but it was immediately clear that they could not be Yukawa's mesons because they travelled far too easily through matter, they can easily penetrate many metres of rock, while Yukawa's mesons were expected to interact so strongly with matter, via the strong force, that they should be stopped very quickly.[16] These new particles were given the Greek symbol μ and called μ-mesons, or muons for short (nowadays they would not be classified as mesons, but are still called muons). Muons have electric charge -1 and a mass $m_\mu = 106\, MeV/c^2$, about 200 times the mass of an electron. Indeed a muon looks exactly like a heavy clone of an electron — to this day it is not really understood why they should exist, though they are probably crucial for consistency of any final mathematical model of particle physics. Muons also have their own version of neutrinos to go with them — there are at least two different types of neutrinos: the *muon neutrino*, ν_μ, and the *electron neutrino*, ν_e, together with their respective anti-neutrinos, $\bar{\nu}_\mu$ and $\bar{\nu}_e$.

Both the muon and the pion are unstable under radioactive decay (due to the weak nuclear force in this instance), the muon has a lifetime of about a microsecond and a charged pion has a lifetime about 100 times less than this:

$$\begin{array}{lllll} \mu^- & \longrightarrow & e^- + \text{neutrinos}, & & \tau_\mu = 2.197 \times 10^{-6}\, s \\ \pi^\pm & \longrightarrow & \mu^\pm + \text{neutrinos}, & & \tau_\pi = 2.603 \times 10^{-8}\, s \end{array}$$

(the lifetime of the pion $10^{-8}\, s$ should not be confused with the time-scale of the strong force $10^{-23}\, s$, $\pi^\pm$ decay is due to the weak nuclear force, not the strong nuclear force). Pions carry electric charge of ± 1, in fact the π^+ is the anti-particle of the π^-, there is also a neutral pion π^0 which is its own anti-particle; the μ^+ is the anti-particle of the μ^-, like a heavier clone of the positron. From a modern perspective the pion is not really fundamental, like the proton and the neutron it is made up of quarks. Unlike the proton and the neutron it is made up of a quark and an anti-quark: the π^+ consists of a u-quark and an anti-d-quark (denoted $\bar{d}$), giving a total charge of $+2/3 + 1/3 = 1$ (the $\bar{d}$ has electric charge $+1/3$, opposite in sign to that of the d); the π^- is made up of an anti-u-quark (denoted $\bar{u}$) and a d-quark, giving a total charge of $-2/3 - 1/3 = -1$.

The number of particles is beginning to get rather large so we need a classification scheme. All matter is divided into two basic categories: particles which "feel" the strong force are called *hadrons* and particles that do not feel the strong force are called *leptons*.

[16] The fact that a particle with a mass intermediate between an electron and a proton can travel many meters through rock does not contradict Yukawa's reasoning explained earlier, which only applied to virtual particles. A real physical particle does not have to worry about time constraints on its existence imposed by the uncertainty principle, though it may have other worries — it may not live for long if it decays radioactively.

Hadrons are not really fundamental, they are made up of quarks, while leptons are believed to be fundamental. Hadrons are further subdivided into *baryons* which are made up of three quarks, and *mesons* which are made up of a quark and an anti-quark.[17] The classification looks like this:

$$\text{Matter} \quad \begin{cases} \text{Hadrons} & \begin{cases} \text{Mesons } (q\bar{q}) & (\pi^{\pm}, \pi^0, \ldots) \\ \text{Baryons } (qqq) & (n, p, \ldots) \end{cases} \\ \\ \text{Leptons} & (e^-, \mu^-, \nu_e, \nu_\mu, \ldots) \end{cases}$$

The strong nuclear force binds protons and neutrons inside atomic nuclei by exchanging virtual pions between them — the pion can be said to mediate the strong force between p and n. But the strong force also binds quarks inside protons, neutrons and pions. The situation here is again somewhat analogous to electromagnetism. Electrons are bound inside atoms, and atoms are bound together to form molecules, by the electromagnetic force, which is mediated by particles called photons. There is a much weaker force, the van der Waals force, between molecules which holds them together to make solids and liquids — the van der Waals force is an external residue of the internal electromagnetic forces in atoms and molecules, which are electrically neutral overall. In a somewhat analogous way the strong force between protons and neutrons, mediated by pions, is an external residue of the strong force that holds quarks inside hadrons. The particles that bind quarks in hadrons, the glue that holds hadrons together, are called *gluons*. It is gluons that are the strong force version of photons and not pions — pions, it turns out, are not fundamental, contrary to Yukawa's original thinking. Like the photon, a gluon is massless. The fact that the force mediated by massless gluons is not infinite in range is a poorly understood consequence of the property of confinement mentioned earlier, gluons are somehow confined inside protons and neutrons, just like quarks are.

We shall see later that there are also particles that mediate the weak force, known as W and Z-bosons (physicists were running out of letters by the time these particles were postulated). Unlike photons and gluons, the W and Z-bosons are massive. Very massive in fact: the W-boson was first observed directly in 1984 and weighs in at $80.4\,GeV/c^2$ over eighty times the mass of a proton. W-bosons carry electric charge $W^{\pm}$ while the Z-boson is electrically neutral, Z^0.

There is also a hypothetical particle that would be associated with quantum effects of the gravitational field, called the *graviton* g, but this is so far from any conceivable experimental detection that we only mention it here to complete the picture.[18]

[17] Originally the distinction between hadrons and leptons was one of mass: *hadron* comes from the Greek $\theta\alpha\delta\rho o\sigma$ meaning "heavy" while *lepton* comes from the Greek $\lambda\epsilon\pi\tau o$ meaning "light". Once the particles became better understood it was realised that hadrons are made up of quarks (and anti-quarks) while leptons are independent of quarks and do not appear to have any substructure. Originally hadrons were also subdivided in to the heavier hadrons, the baryons, *baryon* comes from $\beta\alpha\rho\nu\sigma$ (another Greek word for heavy), and the intermediate mass mesons, *meson* comes from the Greek $\mu\epsilon\sigma o$ meaning middle. It was only later, with the advent of the quark model, that baryons were classified as having three quarks while mesons have a quark and an anti-quark. Nowadays mesons are routinely produced which are much heavier than the lightest baryons!

[18] Gravitational waves (the gravitational analogue of radio waves) were predicted by Einstein in 1915 and were

So, as well as the matter particles above, we also have force particles:

$$\text{Forces}\begin{cases} \left.\begin{array}{c}\text{Electricity}\\ \text{Magnetism}\end{array}\right\}\text{Electromagnetism: }\gamma & \begin{array}{l}\text{(holds molecules together,}\\ \text{ binds electrons in atoms)}\end{array}\\ \text{Weak Nuclear Force: } W^{\pm}, Z^0 & \text{(causes } \beta\text{-decay, } \pi \text{ decay, } \mu \text{ decay)}\\ \text{Strong Nuclear Force: } G & \begin{array}{l}\text{(holds } p \text{ and } n \text{ in nuclei,}\\ \text{ binds quarks in nucleons)}\end{array}\\ \text{Gravity: } g\ (?) & \end{cases}$$

Particles can be further categorised by physical properties other than their mass, such as electric charge and intrinsic angular momentum (usually called *spin*). These extra numbers are called *quantum numbers* of the particles because they take discrete values. Particles whose intrinsic spin, S, is an integral multiple of $\hbar$, $S = s\hbar$ with s a non-negative integer, are called *bosons*: examples are $\pi^{\pm}$, π^0 ($s = 0$) and γ, G, $W^{\pm}$, Z^0 ($s = 1$). Particles whose intrinsic spin is a half-integral multiple of $\hbar$ are called *fermions*: examples are $e^{\pm}$, $\mu^{\pm}$, ν_μ, ν_e, p, n ($s = 1/2$). Fermions must obey the Pauli exclusion principle but bosons do not — as many bosons as you like can fit into the same quantum state.

This chapter closes with a summary of all the known fundamental particles to date, including some new ones that have not yet been mentioned: there are no less than *three* copies, or families, of leptons. The muon, the mysterious heavy electron, was just the first hint of things to come. There is yet a third copy of the electron, known as the tau-lepton, τ^-, and it also has its own neutrino, ν_τ, as a partner. In addition there are also two copies of the (u, d) pair of quarks, giving three families of quarks too. The first copy are poetically called *charm* and *strange* quarks, (c, s), and the second copy are given the rather less interesting names of *top* and *bottom*, (t, b).[19] So there are three families of quarks and three of leptons, which we can group like this:

$$\begin{array}{ccc} \begin{pmatrix} u \\ d \end{pmatrix} & \begin{pmatrix} c \\ s \end{pmatrix} & \begin{pmatrix} t \\ b \end{pmatrix} \\ \begin{pmatrix} \nu_e \\ e^- \end{pmatrix} & \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix} & \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix} \end{array}.$$

Physical properties of all of these particles are collected together in the following table (protons, neutrons and Yukawa's π-mesons are not included here because they are not considered to be truly fundamental — they are made up of quarks — and the electric charge Q is quoted in units in which the charge on an electron is $Q_e = -1$):

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detected for the first time 100 years later, at the end of 2015, after a 50 year search. A short pulse of waves was detected in laser interferometers in the US with arms 4 km long. The pulse came from two black holes, each with a mass some 30 time the mass of the Sun, losing energy through gravitational radiation and spiraling in towards each other to merge into a single black hole in a cataclysmic event which, for a brief period lasting only 0.02 seconds, radiated more power than all the stars in the Universe combined! To detect a graviton quantum of a gravitational wave, is simply inconceivable with current technology.

[19] The rather more poetic names *truth* and *beauty*, which were in fashion for a time, do not seem to have stuck.

Fundamental Particles

Matter: $s = 1/2$

LEPTONS			
$Q=-1$		$Q=0$	
e^-	$0.511\, MeV/c^2$	ν_e	$< 1\, eV/c^2$
μ^-	$106\, MeV/c^2$	ν_μ	$< 0.19\, MeV/c^2$
τ^-	$1777\, MeV/c^2$	ν_τ	$< 18.2\, MeV/c^2$

QUARKS			
$Q=2/3$		$Q=-1/3$	
u (up)	$2.3\, MeV/c^2$*	d (down)	$4.8\, MeV/c^2$
c (charm)	$1.3\, GeV/c^2$	s (strange)	$95\, MeV/c^2$
t (top)	$174\, GeV/c^2$	b (bottom)	$4.2\, GeV/c^2$

*Since quarks have never been seen in isolation it is not possible to measure their mass directly. The values quoted here are obtained indirectly and, especially for the lighter quarks, are rather uncertain. Note that the proton and neutron weigh considerably more than three times the mass of the u or d quark: most of the mass of the proton and the neutron is due to the energy of the strong force and $m=E/c^2$.

Force particles: (s=1)

Photon γ		$< 2 \times 10^{-16}\, eV/c^2$	$Q=0$
Gluons G		Massless (?)	$Q=0$
Weak bosons	$W^\pm$	$80.4\, GeV/c^2$	$Q=\pm 1$
	Z^0	$91.2\, GeV/c^2$	$Q=0$

In addition to all of the particles above there is one more boson to add to the list, called the *Higgs boson*, which is expected to have spin $s = 0$ and charge $Q = 0$. It is essential for the internal consistency of our current understanding of particle physics and a particle that appears to have some of the expected properties of the Higgs boson was found only this year, some 48 years after its was first predicted in 1964. In 2012 the Large Hadron Collider (LHC) at CERN (the European particle physics facility near Geneva), running at an energy of 8 TeV discovered the Higgs boson with a mass of 126 GeV/c^2. As a consequence the 2013 Nobel prize for physics was awarded to two of the people who did the fundamental theoretical work that predicted its existence: Peter Higgs and Francois Englert.

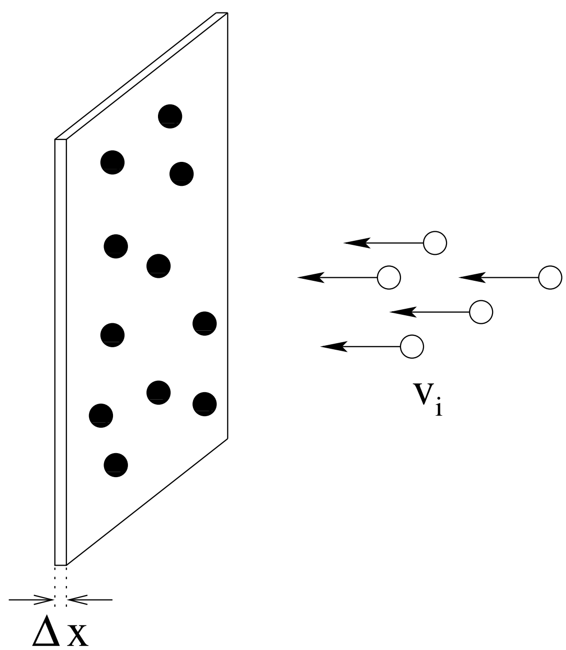
2. Basic concepts

i) **Cross-section:** cross-sections are the basic quantities that are extracted from measurements in real experiments. Elementary particles are so small that we cannot pick them up and manipulate them individually. Instead beams of particles are fired at targets made up of other types of particles and we measure the speed, direction and other physical properties, such as electric charge, of what comes out. In this way we can get information about the nature of the interactions that occur during the collisions.

The cross-section of a target is the area presented to incoming projectiles (such as particles fired at the target). Consider an incoming beam of particles, with n_i particles per unit volume each moving with speed v_i, impinging on a target of stationary particles where each of the target particles presents cross-sectional area σ to the beam. If there is a total of N_t particles in the target and we take the geometry of the target to be a slab of area A and thickness Δx then the probability of a beam particle hitting a target particle is

$$P = \frac{N_t \sigma}{A}$$

(this assumes that no target particle is hidden behind any other, which requires $N_t << A/\sigma$: this can always be arranged by making the target very thin, *i.e.* taking Δx small enough).



The incident flux Φ_i is defined as being the number of particles in the incident beam passing unit area perpendicular to the beam each second,

$$\Phi_i = n_i v_i.$$

We expect the number of 'hits' per second to be

$$\Phi_i A P = n_i v_i N_t \sigma := \mathcal{L}\sigma$$

where

$$\mathcal{L} := n_i v_i N_t$$

is called the *luminosity* of the experiment, it has dimensions of inverse area per unit time ($m^{-2}\,s^{-1}$). The number of hits per second is called the reaction rate and we denote it by W so

$$W = N_t n_i v_i \sigma = \mathcal{L}\sigma. \tag{8}$$

The luminosity is a kinematic quantity, it contains no information about the underlying dynamics of the particle collisions, it is determined purely by geometry and the speed of the incoming particles which to a large extent are under the control of the experimentalists. A large luminosity means a very tightly focused beam, producing a greater chance of collisions. Integrating $\mathcal{L}$ over time gives the *integrated luminosity*, $\int \mathcal{L} dt$, which has dimensions of inverse area, which is an important parameter for running a particle accelerator. A more tightly focused beam, with a smaller cross-sectional area, leads to a larger luminosity. For example, the integrated luminosity of the LHC is measured in inverse femtobarns, with $1\ femtobarn = 10^{-15}\ barns = 10^{-43}\ m^2$, a beam of light focused into such a tiny area would have a large luminosity, and the terminology is taken over from light (photons) to any other kind of particle beam.[20]

The reaction rate W factorises into the product of $\mathcal{L}$ and σ and the dynamics of the underlying collisions is contained in σ. Experimentally it is difficult to measure σ directly, it is W that is measured. Assuming that N_t, n_i and v_i are known it is usual to *define* the cross-section using equation (8) as

$$\sigma := \frac{W}{\mathcal{L}}. \tag{9}$$

Thus the total cross-section per target particle is the reaction rate divided by the luminosity. The cross-section σ has dimensions of area and in particle physics typical cross-sections are so small that it is conventional to measure them in units called *barns* where $1\ barn = 10^{-28}\ m^2$. Cross-sections can vary strongly with energy, but typical centre of mass cross-sections, at around $1\ GeV$, are: $\sigma \approx 10^{-2}\ b$ for strong interactions; $\sigma \approx 10^{-8}\ b = 10\ nb$ for electromagnetic interactions and $\sigma \approx 10^{-14}\ b = 10^{-2}\ pb$ for weak interactions where b denotes barn, nb denotes *nanobarns* ($1\ nb = 10^{-9}\ b$) and pb denotes *picobarns* ($1\ pb = 10^{-12}\ b$).[21]

The incoming particles scatter off the target particles at various angles and we can learn a lot about the microscopic forces controlling the collisions by measuring these

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[20] A more familiar quantity with dimensions of inverse area is the fuel consumption of a car. A consumption of $10\ km/litre{=}10^7\ m^{-2}$ and an inverse femtobarn translates to $10^{37}\ km/litre$!

[21] The LHC at CERN in Geneva is the largest particle accelerator ever built. It collides protons on protons and its first run, from 2010 to 2013, was at energy $7{\sim}8\ TeV$. The cross-section for proton-proton collisions was measured to be 100 millibarns or $\frac{1}{10}\ b$ and the integrated luminosity over 3 years was 30 inverse femptobarns or $3{\times}10^{16}$ inverse barns. The total number of collisions in the first run was therefore about $\sigma \int^t \mathcal{L} dt{=}3{\times}10^{15}$, or about 30 million collisions per second over 3 years. Of course the machine was not running all the time, there was some down-time, so the collision rate during the time the machine was running is somewhat larger than this, at least 10^8 collisions per second!

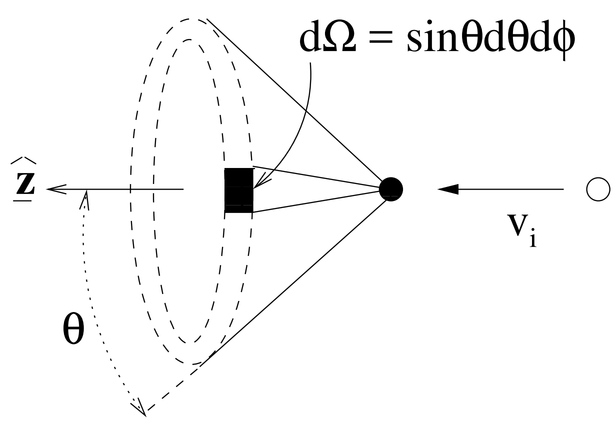
angles. We denote the reaction rate for particles scattering into the solid angle range $d\Omega = \sin\theta d\theta d\phi$ by

$$dW(\theta, \phi) = \frac{dW(\theta, \phi)}{d\Omega} d\Omega = \mathcal{L} \frac{d\sigma(\theta, \phi)}{d\Omega} d\Omega.$$

$\frac{d\sigma(\theta,\phi)}{d\Omega}$ is called the *differential cross-section*. The total cross-section is the integral of the differential cross-section over all directions,

$$\sigma = \int_0^{\pi} \int_0^{2\pi} \frac{d\sigma}{d\Omega} \sin\theta d\theta d\phi$$

(polar co-ordinates are defined here by taking the target to be at the origin and the z-axis parallel to the incoming beam).



ii) **Scattering amplitude:** this is a quantum mechanical amplitude from which collision probabilities can be calculated. Consider a single beam particle a colliding with a single target particle b, the whole process taking place in a cube of size L and volume $V = L^3$. For a single particle in the beam $n_i = 1/V$ and the incident flux and the luminosity are the same, since $N_t = 1$,

$$\Phi_i = n_i v_i = \frac{v_i}{V} = \mathcal{L}.$$

The differential reaction rate for incoming particles to scatter into the outgoing solid angle $d\Omega$ is

$$dW(\theta, \phi) = \mathcal{L} \frac{d\sigma(\theta, \phi)}{d\Omega} d\Omega = \frac{v_i}{V} \frac{d\sigma(\theta, \phi)}{d\Omega} d\Omega. \quad (10)$$

Understanding the details of scattering process requires using the theory of quantum mechanics. Suppose first of all that b is so massive that it is not affected by the collision and stays in the same place. Let the incoming particle a be described by a quantum mechanical wave-function $\psi_i(r)$ which is a plane wave and the collision process results in a final state with wave-function $\psi_f(r)$, then the quantum amplitude M_{fi} for the scattering process can be calculated from the potential energy, U, for the forces involved, using perturbation theory. At lowest order in perturbation theory the amplitude is

$$M_{fi} = \int \psi_f^*(r) U(r) \psi_i(r) d^3r.$$

Since ψ_i and ψ_f are dimensionless we chose to normalise them so that $\int \psi_i^* \psi_i d^3r = \int \psi_f^* \psi_f d^3r = V$. In the usual principles of quantum mechanics a quantum mechanical amplitude M_{fi} is a complex number from which the probability of a final state f emerging from an initial state i is $|M_{fi}|^2$ (an example of the calculation of M_{fi} for a specific process will be given later). The transition rate produced by an interaction amplitude $|M_{fi}|^2$ is affected by the number of quantum mechanical states available to the outgoing particles — the more quantum states that are available the more likely the process is to occur — and the number of final states can depend on the total final state energy E and also on the direction of the outgoing particle. We shall denote the number of final states available to the outgoing particle by $N_f(E, \theta, \phi)$ (in classical physics it would be infinite, but in quantum mechanics it is finite). The number of final states available in the energy range dE and solid angle $d\Omega$ is

$$d^2N_f = \frac{d^2N_f}{dEd\Omega} d\Omega dE = \rho(E) d\Omega dE,$$

where $\rho(E) = \frac{d^2N_f}{dEd\Omega}$ is called the **density of states** and has dimensions of $(energy)^{-1}(radians)^{-2}$.

The transition rate is proportional to

$$dW \propto \frac{|M_{fi}|^2}{V^2} \rho(E) d\Omega, \tag{11}$$

where the $1/V^2$ is introduced to cancel the normalisation $\int \psi_i^* \psi_i d^3r = \int \psi_f^* \psi_f d^3r = V$. The left-hand side of (11) is a rate, a probability per unit time, and so it has dimensions of $(time)^{-1}$; while the right-hand side has dimensions of $(energy)^{+1}$, since M_{fi} has dimensions $(energy) \times (volume)$ and $\rho(E)$ dimensions of $(energy)^{-1}$. The dimensions are balanced by a factor of $1/\hbar$ in the proportionality factor on the right-hand side,

$$dW \propto \frac{1}{\hbar} \frac{|M_{fi}|^2}{V^2} \rho(E) d\Omega,$$

where the proportionality factor is now a pure number. A full quantum mechanical treatment produces a proportionality factor of 2π, giving

$$dW = \frac{2\pi}{\hbar} \frac{|M_{fi}|^2}{V^2} \rho(E) d\Omega. \tag{12}$$

This is a central result from time-dependent perturbation theory in quantum mechanics: the probability of a quantum mechanical transition between two energy eigenstates, when the Hamiltonian is perturbed by a potential U, is independent of time to first order in U and is given by (12), which is called *Fermi's Golden Rule*.

To relate the cross-section $\sigma(E, \theta, \phi)$ to M_{fi} we need to know the density of states

$$\rho(E) = \frac{d^2N_f}{dEd\Omega}$$

where $N_f(E, \theta, \phi)$ is the number of final states available to the outgoing particle with energy E in the (θ, ϕ) direction, it is purely kinematical, independent of the dynamics of the forces governing the collision. Then $\rho(E)$ can be calculated as follows. The possible momentum states for one of the final state particles in the volume V are

$$p_{f,x} = \frac{2\pi}{L}\hbar n_x, \qquad p_{f,y} = \frac{2\pi}{L}\hbar n_y, \qquad p_{f,z} = \frac{2\pi}{L}\hbar n_z,$$

where n_x, n_y and n_z are integers. A small cube in momentum space with volume

$$d^3p_f = dp_{f,x}dp_{f,y}dp_{f,z} = p_f^2 dp_f d\Omega$$

therefore contains

$$\left(\frac{L}{2\pi\hbar}\right)^3 p_f^2 dp_f d\Omega = \frac{V}{(2\pi\hbar)^3} p_f^2 dp_f d\Omega = d^2N_f$$

allowed quantum states, so

$$\frac{dN_f}{dp_f} = \frac{V}{(2\pi\hbar)^3} p_f^2 d\Omega$$

is the number of final states available with momentum between p_f and $p_f + dp_f$ in the direction defined by the solid angle cone $d\Omega$. For non-relativistic energies $E = \frac{1}{2}m_a v_f^2 = \frac{1}{2}\frac{p_f^2}{m_a}$, so $\frac{dE}{dp_f} = \frac{p_f}{m_a} = v_f$. So

$$\frac{dN_f}{dp_f} = \frac{dN_f}{dE}\frac{dE}{dp_f} = \frac{1}{v_f}\frac{dN_f}{dE}$$

and

$$\rho(E) = \frac{d^2N_f}{dEd\Omega} = \frac{V}{(2\pi\hbar)^3}\frac{p_f^2}{v_f}. \tag{13}$$

Now combining (10) and (12) gives the differential cross-section

$$\frac{d\sigma}{d\Omega} = \frac{V}{v_i}\frac{dW}{d\Omega} = \frac{V}{v_i}\frac{2\pi}{\hbar}\frac{|M_{fi}|^2}{V^2}\rho$$

and substituting (13) the volumes cancel leaving

$$\frac{d\sigma}{d\Omega} = \frac{1}{4\pi^2\hbar^4}\frac{p_f^2}{v_f v_i}|M_{fi}|^2. \tag{14}$$

The same formula works even if we relax the assumption that the target particle is infinitely massive, provided we work in the centre of mass frame and interpret the symbols correctly. We can even allow the particles to change their nature and two particles, say c and d, emerge, which are not the same as the original particles a and b. In this more general situation the final energy is

$$E = \frac{1}{2m_c}p_f^2 + \frac{1}{2m_d}p_f^2 = \frac{m_c + m_d}{2m_c m_d}p_f^2$$

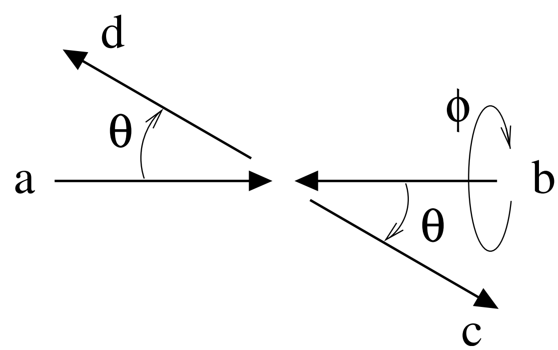
from which

$$\frac{dE}{dp_f} = \frac{m_c + m_d}{m_c m_d} p_f = v_d + v_c := v_f$$

where v_c and v_d are the final speeds of particles c and d and v_f is their relative velocity. Inverting this

$$\frac{dp_f}{dE} = \frac{1}{v_f}, \tag{15}$$

the same formula as before. Also in the centre of mass frame we must interpret v_i as the relative incoming velocity: $v_i = v_a + v_b$. The formula (14) for the cross-section for the process $a + b \to c + d$ is then valid in the centre of mass frame, with p_f the magnitude of the momentum of either of the final state particles c or d (they are the same in the centre of mass frame). The angles θ and ϕ are measured relative the direction of the incoming particles,

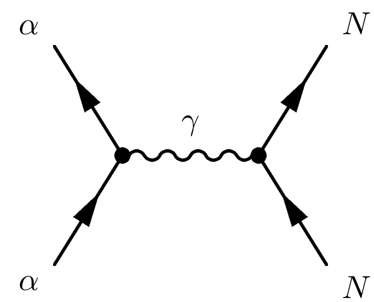


Although this formula has been calculated using non-relativistic kinematics it is also true, in the centre of mass frame, for collisions at relativistic velocities — although there are subtleties with wave function normalisation due to Lorentz contraction reducing the volume. This will not be proven here but it will be assumed when necessary.[22]

Equation (14) is a very important formula in particle physics. What experiments measure is cross-sections (actually reaction rates, from which cross-sections are inferred using (9)), often differential cross-sections. The factor $\frac{p_f^2}{v_f v_i}$ is purely kinematical and gives no information about the underlying physics of the scattering process. The real meat of the problem lies in the quantum mechanical amplitude M_{fi}. We can obtain experimental information about M_{fi} by measuring cross-sections and using equation (14). If we have an underlying theory which specifies the dynamics of the interaction we can attempt to calculate the transition amplitudes M_{fi} and compare our theoretical predictions with the experimentally measured data to see if our theory correctly reproduces the experimentally numbers.

As an example of the application of the formula (14) consider an α-particle, with charge $2e$, scattering off a massive positively charged particle N such as a nucleus with charge Ze, $\alpha + N \to \alpha + N$, by exchanging a photon

[22] As an exercise you can check equation (15) is also true using the fully relativistic expression $E = \sqrt{p_f^2 + m_c^2 c^4} + \sqrt{p_f^2 + m_d^2 c^4}$ for the final state energy.



Suppose the α-particle is non-relativistic and has incoming momentum $\mathbf{q}_{in} = \hbar\mathbf{k}_{in}$ and outgoing momentum $\mathbf{q}_{out} = \hbar\mathbf{k}_{out}$, and incoming energy $E_{in} = q_{in}^2/2m_\alpha$ and outgoing energy $E_{out} = q_{out}^2/2m_\alpha$. For simplicity further suppose that N is so massive that its recoil velocity under the impact can be ignored and is essentially stationary in the centre of mass frame. Conservation of energy then imposes $E_{out} = E_{in} \Rightarrow q_{out} = q_{in}$. In formula (14) we therefore have $v_i = v_f = q_{in}/m_\alpha$, so $v_f v_i = q_{in}^2/m_\alpha^2 = 2E_{in}/m_\alpha$ and $p_f^2 = q_{out}^2 = 2m_\alpha E_{in}$, which imply $\frac{p_f^2}{v_i v_f} = m_\alpha^2$, giving

$$\frac{d\sigma}{d\Omega} = \frac{m_\alpha^2}{4\pi^2\hbar^4}|M_{fi}|^2.$$

It remains to calculate the quantum mechanical amplitude M_{fi} for this process. The amplitude is

$$M_{fi} = \int d^3x\, \psi_f^*(\mathbf{x})U(\mathbf{x})\psi_i(\mathbf{x})$$

where $U(r) = \frac{2Ze^2}{4\pi\epsilon_0 r}$ is the Coulomb potential between the α-particle, with charge $2e$, and N with charge Ze, which is taken to sit at the origin. Represent the incoming and outgoing α-particle wave-functions by plane waves (the massive particle N is essentially being treated as classical here)

$$\psi_i(\mathbf{x}) = e^{-i\mathbf{k}_{in}.\mathbf{x}} \qquad \text{and} \qquad \psi_f(\mathbf{x}) = e^{-i\mathbf{k}_{out}.\mathbf{x}}.$$

Then it is convenient to use Fourier transforms, where the Fourier transform of a function $f(x)$ is defined to be

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx f(x)e^{ikx}$$

with the inverse transform

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\tilde{f}(k)e^{-ikx}$$

(the inverse transform can be calculated using the integral representation of the Dirac δ-function derived in the mathematical methods course, $\delta(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}dx$). Applying this to $U(\mathbf{x})$, which is a function of three Cartesian co-ordinates, gives[23]

$$\widetilde{U}(k) = \int d^3x\; U(\mathbf{x}) e^{i\mathbf{k}.\mathbf{x}} = \frac{2Ze^2}{\epsilon_0 k^2}$$

(the calculation of $\widetilde{U}(k)$ is left as an exercise[24]) and

$$U(r) = \frac{1}{(2\pi)^3} \int d^3k\, \widetilde{U}(k) e^{-i\mathbf{k}.\mathbf{x}}.$$

So

$$\begin{aligned} M_{fi} &= \int d^3x\, \psi_f^*(\mathbf{x}) U(\mathbf{x}) \psi_i(\mathbf{x}) = \frac{1}{(2\pi)^3} \int d^3x \int d^3k\, e^{i\mathbf{k}_{out}.\mathbf{x}} \widetilde{U}(k) e^{-i\mathbf{k}.\mathbf{x}} e^{-i\mathbf{k}_{in}.\mathbf{x}} \\ &= \int d^3k\, \widetilde{U}(k) \delta^{(3)}(\mathbf{k}_{out} - \mathbf{k}_{in} - \mathbf{k}) = \widetilde{U}(\mathbf{k}_{out} - \mathbf{k}_{in}) = \frac{2Ze^2}{\epsilon_0 |\mathbf{k}_{out} - \mathbf{k}_{in}|^2}. \end{aligned}$$

Defining

$$\mathbf{q} = \hbar(\mathbf{k}_{in} - \mathbf{k}_{out})$$

to be the momentum carried by the photon, we have

$$M_{fi} = \frac{2Ze^2\hbar^2}{\epsilon_0 q^2} \tag{16}$$

with

$$q^2 = q_{in}^2 + q_{out}^2 - 2\mathbf{q}_{out}.\mathbf{q}_{in} = 2q_{in}^2(1-\cos\theta) = 4m_\alpha E_{in}(1-\cos\theta) = 8m_\alpha E_{in}\sin^2(\theta/2)$$

where θ is the angle between $\mathbf{k}_{in}$ and $\mathbf{k}_{out}$, $\mathbf{k}_{in}.\mathbf{k}_{out} = k_{in}k_{out}\cos\theta$, that is the angle through which the α-particle is scattered.

The cross-section is finally

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \frac{Z^2 e^4}{\epsilon_0^2 E_{in}^2 \sin^4(\theta/2)}.$$

Note that $\hbar$ has disappeared from this equation — the result is purely classical. In fact this is the Rutherford scattering formula, describing how α-particles scatter off atomic nuclei.

[23] There is a subtlety with the volume here. Because the photon is massless the electromagnetic force has infinite range and we must take $V \to \infty$ to calculate the Fourier transform, so one might worry about the normalisation $\int |\psi(x)|^2 d^3x = V$. Actually V cancels out in the cross-section (14) so we can keep it finite during the calculation of the cross-section and let $V \to \infty$ at the end. Alternatively use a finite volume with $\psi_i(\mathbf{x}) = \frac{1}{\sqrt{V}} e^{-i\mathbf{k}_{in}.\mathbf{x}}$ and $\psi_f(\mathbf{x}) = \frac{1}{\sqrt{V}} e^{-i\mathbf{k}_{out}.\mathbf{x}}$ normalised to unity, then take $M_{fi} = V \int d^3x\, \psi_f^*(\mathbf{x}) U(\mathbf{x}) \psi_i(\mathbf{x})$ (M_{fi} has dimensions of *Energy* $\times$ *Volume*) and then let $V \to \infty$. The final answer is the same.

[24] Hint: first calculate the Fourier transform of the Yukawa potential (7), using spherical polar co-ordinates, and then let $\kappa \to 0$.

iii) **Spin and helicity**: many fundamental particles have an intrinsic spin associated with them. In a classical world view it looks as though fundamental particles are spinning, like little balls. Although a truly point particle cannot be spinning in the usual sense nevertheless quantum mechanically particles can have intrinsic spin $\mathbf{s}$, or angular momentum $\mathbf{s}\hbar$. If a specific direction is chosen, the z-direction say, and the spin of massive particle with intrinsic spin $\mathbf{s}$ is measured in this direction the result is quantised: a measurement of the z-component of the spin can be any one of $2s+1$ possible values: $-s, -s+1, \ldots, s-1, 1$. An electron, for example, has intrinsic spin $1/2$ and measuring the component of the spin in any direction can only give $+1/2$ or $-1/2$.

The intrinsic spin of a particle is an example of a quantum number that is used to classify particles — all fundamental particles of the same type have the same intrinsic spin. A particle, such as the electron, with half-integral spin are called *fermions* while particles with integral spin are called bosons, an example of a boson is a photon which has spin 1.

There is a subtlety for massless particles, such as the photon, though: some of the spin states are missing! A massive spin 1 particle has three possible spin states, $-1, 0$ and $+1$, but a photon only has two spin states, ± 1, the 0 is missing.

When a particle is moving we can use the direction of motion to define the spin state. The *helicity* h of a particle moving with momentum $\mathbf{p}$ is defined as

$$h := \frac{\mathbf{p}.\mathbf{s}}{|\mathbf{p}|}.$$

For a massive particle there are thus $2s+1$ helicity states. A photon has two helicity states.

For photons the concept of helicity is related to the familiar notion of polarisation of a light beam via particle-wave duality: helicity is to a particle what polarisation is to a wave. A changing magnetic field generates an electric field (Faraday's law of electromagnetic induction), conversely a changing electric field also generates an electric field, and a classical electromagnetic wave consist of self-sustaining oscillating electromagnetic fields. Electromagnetic waves are transverse and can be polarised, a beam of light moving in the z-direction has oscillating electric and magnetic fields in the $x-y$ plane (hence transverse), for example the electric field with wavenumber k and angular frequency ω could be of the from

$$\mathbf{E} = E_0 \cos(kz - \omega t)\hat{\mathbf{x}},$$

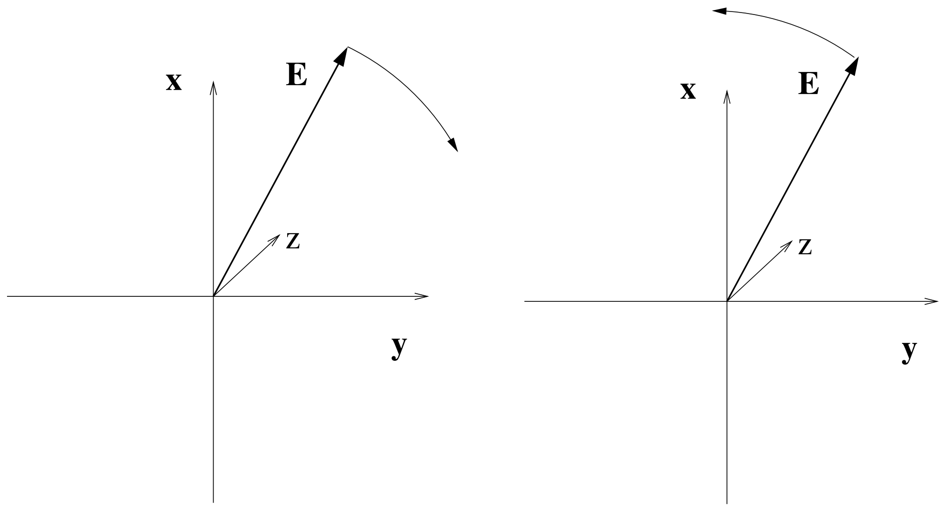
where E_0 is a constant. This field always points in the x-direction and oscillates in magnitude: it is said to be *linearly polarised* in the x-direction. A more compact way of writing it is to use a complex notation and write

$$\mathbf{E} = E_0 Re\left(e^{i(kz-\omega t)}\hat{\mathbf{x}}\right).$$

Another possibility is

$$\begin{aligned}\mathbf{E} &= E_0 Re\left(e^{i(kz-\omega t)}(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})\right) \\ &= E_0\big(\cos(kz-\omega t)\hat{\mathbf{x}} \mp \sin(kz-\omega t)\hat{\mathbf{y}}\big),\end{aligned}$$

which corresponds a rotating electric field of constant magnitude in the $x - y$ plane. If we visualise the $x - y$ plane with the z-axis pointing away from is, the upper sign corresponds to clockwise rotation and minus sign to anti-clockwise rotation. This electromagnetic wave is said to be *circularly polarised*: if we imagine travelling along with the wave in the z-direction the clockwise rotation is like a right-handed corkscrew and is called *right circularly polarised* and the anti-clockwise rotation is called *left circularly polarised*.



Right circularly polarised   Left circularly polarised

In summary, if the wave is moving in the z-direction, $\hat{\mathbf{x}} + i\hat{\mathbf{y}}$ represents right circular polarisation and $\hat{\mathbf{x}} - i\hat{\mathbf{y}}$ left circular polarisation. Note that, for a wave moving in the opposite, $-z$, direction it is the other way around: $\hat{\mathbf{x}} - i\hat{\mathbf{y}}$ represents right circular polarisation and $\hat{\mathbf{x}} + i\hat{\mathbf{y}}$ left circular polarisation.

Quantum mechanically the photon has two helicity states, $h = \pm 1$, and these correspond respectively to the right and left circular polarisation of a classical electro-magnetic wave. There is no classical analogue of the helicity of an electron.

There is a modification of equation (14) if the particles involved have intrinsic spin. As before label the two colliding particles as particle a and particle b and suppose two particles c and d come out.[25] If the outgoing particles have intrinsic spins s_c and s_d, with helicities h_c and h_d, then there are more quantum mechanical states available for outgoing particles to scatter into and, if the final spins are not measured, we should sum over them in calculating the cross-section (there are $(2s_c + 1)(2s_d + 1)$ possibilities). If the incoming particles have spins s_a and s_b respectively then there are $(2s_a + 1)(2s_b + 1)$ possible initial

[25] It is not always the case that the particles coming out are the same as the particles going in — sometimes the colliding particles can transfer electric charge, or other quantum numbers, during a collision. So, for example, an ingoing proton might might lose its charge and emerge as a neutron (though the *total* electric charge, when all other particles are accounted for, must remain unchanged). In particle physics a process in which the outgoing particles are the same as the ingoing particles is called an *elastic* collision. If the emerging particles differ from the ingoing particles the process is termed *inelastic*. This is a different use of the words elastic and inelastic as applied to non-relativistic collisions, where the words refer to whether or not energy is conserved in the collision. In a relativistic collision energy is *always* conserved.

states and, if none of these is fixed by the experiment (*i.e.* the initial particle states are unpolarised), then we should average over the initial spin states, that is the initial helicities. The net result is that we should average over the initial helicities and sum over the final helicities. In general amplitudes can depend on the helicities of the incoming and outgoing particle and we denote this dependency by $M_{fi}^{h_c h_d; h_a h_b}$ so this prescription gives [26]

$$\frac{d\sigma}{d\Omega} = \frac{1}{4\pi^2\hbar^4}\frac{p_f^2}{v_f v_i}\frac{1}{(2s_a+1)(2s_b+1)}\sum_{h_a,h_b,h_c,h_d}|M_{fi}^{h_c h_d; h_a h_b}|^2. \tag{17}$$

The simplest possibility is that M_{fi} is independent of spins, in which case this reduces to

$$\frac{d\sigma}{d\Omega} = \frac{1}{4\pi^2\hbar^4}\frac{p_f^2}{v_f v_i}(2s_c+1)(2s_d+1)|M_{fi}|^2 \tag{18}$$

and the cross-section just increases by a factor corresponding to the number of spin states available to the emerging particles, but this is not always the case.

iv) **Resonances:** we can use the concept of reaction rate W even for a single particle decaying into daughter particles: the mean lifetime of a decaying particle is

$$\tau = \frac{1}{W}.$$

For particles that decay due to strong interactions typical lifetimes are extremely short, $\tau \approx 10^{-23}$ s, which is too short to measure directly. For a given lifetime an energy Γ can be defined, using the uncertainty principle $\tau\Delta E \approx \hbar$, as

$$\Gamma = \frac{\hbar}{\tau}.$$

Γ represents an uncertainty, or a spread, in the energy of the decaying state. Starting with a sample of N_0 particles at time $t = 0$ the number left after a time $t > 0$ is

$$N(t) = N_0 \mathrm{e}^{-t/\tau} = N_0 \mathrm{e}^{-\Gamma t/\hbar}.$$

For a decaying particle created at time $t = 0$ with mass m, and so rest energy $E_0 = mc^2$, and initial wave-function $\psi(0)$, the wave-function after as a function of t will be

$$\psi(t) = \begin{cases} e^{-(i\frac{E_0 t}{\hbar} + \frac{\Gamma t}{2\hbar})}\psi(0) & t \geq 0, \\ 0 & t < 0 \end{cases}$$

in the rest frame of m. The Γt term in the exponent represents the fact that the particle can decay and the number of particles left after a time t is

$$N(t) \propto \psi^*(t)\psi(t) = e^{-\frac{\Gamma t}{\hbar}}|\psi(0)|^2.$$

[26] Note that we use $\sum_{h_a,h_b,h_c,h_d}|M_{fi}^{h_c h_d; h_a h_b}|^2$ and not $|\sum_{h_a,h_b,h_c,h_d} M_{fi}^{h_c h_d; h_a h_b}|^2$: the final state particles do have definite helicities, we are just choosing not to measure them.

Take the wave function to vanish for $t < 0$ as this is consistent with the particles being created at $t = 0$.

It is convenient to work with the Fourier transformed wave-function

$$\widetilde{\psi}(\omega) = \int_{-\infty}^{\infty} \psi(t) e^{i\omega t} dt = \int_{0}^{\infty} \psi(t) e^{i\omega t} dt.$$

Using Einstein's relation $E = \hbar\omega$ this can be written in terms of energy rather than frequency

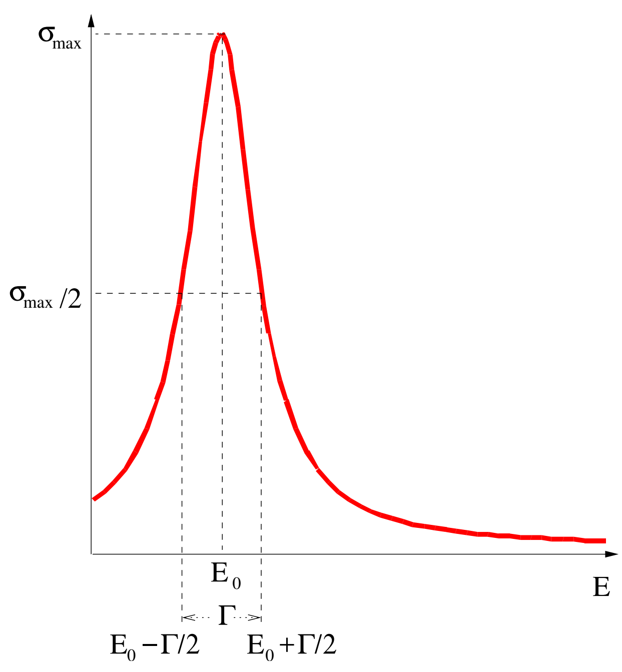
$$\widetilde{\psi}(E) = \int_{0}^{\infty} \psi(t) \mathrm{e}^{\frac{iEt}{\hbar}} dt = \psi(0) \int_{0}^{\infty} \mathrm{e}^{\left(i\frac{(E-E_0)}{\hbar} - \frac{\Gamma}{2\hbar}\right)t} dt = \frac{i\hbar\psi(0)}{(E - E_0) + i\frac{\Gamma}{2}},$$

giving the probability

$$|\widetilde{\psi}(E)|^2 = \widetilde{\psi}^*(E)\widetilde{\psi}(E) = \frac{\hbar^2 |\psi(0)|^2}{(E - E_0)^2 + \frac{\Gamma^2}{4}}$$

which is directly proportional to a cross-section.

This is called the *Breit-Wigner* resonance formula and it is essentially the same formula as that for the amplitude of oscillations as a function of driving frequency for a damped harmonic oscillator near resonance.



The peak at $E = E_0$ is called a *resonance* of the cross-section and Γ determines the *width* of the resonance, as a function of energy. The above form of the cross-section near a resonance is non-relativistic, the special relativistic generalisation is

$$\sigma(E) = \frac{\sigma_{max} E_0^2 \Gamma^2}{(E^2 - E_0^2)^2 + \Gamma^2 E_0^2}$$

where E^2 is defined here as $E^2 = -\underset{\sim}{P}.\underset{\sim}{P}/c^2$ with $\underset{\sim}{P}$ is the total initial 4-momentum. Thus $E^2 = m^2c^4$ for a single particle decay, but is more complicated for a two-particle collision. Conventionally the symbol s is used for E^2 where

$$s := -\underset{\sim}{P}.\underset{\sim}{P}/c^2$$

is the total centre of mass energy.

Very short lived particles have very large Γ and appear as broad peaks in the cross-section as a function of energy, indeed they are hardly particles at all and are usually referred to as *resonances*. These states are more appropriately described by the energy $E = mc^2$ and the width Γ rather than the mass m and the lifetime τ, the latter are more usually applied to long lived particles. For example there is a baryonic state with spin 3/2, given the symbol Δ^+, which is like an excited version of the proton and, like the proton, has quark content uud. This state has $E_0 = 1232\ MeV$ and $\Gamma = 120\ MeV$ and is often referred to as a *hadronic resonance* rather than a particle. Charged pions on the other hand, $\pi^\pm$, have

$$\Gamma = \hbar/(2.6 \times 10^{-8}\ s) = 2.5 \times 10^{-8}\ eV << m_\pi c^2$$

and would usually be called particles rather than resonances.

The quantity that really distinguishes between a 'particle' and a 'resonance' is the dimensionless ratio $\Gamma/E_0 = \Gamma/(mc^2)$: if this is small the state is best thought of as a particle, with a relatively long life-time; if it is large the state is best thought of as a resonance. For example the Z^0 bosons mentioned earlier have $E_0 = mc^2 = 91.2\ GeV$ and $\Gamma = 2.5\ GeV$ which translates to an immeasurably short lifetime of $\tau_{Z^0} = 2.6 \times 10^{-25}\ s$. Such a short lifetime could never be measured directly in the laboratory. Nevertheless the Z^0 is usually thought of as a particle since $\Gamma/(mc^2) \approx 0.02 << 1$, even though its width is large compared to that of the Δ^+ resonance mentioned above.

3. Symmetries and Conservation Laws

In Lagrangian dynamics symmetries of the action imply conservation laws. Suppose a physical system has generalised co-ordinates $q^i(t)$ and velocities $\dot{q}^i(t)$ whose dynamics is governed by a Lagrangian $L(q,\dot{q})$. Suppose that the Lagrangian (and hence the action) is invariant under a change $\delta q^i = \epsilon f^i(q)$ for some definite functions $f^i(q)$, with ϵ an infinitesimal constant. Now under *any* infinitesimal variation $q^i \to q^i + \delta q^i$ the Lagrangian changes by

$$L(q,\dot{q}) \quad \longrightarrow \quad L(q,\dot{q}) + \delta L(q,\dot{q})$$

with

$$\delta L(q,\dot{q}) = \sum_i \frac{\partial L}{\partial \dot{q}^i}\delta\dot{q}^i + \sum_i \frac{\partial L}{\partial q^i}\delta q^i = \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\delta q^i\right) - \sum_i \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i}\right]\delta q^i.$$

If $q(t)$ is a solution of the Lagrangian equations of motion then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0$$

and so

$$\delta L(q,\dot{q}) = \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\delta q^i\right) = \frac{d}{dt}\left(\sum_i \frac{\partial L}{\partial \dot{q}^i}\delta q^i\right).$$

Now, if $\delta q^i(t) = \epsilon f^i(q)$ is a symmetry of $L(q,\dot{q})$ then $L(q^i,\dot{q}^i) = L(q^i,\dot{q}^i) + \delta L(q^i,\dot{q}^i)$ so $\delta L(q,\dot{q}) = 0$, because that is what we mean by a symmetry. We conclude that, if $q^i(t)$ is a solution of the equations of motion *and* $\delta q^i = \epsilon f^i(q)$ is a symmetry, then

$$\frac{d}{dt}\left(\sum_i \frac{\partial L}{\partial \dot{q}^i}\delta q^i\right) = 0 \qquad \Rightarrow \qquad \sum_i \frac{\partial L}{\partial \dot{q}^i} f^i(q) = C_f$$

is a constant of the motion, independent of time. Hence symmetries imply conservation laws.

As an example consider a free particle with mass m and position described by Cartesian co-ordinates $x = q^1$, $y = q^2$ and $z = q^3$. The Lagrangian is

$$L(q,\dot{q}) = \frac{1}{2}m\sum_{i=1}^{3}\dot{q}^i\dot{q}^i.$$

From translational invariance we see that $L(q,\dot{q})$ is invariant under $x \to x + \epsilon$ where ϵ is a constant. So let $f^1(q) = 1$ and $f^2(q) = f^3(q) = 0$ and then

$$\sum_i \frac{\partial L}{\partial \dot{q}^i} f^i(q) = m\dot{q}^1 = p^1$$

is the x-component of the particle's momentum, which must be constant if $q^i(t)$ are a solution of the equations of motion. Similarly $p^2 = m\dot{q}^2(t)$ and $p^3 = m\dot{q}^3(t)$ are also constants. In other words translational invariance implies conservation of linear momentum.

This is a very powerful result in mechanics. Invariance under translations in time gives rise to conservation of energy and any system whose dynamics is invariant under rotations must have constant angular momentum. The fact that energy, linear momentum and angular momentum are conserved is directly traceable to known symmetries: invariance under time translations, space translations and rotations.

To summarise:

Translational Invariance	$\Leftrightarrow$	Conservation of Linear Momentum
Rotational Invariance	$\Leftrightarrow$	Conservation of Angular Momentum
Time Independence	$\Leftrightarrow$	Conservation of Energy

In particle physics much can be understood by finding things that seem to be conserved in the phenomena which then imply symmetries in the underlying dynamics and this often allow us to guess what the dynamics might be.

Conservation of Energy and Momentum

In relativistic physics energy and momentum are unified — they are just different components of the relativistic 4-momentum. The law of conservation of 4-momentum (which is a consequence of the symmetry of Lorentz invariance in relativistic dynamics) is often expressed as conservation of energy in the rest frame of a collision.

We have already used the law of conservation of energy in analysing particle decays. A neutron can decay into a proton, an electron and an anti-neutrino

$$n \to p + e^- + \bar{\nu}_e$$

only because there is excess energy available

$$m_n c^2 = (m_p + m_e + m_{\nu_e})c^2 + E_{Kin} > (m_p + m_e + m_{\nu_e})c^2,$$

where $E_{Kin} > 0$ is the total kinetic energy of the final state particles in the rest frame of the neutron. The inverse process

$$p \to n + e^+ + \nu_e \tag{19}$$

is forbidden by conservation of energy because

$$m_p c^2 < (m_n + m_e + m_{\nu_e})c^2. \tag{20}$$

As already mentioned quantum mechanically processes like (19) can happen if the final states are virtual off-shell states and do not survive indefinitely. One still has conservation of 4-momenta and, in particular, energy even in a virtual process, but the kinetic energy E_{Kin} can be negative for a time less than $\Delta t \approx \frac{\hbar}{E_{Kin}}$.

Conservation of Angular Momentum

As another an example of the use of conservation laws in particle physics, consider the emission of a photon by an electron in an atom via an electric dipole transition. In the process the electron jumps from a higher orbital to a lower orbital in the atom. If the initial electron wave-function was $\psi_l \propto P_l(\cos\theta)$, where l denotes the orbital angular momentum of the wave-function, then the final electron wave-function has angular momentum $l - 1$. The process is

$$\psi_l \to \psi_{l-1} + \gamma.$$

The initial angular momentum of the electron is $l\hbar$. If the spin of the photon is s_γ then the final total angular momentum of the electron plus the photon is $(l-1)\hbar + s_\gamma\hbar$. Conservation of angular momentum then requires that

$$l\hbar = (l-1)\hbar + s_\gamma\hbar,$$

so we deduce that the spin of the photon is

$$s_\gamma = 1.$$

In order to utilise the full power of the principle of angular-momentum conservation it will be necessary to understand how to add two angular momenta in quantum physics, which is very different to the familiar process of vector addition in classical physics. In classical physics angular momentum is a vector $\mathbf{J}$ with a magnitude J and a direction specified by three components (J_x, J_y, J_z). Of course these are not independent, since

$$J^2 = \mathbf{J}.\mathbf{J} = J_x^2 + J_y^2 + J_z^2,$$

and angular momentum is specified by three real numbers. Vectors are added using the usual triangle law from which

$$|J_1 - J_2| \leq |\mathbf{J}_1 + \mathbf{J}_2| \leq J_1 + J_2. \quad (21)$$

Quantum mechanically things are different in a number of ways:

- Angular momentum is *quantised*. J is not a continuous variable but can take only discrete values which are multiples of the fundamental quantum unit $\hbar$. In fact

  $$J^2 = j(j+1)\hbar^2$$

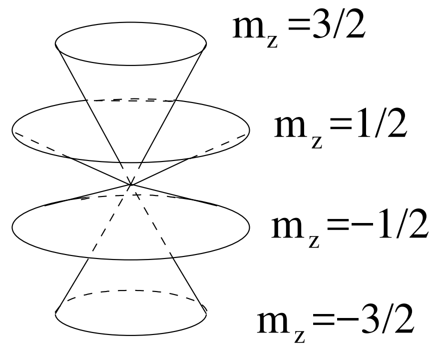
  where j is either a non-negative integer, $j = 0, 1, 2, \ldots$ or a positive half-integer $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$.
- The uncertainty principle says that J_x, J_y and J_z cannot be simultaneously specified — there are only two degrees of freedom in quantum angular momenta. If J is specified then we can measure only one linear combination of J_x, J_y or J_z, not all three independently. For example if J and J_z are known for a given quantum state then J_x and J_y are completely undetermined and have no physical value. Furthermore J_z is also quantised and has only $2j + 1$ possible values

  $$J_z = m_z \hbar$$

  where

  $$m_z = -j, -j+1, \ldots, j-1, j. \quad (22)$$

  For $j = \frac{3}{2}$, for example, there are 4 possible values of J_z, given by $m_z = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$, but J_x and J_y are completely undetermined. Pictorially the situation looks something like this:



  where $\mathbf{J}$ is represented by a cone whose height is determined by m_z.

- Because of the above two restrictions, addition of angular momenta in quantum mechanics is also different to the classical case. If two angular momenta $\mathbf{J}_1$ and $\mathbf{J}_2$, are added to give a third $\mathbf{J}$ then the triangle inequality (21) still holds, but J can only take the discrete values

  $$J = |J_1 - J_2|, |J_1 - J_2| + \hbar, \ldots, J_1 + J_2 - \hbar, J_1 + J_2,$$

  or equivalently

  $$j = |j_1 - j_2|, |j_1 - j_2| + 1, \ldots, j_1 + j_2 - 1, j_1 + j_2.$$

  If the z-components are known there are further constraints on the sum: if $\mathbf{J}_1$ has third component $J_{1,z}$ and $\mathbf{J}_2$ has third component $J_{2,z}$ then these do just add like the classical case and $\mathbf{J}$ will have third component

  $$J_z = J_{1,z} + J_{2,z}$$

  or $m = m_1 + m_2$. This requires $j \geq |m|$ because of (22).

In general a particle need not be in a definite state of angular momentum, it may be in a linear superposition of different j and m. A general state is a vector in a Hilbert space and we can use definite states of j and m as basis vectors, denoted by $|j; m>$.[27] This basis can be chosen so that it is orthonormal with

$$< j; m|j'; m' >= \delta_{jj'}\delta_{mm'}.$$

A general state is then a linear sum of basis vectors

$$\Psi = \sum_j \sum_{m=-j}^{j} \psi_{jm}|j; m>$$

where ψ_{jm} are complex numbers. For example adding a state with definite angular momentum $|j_1; m_1>$ to a second state with definite angular momentum $|j_2; m_2>$ produces a linear superposition of states with angular momenta j between $|j_2 - j_1|$ and $j_2 + j_1$. A standard notation for this is

$$|j_1; m_1 > \otimes |j_2; m_2 >= \sum_{j=|j_2-j_1|}^{j_2+j_1} C^{j;m}_{j_1 j_2; m_1 m_2} |j; m> \tag{23}$$

where the numbers $C^{j;m}_{j_1 j_2; m_1 m_2}$ are called *Clebsch-Gordon coefficients*. Conservation of angular momentum requires that only states with $m = m_1 + m_2$ appear in the sum on the right-hand side, so $C^{j;m}_{j_1 j_2; m_1 m_2} = 0$ unless $m = m_1 + m_2$.

[27] It is common in quantum mechanics to denote quantum states associated with some property a by $|a>$ and the complex conjugate state by $<a|$.

The reason for the $\otimes$ symbol is that, for fixed j_1 and j_2, there are $2j_1+1$ possible values of m_1 and $2j_2+1$ possible values of m_2. Thus the set

$$\{|j_1; m_1>; m_1 = -j_1, -j_1+1, \ldots, j_1-1, j_1\}$$

is a basis for a $2j_1+1$ dimensional vector space and

$$\{|j_2; m_2>; m_2 = -j_2, -j_2+1, \ldots, j_2-1, j_2\}$$

is a basis for a $2j_2+1$ dimensional vector space. There are therefore $(2j_1+1)(2j_2+1)$ possible states on the left-hand side of equation (23) so vector addition in quantum mechanics is like multiplication in terms of Hilbert space dimensions.[28] A slightly more compact notation, which we shall often use, is

$$|j_1, j_2; m_1, m_2 > := |j_1; m_1 > \otimes |j_2; m_2 > .$$

For each value of j on the right-hand side of (23) there are $2j+1$ possible values of m and j runs from $|j_2 - j_1|$ to $j_2 + j_1$ giving, assuming $j_2 \geq j_1$ for example,

$$\sum_{j=j_2-j_1}^{j_2+j_1} (2j+1) = \big\{(j_2+j_1) + (j_2-j_1)\big\}(2j_1+1) + (2j_1+1) = (2j_1+1)(2j_2+1)$$

different $|j; m>$.

The quantum Hilbert space associated with these angular momentum states is a finite dimensional complex vector space with dimension $(2j_1+1)(2j_2+1)$ and either

$$\{|j; m>; j = |j_1 - j_2|, |j_1 - j_2| + 1, \ldots, j_1 + j_2 - 1, j_1 + j_2, m = -j, -j+1, \ldots, j-1, j\}$$

or

$$\{|j_1, j_2; m_1, m_2 >; m_1 = -j_1, -j_1+1, \ldots, j_1-1, j_1, m_2 = -j_2, -j_2+1, \ldots, j_2-1, j_2\}$$

can be used as an orthonormal basis for the space. The Clebsch-Gordon coefficients just give a linear transformation between two different orthonormal bases.

Consider for example combining two spin-1/2 electrons, so $j_1 = j_2 = 1/2$. Each electron has a spin which is a vector in a two-dimensional Hilbert space of possible spins, its spin is some linear combination of two possible spin states, often called spin up ($|\uparrow>$) and spin down ($|\downarrow>$) relative to some fiducial direction such as the z-axis of a Cartesian co-ordinate system. In our notation

$$\Big|\frac{1}{2}; \frac{1}{2} >= |\uparrow>, \qquad \text{and} \qquad \Big|\frac{1}{2}; -\frac{1}{2} >= |\downarrow> .$$

[28] For historical reasons the $\otimes$ operation is called a *tensor product* — a term that comes originally from the theory of elasticity in solids.

There are only two possible results for the total angular momentum when the electron spins are combined, $j = \frac{1}{2} - \frac{1}{2} = 0$ or $j = \frac{1}{2} + \frac{1}{2} = 1$. If both electrons are spin up, $m_1 = m_2 = 1/2$, the combined spin must have $m = 1$ and only $j = 1$ is allowed in the combined state:

$$\Big|\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2}>=|1;1> \qquad \Leftrightarrow \qquad |\uparrow>\otimes|\uparrow>=|\uparrow\uparrow> \qquad \Leftrightarrow \qquad C^{1,1}_{\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}} = 1.$$

Similarly if both electrons are spin down, $m_1 = m_2 = -1/2$, the combined spin must have $m = -1$ and again only $j = 1$ is allowed in the combined state:

$$\Big|\frac{1}{2},\frac{1}{2};-\frac{1}{2},-\frac{1}{2}>=|1;-1> \qquad \Leftrightarrow \qquad |\downarrow>\otimes|\downarrow>=|\downarrow\downarrow> \qquad \Leftrightarrow \qquad C^{1,-1}_{\frac{1}{2}\frac{1}{2};-\frac{1}{2}-\frac{1}{2}} = 1.$$

Obviously these two states are symmetric under interchange of the two electrons.[29]

If however one electron has $m_1 = 1/2$ and the other $m_2 = -1/2$, or *vice versa*, then the combination has $m = 0$ and this could be either $j = 1$ or $j = 0$: $j = 1$ is the symmetric combination

$$|1;0>=\frac{1}{\sqrt{2}}\big(|\uparrow\downarrow>+|\downarrow\uparrow>\big)$$

and $j = 0$ the orthogonal combination

$$|0;0>=\frac{1}{\sqrt{2}}\big(|\uparrow\downarrow>-|\downarrow\uparrow>\big),$$

which is necessarily anti-symmetric.

These can be inverted to give

$$|\uparrow\downarrow>=\Big|\frac{1}{2},\frac{1}{2};\frac{1}{2},-\frac{1}{2}>=\frac{1}{\sqrt{2}}\big(|1;0>+|0;0>|\big),$$
$$|\downarrow\uparrow>=\Big|\frac{1}{2},\frac{1}{2};-\frac{1}{2},\frac{1}{2}>=\frac{1}{\sqrt{2}}\big(|1;0>-|0;0>\big),$$

from which the Clebsch-Gordon coefficients can be read off

$$C^{1;0}_{\frac{1}{2},\frac{1}{2};\frac{1}{2},-\frac{1}{2}} = \frac{1}{\sqrt{2}}, \qquad C^{0;0}_{\frac{1}{2},\frac{1}{2};\frac{1}{2},-\frac{1}{2}} = \frac{1}{\sqrt{2}}, \qquad C^{1;0}_{\frac{1}{2},\frac{1}{2};-\frac{1}{2},\frac{1}{2}} = \frac{1}{\sqrt{2}}, \qquad C^{0;0}_{\frac{1}{2},\frac{1}{2};-\frac{1}{2},\frac{1}{2}} = -\frac{1}{\sqrt{2}}.$$

The total Hilbert space for the electrons' spin is 4-dimensional and one can use either

$$\left\{\Big(\Big|\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2}>\Big),\ \Big(\Big|\frac{1}{2},\frac{1}{2};\frac{1}{2},-\frac{1}{2}>\Big),\ \Big(\Big|\frac{1}{2},\frac{1}{2};-\frac{1}{2},\frac{1}{2}>\Big),\ \Big(\Big|\frac{1}{2},\frac{1}{2};-\frac{1}{2},-\frac{1}{2}>\Big)\right\}$$

[29] The total quantum state corresponding to two Fermions should of course be anti-symmetric under interchange of the two particles. Spin is only part of the story, a complete quantum description should also include position. For example electrons with j=1 could have a relative orbital angular momentum of l=1 making the total quantum state antisymmetric under interchange of the two electrons.

or

$$\left\{|1;1>, |1;0>, |1;-1>, |0;0>\right\}$$

as a set of basis vectors. The decomposition into $j = 1$ (triplet) and $j = 0$ (singlet) sectors is

$$\begin{aligned} |1;1> &= |\uparrow\uparrow> \\ |1;0> &= \frac{1}{\sqrt{2}}\big(|\uparrow\downarrow> + |\downarrow\uparrow>\big) \\ |1;-1> &= |\downarrow\downarrow> \end{aligned} \tag{24}$$

and

$$|0;0>= \frac{1}{\sqrt{2}}\big(|\uparrow\downarrow> - |\downarrow\uparrow>\big). \tag{25}$$

It is often useful to exhibit the linear transformation that the Clebsch-Gordon coefficients represent in the form of a table.

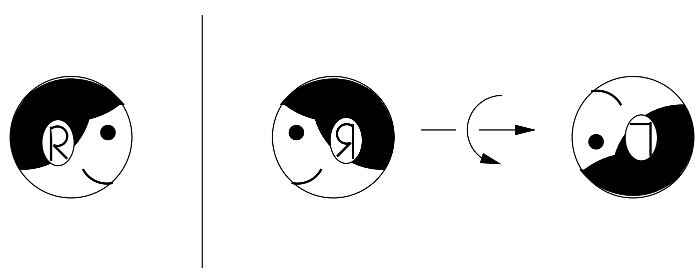
		$j=1$			$j=0$
m_1	m_2	$m=1$	$m=0$	$m=-1$	$m=0$
$1/2$	$1/2$	1	0	0	0
$1/2$	$-1/2$	0	$1/\sqrt{2}$	0	$1/\sqrt{2}$
$-1/2$	$1/2$	0	$1/\sqrt{2}$	0	$-1/\sqrt{2}$
$-1/2$	$-1/2$	0	0	1	0

A table for $j = \frac{1}{2}$ combined with $j = \frac{3}{2}$ will be derived later.

Parity

Translational and rotational invariance, as well as time translational invariance, are examples of *continuous* symmetries. There is no limit to how small a translation can be and any translation can be generated by repeated application of a very small, or indeed infinitesimal, translation. There are also symmetries in nature that are *discrete*, such as reflection in a mirror.

The operation of reflecting all points in the origin so that, in Cartesian co-ordinates, $(x, y, z) \to (-x, -y, -z)$ is called spatial inversion, often denoted by $\mathbf{P}$, for *Parity*. It is equivalent to reflection in a mirror, which interchanges 'back' and 'front', followed by a rotation through π radians about an axis normal to the mirror:



In quantum mechanics this process has the following effect on wave-functions

$$\mathbf{P}\psi(\mathbf{r}) \to \psi(-\mathbf{r}).$$

Since $\mathbf{P}^2 = 1$ it follows that, if an eigenfunction of $\mathbf{P}$ exists with eigenvalue $\mathcal{P}$ so that $\mathbf{P}\psi = \mathcal{P}\psi$, then $\mathcal{P} = \pm 1$. For example, in one dimension,

$$\begin{aligned} \psi(x) = \cos(x) \quad &\Rightarrow \quad \mathbf{P}\psi = \cos(-x) = \cos(x) = \psi \quad \Rightarrow \mathcal{P} = +1 \\ \psi(x) = \sin(x) \quad &\Rightarrow \quad \mathbf{P}\psi = \sin(-x) = -\sin(x) = -\psi \quad \Rightarrow \mathcal{P} = -1 \\ \psi(x) = \cos(x) + \sin(x) \quad &\Rightarrow \quad \mathbf{P}\psi = \cos(x) - \sin(x) \quad \Rightarrow \quad \text{not an eigenstate.} \end{aligned}$$

While in 3-dimensions

$$\begin{aligned} \psi(\mathbf{r}) = r^{-(l+1)} P_l(\cos(\theta)) \quad &\Rightarrow \quad \mathbf{P}\psi = r^{-(l+1)} P_l(\cos(\pi - \theta)) = (-1)^l r^{-(l+1)} P_l(\cos(\theta)) \\ &\Rightarrow \quad \mathcal{P} = (-1)^l. \end{aligned}$$

States with parity $+1$ are called *even* parity states while state with parity -1 are called *odd* parity sates. Parity eigenvalues are *multiplicative*, *i.e.* if ψ_a is an eigenstate with eigenvalue $\mathcal{P}_a$ and ψ_b is an eigenstate with eigenvalue $\mathcal{P}_b$ then $\psi_a \psi_b$ is also an eigenstate with eigenvalue $\mathcal{P}_a \mathcal{P}_b$.

Some vector quantities, *e.g.* position $\mathbf{r} \to -\mathbf{r}$ and momentum $\mathbf{p} \to -\mathbf{p}$ change sign under $\mathbf{P}$ while others, such as angular momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \to (-\mathbf{r}) \times (-\mathbf{p}) = \mathbf{r} \times \mathbf{p},$$

do not. Vectors which do not change sign under the parity operation are called *axial vectors* in mechanics (the term *pseudo-vector* is also commonly used). A magnetic field $\mathbf{B}$ is another example of an axial vector.

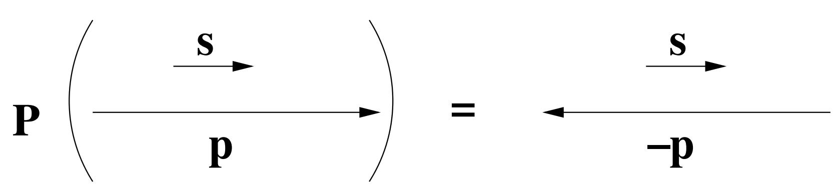
Now spin, like angular momentum, is an axial vector so it does not change under $\mathbf{P}$. The helicity h of a particle moving with momentum $\mathbf{p}$ is defined as

$$h := \frac{\mathbf{p}.\mathbf{s}}{|\mathbf{p}|}.$$

Now

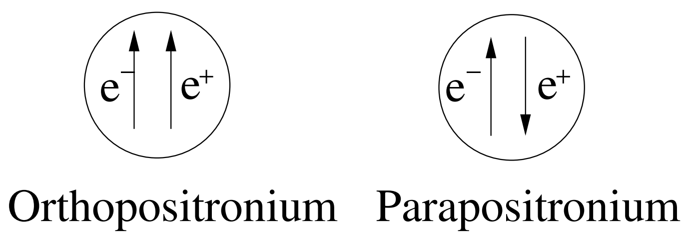
$$\mathbf{P}\left(\frac{\mathbf{p}.\mathbf{s}}{|\mathbf{p}|}\right) = \frac{(-\mathbf{p}).(\mathbf{s})}{|-\mathbf{p}|} = -\frac{\mathbf{p}.\mathbf{s}}{|\mathbf{p}|},$$

so parity changes the helicity. Pictorially this can be visualised as follows: suppose a spin-one particle has spin $\mathbf{s}$ parallel to $\mathbf{p}$, so that $h = +1$, then $\mathbf{P}$ flips the direction of $\mathbf{p}$ without changing the spin, so the helicity is flipped $h = +1 \to h = -1$.



Until the middle of the 20th century it was tacitly assumed that all the laws of nature were symmetric under $\mathbf{P}$, in other words the fundamental laws of physics would look exactly the same when viewed in a mirror,[30] but in 1956 an experiment showed that parity is *not* an exact symmetry of nature. It does seem to be an exact symmetry of the electromagnetic and strong forces, but not of the weak force. Mathematically this means that if an initial wave-function ψ_i, perhaps describing more than one particle, is a parity eigenstate $\mathbf{P}\psi_i = \pm\psi_i$ then after either an electromagnetic or a strong interaction the final wave-function ψ_f, which could be describing different particles, must also be a parity eigenstate, with the same eigenvalue $\mathbf{P}\psi_f = \pm\psi_f$. This is not true of the weak interactions. As an example of the use of parity conservation in an electromagnetic process we shall prove that a positron has the opposite parity to an electron.[31] This is an example of a more general result, first shown by Dirac, that fermions that are parity eigenstates always have the opposite parity to their anti-particles.

It is possible to make a bound state of a positron and an electron, like a Hydrogen atom but with the proton replaced with the positron. Such a state is called *positronium* but, unlike a Hydrogen atom, it is unstable because the positron and the electron are anti-particles and if they meet they annihilate each other, producing pure energy, electromagnetic radiation, in the form of photons. Just like the Hydrogen atom the ground state of positronium is when the two particles are in a s-wave configuration relative to each other, *i.e.* their relative orbital angular momentum is $l = 0$. But electrons and positrons have intrinsic spin, the same intrinsic spin of one-half (particle and anti-particle always have exactly the same intrinsic spin), and the physical characteristics of positronium depends on whether the spins are parallel or anti-parallel — the former is called orthopositronium and the latter is called parapositronium:



If the orbital angular momentum is $l = 0$ then the total angular momentum is $J = 0$ for parapositronium and $J = 1$ for orthopositronium. In the notation of atomic spectroscopy parapositronium is a singlet ${}^1S_0$ state while orthopositronium is a triplet ${}^3S_1$, where the subscript denotes the total angular momentum. Parapositronium decays more quickly than

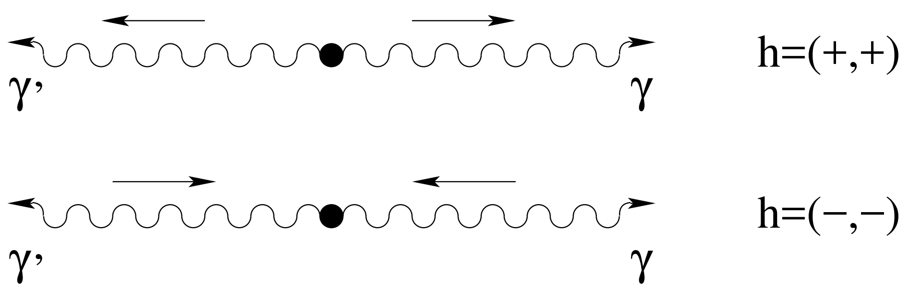
[30] Of course there are many phenomena which are not parity symmetric, such as the right-handed spiral of DNA molecules for example, but, as far as is known, these do not reflect an underlying asymmetry in the fundamental laws of physics, but seem to be a chance result of biological evolution. There is nothing in the laws of chemistry saying that DNA should be right-handed and it could just as easily have evolved as a left-handed molecule. The most likely explanation of this is that once one form has a slight preponderance over the other then natural selection dictates that the minority form is doomed.

[31] As long as we can leave the weak force out of the picture we can assign an intrinsic parity to every particle. For massive particles this means that, in the rest frame of the particle, the wave-function is an eigenstate of $\mathbf{P}$ with eigenvalue either ± 1. This eigenvalue then becomes the intrinsic parity of the particle and is another quantum number that characterises it.

orthopositronium, about 1000 times more quickly in fact since the former has a lifetime of $\tau_{J=0} = 1.2 \times 10^{-10}$ s while the latter has $\tau_{J=1} = 1.4 \times 10^{-7}$ s. Parapositronium also has a different decay mode to orthopositronium, decaying predominantly into two photons while orthopositronium decays predominantly into three photons:

$$^1S_0 \to 2\gamma \qquad ^3S_1 \to 3\gamma.$$

We can determine the relative parity of the positron and the electron by examining the decay of parapositronium in detail. This is an electromagnetic decay so parity is conserved. The initial total angular momentum is $J = 0$ and angular momentum is conserved so the final total angular momentum must also be $J = 0$. This means that, in the rest frame of the positronium, the two photons must come out back to back with the same helicity, either $h = +1$ or $h = -1$ for both photons



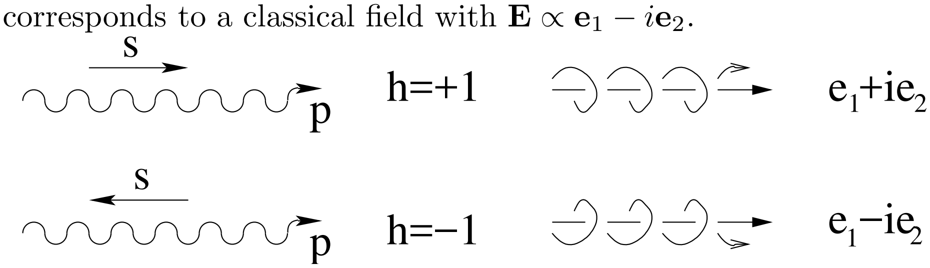
or, more generally, a some linear combination of these two states. Denote these two final state of the two photons by

$$|R, R >:= \psi(\mathbf{p}, -\mathbf{p}; +, +)$$

when $h = +1$ and by

$$|L, L >:= \psi(\mathbf{p}, -\mathbf{p}; -, -)$$

when $h = -1$. The notation here reflects the fact that a positive helicity photon in quantum mechanics corresponds to a right-circularly polarised electromagnetic wave in classical electromagnetism. Let $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$, denote orthonormal unit vectors in the x, y and z-directions respectively. As described in the section on **Spin and helicity** a classical electric field corresponding to positive helicity photons moving in the $\mathbf{e}_3$ direction is the real part of a complex field with $\mathbf{E} \propto \mathbf{e}_1 + i\mathbf{e}_2$; a negative helicity photon in quantum mechanics corresponds to a classical field with $\mathbf{E} \propto \mathbf{e}_1 - i\mathbf{e}_2$.



Note the left-moving photon in the decay of positronium depicted above is moving in the $-\mathbf{e}_3$ direction and a right-circularly polarised left-moving photon corresponds to a classical electromagnetic wave whose electric field is proportional to the real part of $\mathbf{e}_1 - i\mathbf{e}_2$ while a left-circularly polarised photon corresponds to a classical electromagnetic wave whose electric field is proportional to the real part of $\mathbf{e}_1 + i\mathbf{e}_2$: the opposite way round to the right-moving wave. In classical language a right-moving R is equivalent to $\mathbf{e}_1 + i\mathbf{e}_1$ while a left-moving R is equivalent to $\mathbf{e}_1 - i\mathbf{e}_1$, correspondingly a right-moving L is equivalent to $\mathbf{e}_1 - i\mathbf{e}_1$ while a left-moving L is equivalent to$\mathbf{e}_1 + i\mathbf{e}_1$

The parity operator flips helicity and momentum and both $|R, R>$ and $|L, L>$ are symmetric under interchange of the two photon states, because photons are bosons, hence

$$\mathbf{P}(|R, R>) = \mathbf{P}\psi(\mathbf{p}, -\mathbf{p}; +, +) = \psi(-\mathbf{p}, \mathbf{p}; -, -) = \psi(\mathbf{p}, -\mathbf{p}; -, -) = |L, L>$$

$$\mathbf{P}(|L, L>) = \mathbf{P}\psi(\mathbf{p}, -\mathbf{p}; -, -) = \psi(-\mathbf{p}, \mathbf{p}; +, +) = \psi(\mathbf{p}, -\mathbf{p}; +, +) = |R, R>$$

so neither $|R, R>$ nor $|L, L>$ is a parity eigenstate. The final eigenstates of parapositronium decay are actually linear superpositions of $|R, R>$ and $|L, L>$, the parity eigenstates are $|R, R> \pm |L, L>$ since

$$\mathbf{P}(|R, R> \pm |L, L>) = \pm(|R, R> \pm |L, L>).$$

Now $|R, R>$ corresponds to oppositely travelling electromagnetic waves with transverse electric field vectors proportional to $\mathbf{e}_1 + i\mathbf{e}_2$ and $\mathbf{e}_1 - i\mathbf{e}_2$ for right and left-moving waves respectively: denote this as

$$|R, R> \quad \Leftrightarrow \quad (\mathbf{e}_1 - i\mathbf{e}_2; \mathbf{e}_1 + i\mathbf{e}_2) = (\mathbf{e}_1; \mathbf{e}_1) + (\mathbf{e}_2; \mathbf{e}_2) + i(\mathbf{e}_1; \mathbf{e}_2) - i(\mathbf{e}_2; \mathbf{e}_1),$$

while $|L, L>$ corresponds to

$$|L, L> \quad \Leftrightarrow \quad (\mathbf{e}_1 + i\mathbf{e}_2; \mathbf{e}_1 - i\mathbf{e}_2) = (\mathbf{e}_1; \mathbf{e}_1) + (\mathbf{e}_2; \mathbf{e}_2) - i(\mathbf{e}_1; \mathbf{e}_2) + i(\mathbf{e}_2; \mathbf{e}_1).$$

Using the principle of superposition in quantum mechanics this means that, up to a overall phase, $|R, R> + |L, L>$ corresponds to $(\mathbf{e}_1; \mathbf{e}_1) + (\mathbf{e}_2; \mathbf{e}_2)$ while $|R, R> - |L, L>$ corresponds to $(\mathbf{e}_1; \mathbf{e}_2) - (\mathbf{e}_2; \mathbf{e}_1)$. Since the parity operation interchanges the two photons, it is clear from these expressions that $\mathbf{P}(|R, R> \pm |L, L>) = \pm(|R, R> \pm |L, L>)$.

Now an electromagnetic wave with $\mathbf{E}$ in either the $\mathbf{e}_1$ or the $\mathbf{e}_2$ direction corresponds to a plane wave linearly polarised in either the $\mathbf{e}_1$ or the $\mathbf{e}_2$ direction respectively. So a measurement of $|R, R> + |L, L>$ corresponds a linear superposition to both photons plane polarised in the $\mathbf{e}_1$ direction or both in the $\mathbf{e}_2$: either way there are two emergent plane polarised photons moving in opposite directions with the planes of polarisation parallel. On the other hand $|R, R> - |L, L>$ corresponds to the two waves being plane polarised in perpendicular directions.

This can be summarised by saying that the two photons are in an eigenstate of parity if they are linearly polarised, not circularly polarised (so they are not helicity eigenstates, they are linear combinations of helicity eigenstates): $\mathcal{P}_{\psi_f} = +1$ if they are polarised parallel to each other, *e.g.* both in the 1-direction with $\mathbf{e}_1.\mathbf{e}_1 = 1$ or both in the 2-direction, with $\mathbf{e}_2.\mathbf{e}_2 = 1$; $\mathcal{P}_{\psi_f} = -1$ if they are polarised perpendicularly to each other, *e.g.* one photon plane polarised in the $\mathbf{e}_1$-direction and the other in the $\mathbf{e}_2$-direction with $(\mathbf{e}_1.\mathbf{e}_2) = 0$.

It is an experimental observation that the two photons always correspond to the perpendicularly plane polarised case, so the final state is $|R, R> - |L, L>$ and the final state wave-function ψ_f is a parity eigenstate with $\mathcal{P}_{\psi_f} = -1$. The total parity of the final state $\mathcal{P}_f$ is the product of the wave-function parity and the intrinsic parities of the photons $\mathcal{P}_\gamma = \mathcal{P}'_\gamma$

$$\mathcal{P}_f = (\mathcal{P}_\gamma)^2 \mathcal{P}_{\psi_f} = \mathcal{P}_{\psi_f} = -1.$$

Conservation of parity in electromagnetic interactions then implies that the initial state of the parapositronium must also be a parity eigenstate with $\mathcal{P}_i = -1$. But parity is multiplicative, so

$$\mathcal{P}_i = \mathcal{P}_{e^+}\mathcal{P}_{e^-}\mathcal{P}_{\psi_i} = -1$$

where $\mathcal{P}_{e^+}$ and $\mathcal{P}_{e^-}$ are the intrinsic parities of the positron and the electron respectively and $\mathcal{P}_{\psi_i}$ is the parity of the parapositronium wave-function. But the parapositronium wave-function is an s-wave, with orbital angular momentum $l = 0$, so its parity is $\mathcal{P}_{\psi_i} = (-1)^l = +1$. Hence the experiments are telling us that

$$\mathcal{P}_{e^+}\mathcal{P}_{e^-} = -1 \qquad \Rightarrow \qquad \mathcal{P}_{e^+} = -\mathcal{P}_{e^-}$$

and we can conclude that indeed the positron and the electron have opposite intrinsic parities.

Electrons cannot be created singly in particle interactions, they can only be produced together with positrons or in conjunction with anti-neutrinos, so we cannot determine the intrinsic parity $\mathcal{P}_e$ of the electron in an unambiguous way. It is therefore defined, by convention as

$$\mathcal{P}_{e^-} = +1$$

which then implies that

$$\mathcal{P}_{e^+} = -1.$$

Similarly quarks are conventionally given the parity

$$\mathcal{P}_q = +1 \qquad \text{and} \qquad \mathcal{P}_{\bar{q}} = -1$$

and protons and neutrons then have

$$\mathcal{P}_p = \mathcal{P}_n = \mathcal{P}_q^3 = +1.$$

Neutrinos are only produced in weak interactions and are *not* parity eigenstates, an intrinsic parity therefore cannot be defined for neutrinos.

As a second example, using both conservation of angular momentum and parity, we shall determine the spin and parity of the pion. Unlike electrons pions can be produced singly, in strong force reactions like

$$p + p \to p + n + \pi^+$$

where two protons collide, one of them turns into a neutron and a pion is produced. This means that the parity of the pion is not determined by a convention — if the intrinsic parity of the proton and neutron are fixed then the parity of the pion can be measured.

We shall consider the above process when the final proton and neutron are locked together in a bound state called a *deuteron*[32] which will be denoted by D,

$$p + p \to D + \pi^+.$$

This reaction can proceed equally well in both directions, we can fire pions at a deuteron and break it up into two protons,

$$D + \pi^+ \to p + p.$$

The fact that the reaction can proceed in either direction is denoted by the single expression

$$p + p \leftrightarrow D + \pi^+.$$

Since the particles involved have spin we must include the final state spin factors: now the deuteron has spin one,[33] $s_D = 1$ and, for $pp \to D\pi$, the quantum mechanical amplitude M_{fi} is independent of spin so, using (18),

$$\sigma(pp \to D\pi^+) \propto (2s_D + 1)(2s_\pi + 1)p_D p_\pi = 3(2s_\pi + 1)p_D p_\pi$$

where $p_D$and p_π are the magnitudes of the deuteron and the pion momenta respectively ($p_D = p_\pi$ in the centre of mass frame). On the other hand for $D\pi \to pp$, since the proton has spin $s_p = 1/2$,

$$\sigma(D\pi^+ \to pp) \propto \frac{1}{2}(2s_p + 1)^2 p_p^2 = 2p_p^2,$$

where p_p is the outgoing proton momentum in the centre of mass frame.[34] We shall also need the fact that the quantum mechanical probability for this process is the same in either direction,

$$|M_{fi}|^2 = |M_{if}|^2$$

[32] The name deuteron comes from an isotope of Hydrogen in with the nucleus is a proton with a neutron added, called heavy hydrogen or *deuterium*, ^{2}H. A deuteron is an ionised deuterium atom.

[33] In the ground state of the deuteron the proton and the neutron have no relative orbital angular momentum (s-wave) and their intrinsic spins are parallel, $p\uparrow n\uparrow$, giving total spin 1. If the spins were anti-parallel, it e.g. $p\uparrow n\downarrow$, the deuteron spin would be zero, but this is unstable as the neutron could then β-decay to a proton, $n\downarrow \longrightarrow p\downarrow$: this cannot happen for $p\uparrow n\uparrow$ due to the Pauli exclusion principle, as the $p\uparrow$ energy state is already filled.

[34] There is a factor of one-half here because the protons are identical and indistinguishable, so the probability of the final state is symmetric under $\theta \to \pi - \theta$, where θ is the polar angle of one of the emerging protons relative to the incoming directions of the D and π (in the centre of mass frame). So to calculate the total cross-section we only need to integrate from $\theta = 0$ to $\theta = \pi/2$

$$\sigma = \int_0^{\pi/2} \frac{d\sigma}{d\theta} \sin\theta d\theta = \frac{1}{2} \int_0^{\pi} \frac{d\sigma}{d\theta} \sin\theta d\theta.$$

(this is related to another discrete symmetry of the electromagnetic and strong forces — time reversal, which will be described later). This means that the amplitudes, together with other irrelevant factors that are identical in both processes, cancel in the ratio

$$\frac{\sigma(pp \to D\pi^+)}{\sigma(D\pi^+ \to pp)} = \frac{3(2s_\pi + 1)}{2} \frac{p_D p_\pi}{p_p^2}.$$

Experimental measurement of the total cross-section (first performed in 1951) reveals that $\frac{3(2s_\pi+1)}{2} = \frac{3}{2}$, hence the $s_\pi = 0$ and the π^+ has zero intrinsic spin. The π^-, which is the anti-particle of the π^+, must therefore also have zero spin.

The parity of the pion can be determined from the similar, but clearly not identical, reaction of the breakup of a deuteron by a negative pion,

$$\pi^- + D \to n + n.$$

The pion is captured more easily if it is not too energetic, it is best therefore if the pion is slow which means quantum mechanically that the orbital angular momentum of the initial $D\pi^-$ system should be zero — their initial wave-function should be s-wave, *i.e.* the initial orbital angular momentum of the $D - \pi$ pair should be $l_i = l_{D\pi} = 0$. Since the pion has zero spin and the deuteron has spin $s_D = 1$, the total initial angular momentum J_i is just the intrinsic angular momentum of the deuteron $J_i = J_D = \hbar$. Now the deuteron is bound state of a proton and a neutron which are themselves in an s-wave state inside the deuteron. Since parity is multiplicative the intrinsic parity of the deuteron is

$$\mathcal{P}_D = \mathcal{P}_p \mathcal{P}_n = +1$$

and the initial parity is

$$\mathcal{P}_i = \mathcal{P}_\pi \mathcal{P}_D (-1)^{l_{D\pi}} = \mathcal{P}_\pi.$$

The reaction $\pi^- D \to nn$ is a strong interaction process so parity is conserved, $\mathcal{P}_i = \mathcal{P}_f$. If we can determine the final parity of the two neutrons then this must be equal to the intrinsic parity of the pion, so now consider the final state. Neutrons are fermions so the final state wave-function must be anti-symmetric under interchange of the two neutrons, either the spatial part is symmetric and spin part is anti-symmetric or *vice versa*. Consider first the spin part. We know from the quantum mechanical rules for addition of angular momenta that two half-integral spins, four states in all $|\uparrow\uparrow>$, $|\uparrow\downarrow>$, $|\downarrow\uparrow>$ and $|\downarrow\downarrow>$, can be combined into either a symmetric triplet or an anti-symmetric singlet (equations (24) and (25)):

$$s = 1 \qquad \begin{cases} |\uparrow\uparrow> \\ \frac{1}{\sqrt{2}}(|\downarrow\uparrow> + |\uparrow\downarrow>) \quad \text{(symmetric)} \\ |\downarrow\downarrow> \end{cases}$$

$$s = 0 \qquad \frac{1}{\sqrt{2}}(|\uparrow\downarrow> - |\downarrow\uparrow>) \qquad \text{(anti-symmetric)}.$$

When the two final state neutrons are interchanged the spin part of the wave-function is symmetric if $s = 1$ and antisymmetric if $s = 0$, so it picks up a factor $(-1)^{s+1}$. If the final orbital angular momentum of the two neutrons is $l_f = l_{nn}$ then the spatial part of the final state wave-function changes by a factor $(-1)^{l_{nn}}$ when the two neutrons are interchanged[35] so the total effect on the final state, upon interchanging the two neutrons, is to pick up a factor of $(-1)^{l_{nn}+s+1}$. Since the final state consists of two neutrons, which are fermions, the wave-function should change sign when they are interchanged, hence

$$(-1)^{l_{nn}+s+1} = -1$$

and $l_{nn} + s$ is even. Now the total final angular momentum is $\mathbf{J}_f = (\mathbf{L}_f + \mathbf{S})$, where $\mathbf{L}_f$ is the orbital angular momentum and $\mathbf{S}$ the spin angular momentum, and conservation of angular momentum dictates that this equal the total initial angular momentum, which was found above to have $J_i = |\mathbf{J}_i| = \hbar$. So conservation of angular momentum forces $J_f = |\mathbf{J}_f| = \hbar$. Using the quantum rules for angular momentum addition again, the possibilities are

$$|\mathbf{J}_f| = |\mathbf{L}_f + \mathbf{S}| = |l_{nn} - s|\hbar, (|l_{nn} - s| + 1)\hbar, \ldots, |l_{nn} + s|\hbar,$$

and only four of these possibilities are compatible with $|\mathbf{J}_f| = \hbar$ and $s = 0$ or 1, namely

$$l_{nn} = 0, s = 1 \qquad l_{nn} = 1, s = 0 \qquad l_{nn} = 1, s = 1 \qquad \text{or} \qquad l_{nn} = 2, s = 1.$$

Of these four only $(l_{nn} = 1, s = 1)$ has $l_{nn}+s$ even and so this is the only possibility allowed by fermi statistics for the two final state neutrons. In the language of atomic physics the final state neutrons are in a p-wave triplet with total angular momentum $J_f/\hbar = 1$, ${}^3P_1$. The parity of the final state is therefore

$$\mathcal{P}_f = \mathcal{P}_n^2(-1)^{l_{nn}} = (-1)^{l_{nn}} = -1,$$

from which we finally deduce the intrinsic parity of the pion from parity conservation.

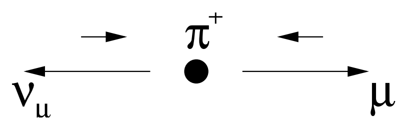
$$\mathcal{P}_\pi = \mathcal{P}_i = \mathcal{P}_f = -1.$$

The pion has intrinsic spin zero and parity -1, usually denoted by $J^P = 0^-$ in the particle physics literature. A spin zero particle is called a *scalar* and a spin zero particle with odd parity is called a *pseudo-scalar* because it transforms differently under $\mathbf{P}$ to the way you would expect, just like a pseudo-vector transforms differently to the way a vector transforms. Pions are therefore sometimes refereed to as *pseudo-scalar mesons*.

Not all of the fundamental interactions of nature are symmetric under the parity operation. It was discovered experimentally in 1957 that weak interactions violate $\mathbf{P}$ invariance. For example consider the weak decay of a π^+

$$\begin{array}{rcl} \pi^+ & \rightarrow & \mu^+ + \nu_\mu \\ & & \hookrightarrow e^+ + \nu_e + \bar{\nu}_\mu . \end{array}$$

[35] As explained in the section on parity, a spatial wave-function corresponding to orbital angular momentum l is proportional to the Legendre polynomial, $P_l(\cos\theta)$ and $\theta \to \pi - \theta$ when the two particles are interchanged. Since $\cos(\pi-\theta) = -\cos\theta$ and P_l are even or odd functions of their argument for l even or odd respectively, we have that, when the particles are interchanged $P_l(\cos(\pi-\theta)) = (-1)^l P_l(\cos\theta)$.



The muon neutrino produced in the first decay here is *always* negative helicity ($h = -1$ or left-handed) and never $h = +1$. Electron neutrinos also are always observed to have $h = -1$. Anti-neutrinos have $h = +1$. [36]

Charge Conjugation

For every fundamental particle in nature there is an anti-particle, with the same mass and opposite electric charge. The existence of anti-particles was predicted by the English physicist Paul Dirac in May 1931 and an anti-particle (a positron) was first observed only 4 months later, in September 1931. Anti-particles are a consequence of marrying the theories of relativity and quantum mechanics and their origin is related to the relativistic formula

$$E^2 = m^2c^4 + (\underline{P}.\underline{P})c^2 \qquad \Rightarrow \qquad E = \pm c\sqrt{m^2c^2 + (\underline{P}.\underline{P})}.$$

The existence of anti-particles is associated with the negative root of this equation. Think about the time dependence of a solution of Schrödinger's equation $\mathrm{e}^{-iEt/\hbar}$. If we change the sign of the energy $E \to -E$ this can be re-expressed as

$$\mathrm{e}^{-iEt/\hbar} \to \mathrm{e}^{-i(-E)t/\hbar} = \mathrm{e}^{-iE(-t)/\hbar}$$

so a negative energy particle moving forwards in time is equivalent to a positive energy particle moving backwards in time. This equivalence goes very deep in the mathematics of relativistic quantum mechanics and the American physicist Richard Feynman first suggested that anti-particles can be viewed, at least from a mathematical point of view, as being like particles moving backwards in time. We shall return to this later.

For the moment we define an operator, called *charge conjugation* $\mathbf{C}$, which interchanges particles with their anti-particles. *e.g.*

$$\mathbf{C}|\pi^+> \quad \to \quad |\pi^-> .$$

Only neutral bosons, for which their is no distinction between particles and anti-particles, can be eigenstates of $\mathbf{C}$, *e.g.*

$$\begin{aligned} \mathbf{C}|\pi^0> &= \mathcal{C}_{\pi^0}|\pi^0> \\ \mathbf{C}|\gamma> &= \mathcal{C}_{\gamma}|\gamma> . \end{aligned}$$

[36] Recently experiments that detect neutrinos coming from the Sun, and neutrinos produced in the atmosphere by cosmic rays, have produced unexpected results the simplest interpretation of which is that right-handed neutrinos might exist. Most people currently believe that this is the correct interpretation of these experiments. But the evidence is indirect and much work remains to be done before the physical properties of neutrinos are completely understood.

A photon is its own anti-particle, though **C** flips helicity — the anti-particle of a positive helicity photon is a negative helicity photon and vice-versa.

Since, from its definition, $\mathbf{C}^2 = 1$ it must be the case that the eigenvalues of **C** are $\mathcal{C} = \pm 1$. For a particle whose wave-function is an eigenstate of **C** the corresponding eigenvalue $\mathcal{C}$ is called the particle's *charge-parity*, and is another example of a quantum number.

For example in classical electromagnetism, changing the sign of all the charges in a given problem also reverses the currents so

$$\mathbf{E} \to -\mathbf{E} \qquad \text{and} \qquad \mathbf{B} \to -\mathbf{B}.$$

We therefore might expect that, at the level of wave-functions in quantum mechanics, $\mathcal{C}_\gamma = -1$. To test this note that **C** is multiplicative in the sense that the eigenvalues of a multi-particle state are products of the eigenvalues of the single particle states. So for a state consisting of n photons $|n\gamma>$, we expect

$$\mathbf{C}|n\gamma >= \mathcal{C}_\gamma^n |n\gamma >= (-1)^n |n\gamma > .$$

Now suppose a neutral pion decays electromagnetically into n photons

$$\pi^0 \to n\gamma.$$

Since electromagnetic interactions preserve charge-parity an even number n immediately implies that $\mathcal{C}_{\pi^0} = +1$, but tells is nothing about $\mathcal{C}_\gamma$. Experimentally the decay

$$\pi^0 \to 2\gamma$$

is often observed, so we conclude that indeed $\mathcal{C}_{\pi^0} = +1$. If $\mathcal{C}_\gamma = -1$ then charge parity-conservation forbids the decay of a neutral pion into an odd number of photons, a π^0 should only ever decay into an even number of photons. If $\mathcal{C}_\gamma$ were $+1$ there would be no such objection to an odd number of photons in the final state. No-one can ever say that a π^0 never decays to three photons, someone might observe such a decay tomorrow. All we can say is that, in experiments to date, all reliably observed decays have been to an even number of photons and, if an occasional π^0 does decay to three photons, the ratio of probabilities is

$$\frac{P(\pi^0 \to 3\gamma)}{P(\pi^0 \to 2\gamma)} < 3 \times 10^{-8}, \qquad (26)$$

i.e. less than about one in 30 million π^0's will decay to three photons.[37] From this we conclude that $\mathcal{C}_\gamma = -1$.

We note in passing that charge conjugation flips helicity. For example it is observed that neutrinos produced in the laboratory always have $h = -1/2$ while anti-neutrinos always have $h = 1/2$.

[37] Ratios like that in equation (26) are called *branching ratios* in particle physics.

A very important consequence of charge conjugation invariance, when it is applicable, is that, if a reaction $a + b \to c + d$ is observed, then $\bar{a} + \bar{b} \to \bar{c} + \bar{d}$ must also happen,

$$a + b \to c + d \qquad \Leftrightarrow \qquad \bar{a} + \bar{b} \to \bar{c} + \bar{d},$$

and the cross-sections are the same.

Electromagnetic and strong interactions conserve **C** but, again, weak interactions do not. It was discovered in 1964 that there are some particle decays in nature, which we shall study later, that even violate **CP** invariance — this phenomenon is called **CP** *violation*.

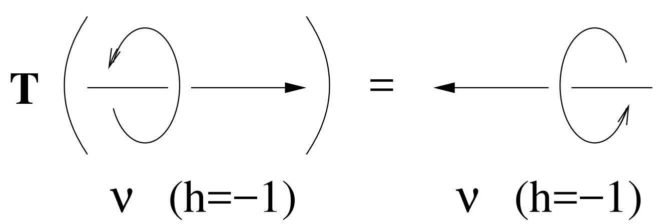
Time Reversal Symmetry

Time reversal was briefly mentioned in the above discussion of charge conjugation. We define an operator **T** that reverses $t \to -t$ in quantum states (we can certainly make such a transformation mathematically on all wave-functions even though we are not capable of reversing the direction of time on real physical clocks). In classical electromagnetism, sending $t \to -t$ will reverse the direction of all currents, but does not change the sign of electric charges, so

$$\mathbf{E} \to \mathbf{E} \qquad \text{and} \qquad \mathbf{B} \to -\mathbf{B}.$$

(As an exercise, check that Maxwell's equation are invariant under the above transformations, provided all currents are reversed at the same time. Newton's 2nd law, with a time independent conservative force, is also invariant under $t \to -t$ since it only involves a second derivative with respect to time.)

Time reversal sends all momenta $\mathbf{p} \to -\mathbf{p}$, does not affect $\mathbf{r}$ and flips spin — it therefore leaves helicity invariant. For example the effect of acting on a left-handed neutrino state with the time reversal operator is the following



Time reversal also has the effect of complex conjugating wave-functions. To see this first complex conjugate Schrödinger's equation with a real time-independent Hamiltonian, with $H^* = H$,

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi \qquad \Leftrightarrow \qquad \left(i\hbar\frac{\partial\psi}{\partial t}\right)^* = (H\psi)^* \qquad \Leftrightarrow \qquad -i\hbar\frac{\partial\psi^*}{\partial t} = H\psi^*,$$

so moving forward in time with wave-function ψ is equivalent to moving backward in time with wave-function ψ^*.

A very important consequence of time reversal invariance, when it is applicable, is that, if a reaction $a + b \to c + d$ is observed, then $c + d \to a + b$ must also happen,

$$a + b \to c + d \qquad \Leftrightarrow \qquad c + d \to a + b.$$

The cross-sections are not necessarily the same, if the masses and spins of a and b are different to those of c and d then the kinematic factors in (18) will be different, but the quantum mechanical amplitudes are related, $M_{a+b\to c+d} = M^*_{c+d\to a+b}$ for any interaction that is symmetric under time-reversal.

The strong and electromagnetic forces are symmetric under time-reversal but, since the experimental observation of **CP** violation in 1964, it has been suspected that small **T** violating effects should also exist in nature (because of the **CPT** theorem described below). Direct experimental verification of **T** violation was however not seen until 1998, in an experiment in the large particle accelerator at the European laboratory CERN near Geneva.

The CPT Theorem

There is an important theorem in the relativistic theory of quantum mechanics (the latter is too advanced for the presentation here) which states that the product **CPT** is an exact symmetry of all phenomena in nature. This is refereed to as the **CPT** *theorem*.[38] Because of the **CPT** theorem it is not necessary to assign to particles a new quantum number associated with time reversal: for **T** eigenstates it would just equal the product $\mathcal{PC}$ of the eigenvalues already assigned to **P** and **C**.

Also, since all three interactions preserve **CP**, the **CPT** theorem states that they must also preserve **T**, as indeed they are observed to do. The existence of some **CP** violating phenomena referred to above must also imply **T** violation — there must be some phenomena in nature that are not symmetric under **T** and for this reason **T** violating effects were suspected to exist 30 years before they were discovered.

The **CPT** theorem can be visualised as follows: the effect of applying first **T** then **P** and finally **C** to, say, a left-handed ($h = -1$) electron moving to the right is: first **T** sends it backwards in time, keeping the helicity the same; then **P** flips both the momentum and the helicity; finally charge conjugation turns the electron into a positron. We end up with a right-handed positron moving moving backward in time and the **CPT** theorem states that this is equivalent to the original state which was a left-handed electron moving forward time. This sequence of steps is shown below:

[38] Like all theorems it depends on certain assumptions, for example it assumed that the underlying physics is invariant under Lorentz transformations. But all the necessary assumptions lie at the core of our understanding, changing any of them would require a radical change of the way we view the world. If anyone observed a violation of **CPT** invariance we would have to change our understanding in quite a drastic way.

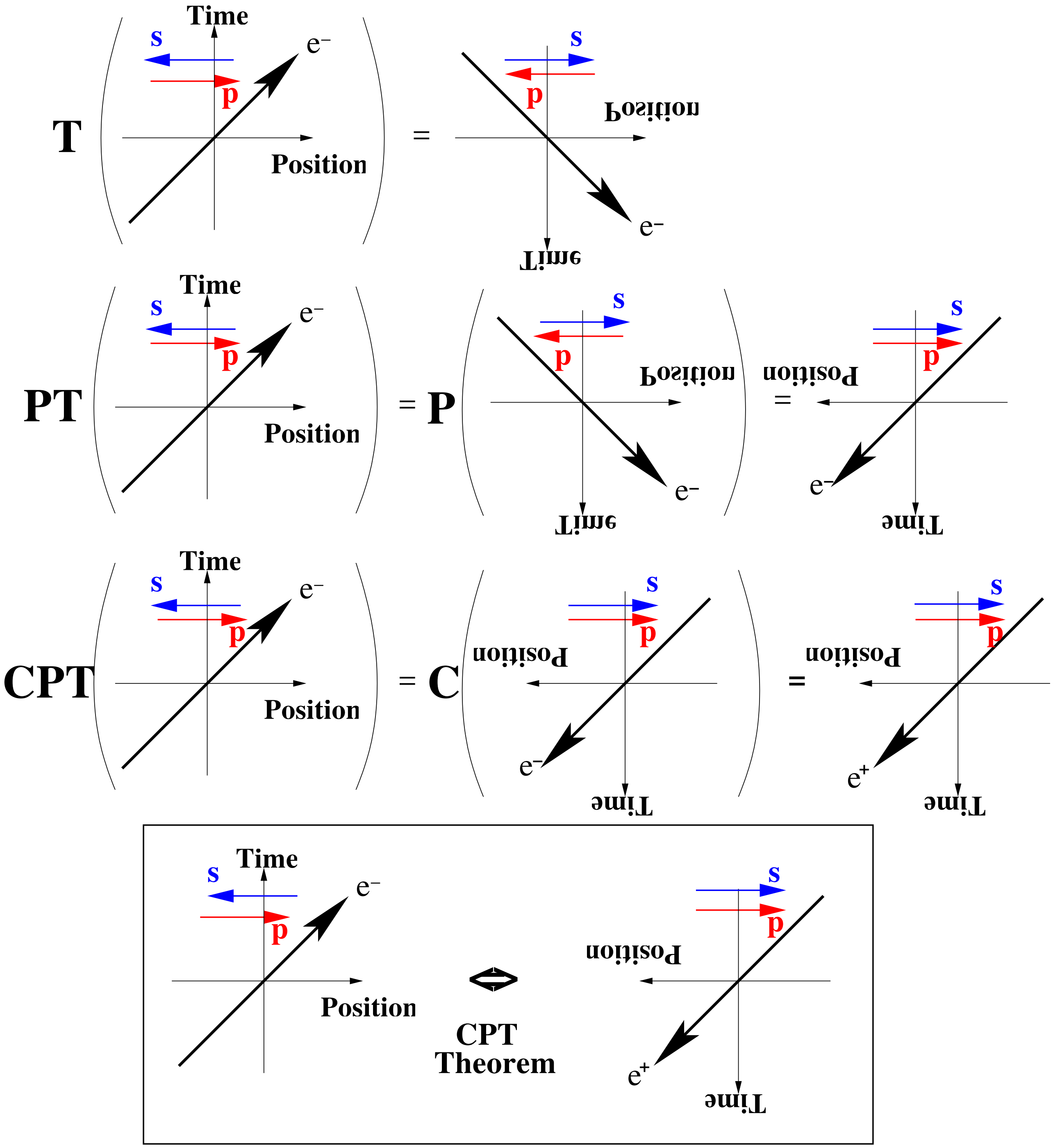


Fig. 1: The CPT Theorem. The *CPT* theorem asserts that all phenomena in nature are invariant under the combined operations of **T**, **P** and **C**. For example a left-handed electron moving forward in time is physically completely equivalent to a right-handed positron moving backwards in time.

Electric Charge

Another conserved quantum number, with which you are already familiar, is electric charge. Electric charge cannot be created or destroyed in any process. This does not mean that charged particles cannot be produced, it just means that, if they are produced, they must be produced in association with other charged particles so that the total electric charge before and after is the same. In other words, if Q_i is the total initial charge and Q_f is the total final charge, then the difference is zero:

$$\Delta Q = Q_f - Q_i = 0.$$

Thus for example

$$p + p \to p + n + \pi^+$$

is allowed by conservation of charge, but

$$p + p \to p + n + \pi^-$$

is forbidden.

Anti-particles always have the opposite electric charge to particles, so the positron e^+ has the opposite charge to the electron e^- and anti-protons $\bar{p}$ have negative charge, opposite to that of protons p.

Although we shall not go into it here, the rule of conservation of charge is associated with a continuous symmetry of the underlying dynamics. In this case it is the freedom to change the electromagnetic potential, the 4-vector A_a with $a = 0, 1, 2, 3$ labelling time and three space directions, by a derivative

$$A_a \to A_a + \partial_a \lambda.$$

This transformation leaves the electric and magnetic fields unchanged and so is an invariance of Maxwell's equations. For historical reasons this transformation is known as a *gauge symmetry* and electromagnetism is an example of a *gauge theory*. Modern approaches to the strong and weak forces are also gauge theories, similar in principle but involving more complicated transformations. Their explicit form is beyond the scope of these lectures.

Baryon Number

Strongly interacting fermions (baryons, such as protons and neutrons which are qqq states) are assigned a quantum number called *baryon number*, denoted by B.

$$B = +1 \quad \text{for} \quad p, n \qquad B = -1 \quad \text{for} \quad \bar{p}, \bar{n}.$$

Experimentally, baryon number appears to be strictly conserved by all three forces,.

$$\Delta B = B_f - B_i = 0.$$

Conservation of baryon number says that the following processes should not happen

$$p \to e^+ + \pi^0 \qquad (B_i = +1,\, B_f = 0 \quad \Rightarrow \quad \Delta B = -1) \tag{27}$$
$$p + p \to p + p + n \qquad (B_i = 2,\, B_f = 3 \quad \Rightarrow \quad \Delta B = +1). \tag{28}$$

In recent years models which unify all three forces, electromagnetism, weak and strong, into a single mathematical framework, called *Grand Unification*, have been put forward. These models allow for baryon number violation and some of them predict that the proton should be unstable under the first process above. Experimentally no-one can ever say that the process (27) never happens, all we can say is that, if it does occur in nature, then the lifetime of the proton is at least

$$\tau_p > 10^{33} \; years.$$

(Of course no-one can sit around for 10^{33} years watching a proton, waiting to see if it decays — instead this bound is achieved by watching 10^{33} protons for one year.)

Baryon number is *additive*: the total baryon number of a bunch of particles is the sum of the individual baryon numbers. Since there are three quarks in a proton quarks have $B = 1/3$. Mesons have baryon number $B = 1/3 - 1/3 = 0$ since they are $q\bar{q}$ bound states and anti-quarks have opposite baryon number to quarks.

Baryon number can be subdivided into individual quark nos., N_u, N_d, N_c, N_s, N_t and N_b, the numbers of up, down, charm, strange, top and bottom quarks respectively, where

$$B = \frac{1}{3}\big(N_u + N_d + N_c + N_s + N_t + N_b\big).$$

While strong and electromagnetic interactions conserve the individual quark numbers, weak interactions do not, but weak interactions still conserve the sum.

Lepton Number

This is similar to baryon number, but for leptons. All leptons are assigned a lepton number L which is ± 1 and lepton number has always been observed to be conserved. Anti-particles have the opposite lepton number to particles. Like baryon number, lepton number is additive. Conventionally electrons have $L_{e^-} = +1$ and positrons have $L_{e^+} = -1$. The existence of β-decay,

$$n \to p + e^- + \bar{\nu}_e,$$

implies that anti-neutrinos have lepton number $L_{\bar{\nu}_e} = -1$, from lepton number conservation. Neutrinos, like electrons, have lepton number $L_{\nu_e} = +1$.

Conservation of lepton number implies that the process

$$\bar{\nu}_e + p \to n + e^+$$

is allowed while

$$\bar{\nu}_e + n \to p + e^-$$

is forbidden.

The situation when the muon μ^- and τ-lepton are included is more complicated. There appear to be three *different* types of lepton number, one for each family $(e^-, \nu_e; e^+, \bar{\nu}_e)$, $(\mu^-, \nu_\mu; \mu^+, \bar{\nu}_\mu)$ and $(\tau^-, \nu_\tau; \tau^+, \bar{\nu}_\tau)$. The evidence for this is that electromagnetic decays like

$$\mu^\pm \to e^\pm + \gamma$$

have never been observed. Muons are always seen to decay via weak process like

$$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu.$$

Experimentally the branching ratio is

$$\frac{P(\mu^- \to e^- + \gamma)}{P(\mu^- \to e^- + \bar{\nu}_e + \nu_\mu)} < 10^{-13}$$

even though electromagnetic decays are usually much faster than weak decays. The assumption is that there is a conserved quantum number which simply forbids the decay $\mu^- \to e^- + \gamma$. It is assumed that μ and e have separately conserved lepton numbers,

$$\begin{array}{ll} L_\mu(\mu^-) = +1, & L_\mu(e^-) = 0 \\ L_e(\mu^-) = 0 & L_e(e^-) = +1. \end{array}$$

There is also a τ-lepton number,

$$L_\tau(\tau^-) = +1 \qquad L_\tau(\tau^+) = -1.$$

To summarise, here is a table of the lepton numbers of all three families of leptons:

	e^-	e^+	ν_e	$\bar{\nu}_e$	μ^-	μ^+	ν_μ	$\bar{\nu}_\mu$	τ^-	τ^+	ν_τ	$\bar{\nu}_\tau$
L_e	+1	−1	+1	−1	0	0	0	0	0	0	0	0
L_μ	0	0	0	0	+1	−1	+1	−1	0	0	0	0
L_τ	0	0	0	0	0	0	0	0	+1	−1	+1	−1

While total lepton number has never been observed to be violated there are observations that indicate violation of individual family lepton number on length scales of hundreds of kilometres. Pions are copiously produced when cosmic rays hit the upper atmosphere (mostly by protons hitting oxygen or nitrogen nuclei). As they travel toward the surface of the Earth the pions decay predominantly through the processes

$$\begin{array}{lll} \pi^+ & \longrightarrow & \mu^+ + \nu_\mu \\ & & \hookrightarrow \quad e^+ + \nu_e + \bar{\nu}_\mu \end{array}$$

or

$$\begin{array}{lll} \pi^- & \longrightarrow & \mu^- + \bar{\nu}_\mu \\ & & \hookrightarrow \quad e^- + \bar{\nu}_e + \nu_\mu \ . \end{array}$$

In either case, if we do not differentiate between neutrinos and anti-neutrinos, we expect the ratio of the number of muon neutrinos to electron neutrinos to be 2. The observed ratio is closer to 1 than 2, indicating that muon neutrinos are disappearing in the journey towards the Earth's surface. The consensus at the moment is that they are probably turning into tau neutrinos on the way, $\nu_\mu \to \nu_\tau$. The total number of all neutrinos is still believed to be conserved.

Recent measurements on neutrinos emitted by the Sun, as nuclear reactions burn material in the solar core, and also on neutrinos produced in the atmosphere by cosmic rays, indicate that family lepton number is not exactly conserved and

$$\nu_e \to \nu_\mu$$

is possible over very large distances (thousands of kilometres).

However, even in the solar neutrino experiments, it is still the case that the total lepton number

$$L_{Total} = L_e + L_\mu + L_\tau$$

appears to be conserved.

Strangeness

Two kinds of hadrons that we have not yet met are a neutral meson called a *kaon* K^0, with mass $m_{K^0} = 498\ MeV/c^2$, and a neutral baryon called a Λ-hyperon,[39] Λ with mass $m_\Lambda = 1116\ MeV/c^2$. When these particles where first observed they appeared to be produced by strong interactions and always in pairs, in reactions like

$$\pi^- + p \to K^0 + \Lambda.$$

(This process preserves baryon number if the meson K^0 is assigned baryon number $B = 0$ and the baryon Λ is assigned $B = 1$.) Despite the fact that these particles are strongly interacting, their decays are surprisingly slow,

$$\Lambda \to \pi^- + p \qquad (\tau_\Lambda \approx 10^{-10}\ s)$$

$$K^0 \to \begin{cases} \pi^+ + \pi^- \\ \pi^0 + \pi^0 \end{cases} \qquad (\tau_{K^0} \approx 10^{-10}\ s).$$

Although $10^{-10}\ s$ is very short on everyday timescales it is extremely long compared to typical strong interaction times which are $\sim 10^{-23}\ s$. In fact $10^{-10}\ s$ is a typical weak interaction time scale. It was proposed, by Gell-Mann and Nishijima, that there is a new quantum number which is conserved by strong interactions but not conserved by weak interactions that would protect the K^0 and Λ against strong decays but allow them to decay weakly. This mysterious new quantum number was called *strangeness* S and the above decays can be accounted for by demanding that

$$S_\pi = S_p = 0, \qquad S_{K^0} = +1, \qquad S_\Lambda = -1,$$

[39] *Hyperon* because it has some properties like a heavy version of the neutron, a hyper-neutron. The kaon and the Λ-hyperon were discovered in the same year as the pion, 1947.

Thus $\Delta S = 0$ in all strong interactions: K^0's and Λ's can only be produced in pairs — a phenomenon known as *associated production*. It is observed that even in weak interactions the strangeness cannot change arbitrarily but seems to obey the rule

$$|\Delta S| = 1.$$

For example

$$\begin{array}{rcc} \bar{K}^0 + n \to \Lambda + \pi^0 & \text{strong interaction,} & \Delta S = 0 \\ K^0 \to \pi^+ + \pi^- & \text{weak decay,} & \Delta S = -1 \\ \Lambda \to p + \pi^- & \text{weak decay,} & \Delta S = +1 \\ \bar{p} + \Lambda \to K^0 + \pi^- & \text{forbidden,} & \Delta S = 2. \end{array}$$

Isospin

The proton and the neutron have very similar masses, $m_p = 938\ MeV/c^2$ and $m_n = 939\ MeV/c^2$. In 1932 Heisenberg suggested that the only real difference between a proton and a neutron is their electric charge and, if the electromagnetic force were ignored, they are essentially the same particle, at least as far as the strong force is concerned. If the electric charge were somehow "turned off" it would be natural to combine the proton wave-function ψ_p and neutron wave-function ψ_n into a doublet

$$\Psi_N = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}. \tag{29}$$

This is reminiscent of spin one-half particles, such as electrons, being put into a doublet of spin-up and spin-down, $\Psi = \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix}$, only the introduction of a magnetic field spoils the rotational symmetry and distinguishes between the 'up' and the 'down' spin-states. Just as for electron spin, we can 'rotate' the ψ_p and ψ_n states with a 2×2 matrix

$$\Psi_N \to U \Psi_N$$

where

$$\begin{aligned} U = \mathrm{e}^{-i(\alpha/2)(\underline{n}.\underline{\sigma})} &= \cos(\alpha/2)\,\mathbf{1} - i(\underline{n}.\underline{\sigma})\sin(\alpha/2) \\ &= \begin{pmatrix} \cos(\alpha/2) - in_3\sin(\alpha/2) & -(n_2 + in_1)\sin(\alpha/2) \\ (n_2 - in_1)\sin(\alpha/2) & \cos(\alpha/2) + in_3\sin(\alpha/2) \end{pmatrix} \end{aligned}$$

is a unitary matrix, $U^\dagger = U^{-1}$, with $\underline{n} = (n_1, n_2, n_3)$ a unit vector in an abstract 3-dimensional space.[40] The three matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

[40] The notation $\underline{n}.\underline{\sigma}$ here stands for $\sum_{i=1}^{3} n_i\sigma_i$, the exponential of a matrix is understood to mean $e^M = \sum_{k=0}^{\infty} \frac{1}{k!} M^k$.

are the three Pauli matrices familiar from quantum mechanics. It can easily be checked that they have the property

$$\sigma_i \sigma_j = \delta_{ij} + i \sum_{k=1}^{3} \epsilon_{ijk} \sigma_k$$

where

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = -\epsilon_{213} = -\epsilon_{321} = -\epsilon_{132} = +1$$

is the totally antisymmetric symbol with three indices and values ± 1 when all three indices differ and zero if any two indices are the same.

Thus for example

$$i\sigma_2 \Psi_N = \begin{pmatrix} \psi_n \\ -\psi_p \end{pmatrix}$$

(which corresponds to $\alpha = -\pi$, $\mathbf{n} = (0, 1, 0)$, a rotation through $-\pi$ about the second axis) interchanges the proton and neutron wave-functions. Heisenberg postulated that this interchange should be a symmetry of the strong interactions, but this is only an approximate symmetry — it is broken by electromagnetic interactions.

In the same way, for the two components of electron spin

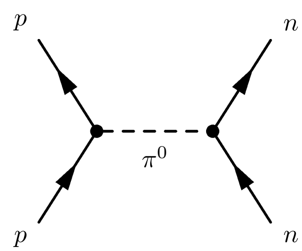
$$i\sigma_2 \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix} = \begin{pmatrix} \psi_\downarrow \\ -\psi_\uparrow \end{pmatrix}$$

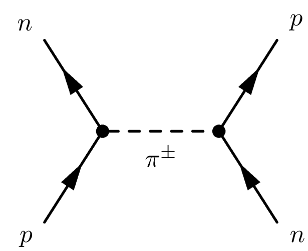
is a symmetry in quantum mechanics when there is no magnetic field.

Heisenberg's symmetry is called *isospin*, in analogy with ordinary spin in quantum mechanics. Mathematically the set of all matrices U with the above form is the set of all unitary 2×2 matrices with the special property that $\det U = 1$. For this reason this set is called the *special* unitary 2×2 matrices, or $SU(2)$ for short.[41] Mathematically there is an exact parallel between isospin and angular momentum in quantum mechanics. The nucleon doublet (29) is said to have isospin one-half, $I = 1/2$, just as the electron doublet has spin one-half $s = 1/2$.

There are other isospin multiplets with more than two components. In general the number of components in a multiplet with total isospin I is $2I + 1$, just as the number of components in a spin j state in quantum mechanics is $2j+1$. The idea that, for the purposes of strong interactions, there is a symmetry between protons and neutrons immediately tells us something about pions. If protons and neutrons are to be indistinguishable, then the Yukawa exchange of pions between a proton and a neutron does not distinguish between protons and neutrons. So the following two processes are identical as far as the strong interactions are concerned:

41 The set of all such matrices constitutes what is called a *group* in mathematics.





The only difference between the above two reactions is in their electromagnetic properties, which are reflected in the difference between the masses of the neutral and charged pions: $m_{\pi^0} = 135\ MeV/c^2$ and $m_{\pi^\pm} = 140\ MeV/c^2$. In fact the three pions fit into an isospin triplet $I = 1$

$$\Pi = \begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix}.$$

The existence of the π^0 was first predicted using isospin symmetry — Yukawa's 1935 prediction did not include three different kinds of pions. This is the second example of a new particle being predicted from a symmetry principle before it was observed, the neutrino was predicted by Pauli in 1930 on the basis of energy conservation (discovered in 1956) and the π^0 was predicted by Kemmer in 1937 on the basis of isospin symmetry (discovered in 1947).

Mathematically isospins are added in just the same way as angular momentum is added in quantum mechanics. Two particles with $I = 1/2$ and wave-functions $\psi_i(x_1)$ and $\psi_j(x_2)$, with $i, j = 1, 2$, can be combined into symmetric and anti-symmetric combinations,

$$\psi_{ij}(x_1, x_2) = \begin{cases} \psi_i(x_1)\psi_j(x_2) + \psi_j(x_1)\psi_i(x_2) : & I = 1 \\ \psi_i(x_1)\psi_j(x_2) - \psi_j(x_1)\psi_i(x_2) : & I = 0, \end{cases}$$

where the symmetric combination is an $I = 1$ triplet and the anti-symmetric combination is an $I = 0$ singlet. In general combining two states with isospins I_1 and I_2 can lead to states with isospins ranging from $|I_1 - I_2|$ up to $|I_1 + I_2|$ in integer steps:

$$|I_1 - I_2|, |I_1 - I_2| + 1, \ldots, |I_1 + I_2| - 1, |I_1 + I_2|.$$

Since this is not the same as ordinary multiplication we often use the notation $I_1 \otimes I_2$ for the combination and write the result as a sum with the special symbol $\oplus$, thus

$$I_1 \otimes I_2 = |I_1 - I_2| \oplus (|I_1 - I_2| + 1) \oplus \ldots \oplus (|I_1 + I_2| - 1) \oplus |I_1 + I_2|.$$

For example π-mesons are bound states of up and down quarks and anti-quarks. Now the u and d quarks themselves form an isospin doublet

$$\mathbf{q} := \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} u \\ d \end{pmatrix},$$

where the notation $q_1 = u$ and $q_2 = d$ is used. Combining two doublets ($I = 1/2$) gives a singlet ($I = 0$) and a triplet ($I = 1$),

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1.$$

Just like electron spin, the singlet is the anti-symmetric combination and the triplet is the symmetric combination of the two $I = 1/2$ indices. The pion wave-function is of the form $q_i(x_1)\bar{q}_j(x_2)$ and we can form the symmetric and anti-symmetric combinations

$$\psi_{ij}(x_1, x_2) = \begin{cases} q_i(x_1)\bar{q}_j(x_2) + q_j(x_1)\bar{q}_i(x_2), & I = 1 \\ q_i(x_1)\bar{q}_j(x_2) - q_j(x_1)\bar{q}_i(x_2), & I = 0. \end{cases}$$

The triplet we have already seen is the pion triplet above, but the isospin $I = 0$ singlet is a new particle, called the η-meson, which has mass $m_\eta = 547\ MeV/c^2$ and usually decays to either two photons or three pions,

$$\eta \to \begin{cases} 2\gamma & 40\% \text{ of the time} \\ 3\pi^0 & 33\% \text{ of the time} \end{cases}$$

with a life-time of $\tau_\eta = 5.5 \times 10^{-19}\ s$. Notice that the η mass is very different from the pion mass. Isospin symmetry demands that particles in the same multiplet should have the same mass. There is nothing to stop particles in different multiplets having very different masses. The pions and the η, for example, are in different isospin multiplets and there is nothing saying that their masses should be the same or even similar.

As a second example of the use of isospin symmetry consider a baryon state, like a proton or a neutron, consisting of three quarks, $q_i q_j q_k$. Using the rules for combination

$$\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = (0 \oplus 1) \otimes \frac{1}{2} = \left(0 \otimes \frac{1}{2}\right) \oplus \left(1 \otimes \frac{1}{2}\right) = \frac{1}{2} \oplus \left(\frac{1}{2} \oplus \frac{3}{2}\right) = \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{3}{2}.$$

There are therefore two doublets and a quadruplet, $I = 3/2$ with $2I + 1 = 4$. One of the doublets is the proton-neutron doublet which started off this discussion of isospin. The quadruplet consists of four baryons labelled $(\Delta^{++}, \Delta^{+}, \Delta^{0}, \Delta^{-})$ with the quark content (uuu, uud, udd, ddd). These four particles all have very similar masses at $m_\Delta = 1232\ MeV/c^2$ (we met the Δ particle before in the discussion on resonances). What about the second doublet? That does not actually exist in nature because it would require putting two fermions in the same quantum state and that is forbidden by the exclusion principle (a full discussion of this point must await the introduction of colour and QCD later).

In analogy with angular momentum in quantum mechanics the isospin I is like total angular momentum, but there is also an isospin analogue of the third component of angular momentum which will be denoted by I_3 here. For a given I labelling a multiplet, the individual particles in that multiplet are characterised by their values of I_3. More accurately I_3 is a linear operator on a vector space and particles are identified with eigenvalues of I_3, characterised by their eigenvalues. For a given I

$$I_3|\psi >= \lambda|\psi >, \qquad \lambda = -I, -I+1, \ldots, I-1, I$$

(often we do not distinguish between I_3 and its eigenvalues, just denoting the eigenvalues themselves by I_3). For $I = 1/2$, which could be proton-neutron or up-down quarks for example, $I_3 = \pm 1/2$ and

$$I_3|p>=\frac{1}{2}|p>, \quad I_3|n>=-\frac{1}{2}|n>, \quad I_3|u>=\frac{1}{2}|u>, \quad I_3|d>=-\frac{1}{2}|d>$$

(in this case the operator I_3 is the Pauli matrix $\frac{1}{2}\sigma_3$ which has eigenvalues $\pm 1/2$). For $I = 1$, pions for example, $I_3 = \pm 1, 0$

$$I_3|\pi^{\pm}>=\pm 1, \qquad I_3|\pi^0>=0.$$

Notice that for protons and neutrons the electric charge is

$$Q_N = I_3 + \frac{1}{2}$$

while for u and d quarks it is

$$Q_{\mathbf{q}} = I_3 + \frac{1}{6}$$

and for pions it is

$$Q_\pi = I_3.$$

Referring back to the discussion on baryon number, these can all be written as

$$Q = I_3 + \frac{B}{2}.$$

When strange quarks are included, they are assigned isospin $I = 0$ and this formula is modified to

$$Q = I_3 + \left(\frac{B+S}{2}\right), \tag{30}$$

this is called the *Gell-Mann – Nishijima* relation.

For example

$$\begin{aligned}
p, &\qquad I_3 = 1/2, B = 1, S = 0, \quad Q = +1 \\
n, &\qquad I_3 = -1/2, B = 1, S = 0, \quad Q = 0 \\
\pi^{\pm}, &\qquad I_3 = \pm 1, B = 0, S = 0, \quad Q = \pm 1 \\
\Lambda, &\qquad I_3 = 0, B = 1, S = -1, \quad Q = 0 \\
K^0, &\qquad I_3 = -1/2, B = 0, S = +1, \quad Q = 0.
\end{aligned}$$

The isospin of multi-particle states can be calculating by adding the isospins of the individual particles and the rules for adding isospin are identical to those of angular momentum: combining a state of total isospin I' and third component I_3' with one of total isospin I'' and third component I_3'' gives a linear superposition of states with total isospin

$$I = |I' - I''|, |I' - I''| + 1, \ldots, I' + I'' - 1, I' + I'' \tag{31}$$

and third component $I_3 = I_3' + I_3''$,

$$|I', I''; I_3', I_3'' >= \sum_{I=|I'-I''|}^{I'+I''} C^{I;I_3}_{I',I'';I_3',I_3''} |I; I_3 >,$$

where $C^{I;I_3}_{I',I'';I_3',I_3''}$ are the same Clebsch-Gordon coefficients as for angular momentum and are non-zero only when $I_3 = I_3' + I_3''$. As an example of these rules we next consider the scattering of a pion off a proton or a neutron.

Isospin of Pion-Nucleon system.

Isospin symmetry can be used to derive quantitative predictions concerning strong interaction cross-sections. As an example we shall examine pion-nucleon scattering. The pions have total isospin $I_\pi = 1$ and the nucleons ($N = \{p, n\}$) have $I_N = 1/2$ and the sum of these is in general a mixture of total isospin $I = 1 + \frac{1}{2} = \frac{3}{2}$ and $I = 1 - \frac{1}{2} = \frac{1}{2}$, from equation (31). The initial and final states are vectors in a 6-dimensional Hilbert space and an orthonormal basis is

$$\Big|1, \frac{1}{2}; 1, \frac{1}{2} > = |\pi^+ p >, \quad \Big|1, \frac{1}{2}; 1, -\frac{1}{2} >= |\pi^+ n >, \quad \Big|1, \frac{1}{2}; 0, \frac{1}{2} >= |\pi^0 p >,$$
$$\Big|1, \frac{1}{2}; 0, -\frac{1}{2} > = |\pi^0 n >, \Big|1, \frac{1}{2}; -1, \frac{1}{2} >= |\pi^- p >, \Big|1, \frac{1}{2}; -1, -\frac{1}{2} >= |\pi^- n > . \quad (32)$$

There are thus 6 possible initial states and 6 final states, so the isospin part of thequantum mechanical amplitude for pion-nucleon scattering M is a 6×6 matrix. However conservation of charge forces many of the entries to vanish, for example $|\pi^+ p >\to |\pi^0 n >$ is forbidden by charge conservation so $M_{\pi^0 n, \pi^+ p} = 0$, only 10 of the 36 entries in M are non-zero, 26 must vanish due to charge conservation. The hypothesis of isospin symmetry puts even stronger constraints on M. The total isospin of any initial or final state is a combination of $I = 3/2$ and $I = 1/2$ and isospin symmetry requires that there are only two independent amplitudes, one for $I = 3/2$ and one for $I = 1/2$. In the total isospin basis, namely

$$\Big|\frac{3}{2}; \frac{3}{2} >, \quad \Big|\frac{3}{2}; \frac{1}{2} >, \quad \Big|\frac{3}{2}; -\frac{1}{2} >, \quad \Big|\frac{3}{2}; -\frac{3}{2} >, \quad \Big|\frac{1}{2}; \frac{1}{2} >, \quad \Big|\frac{1}{2}; -\frac{1}{2} >, \quad (33)$$

M is diagonal and involves only two complex numbers (which depend on the energy) one associated with $I = 3/2$ and one with $I = 1/2$, $M_{3/2}(E)$ and $M_{1/2}(E)$,

$$M = \begin{pmatrix} M_{3/2} & 0 & 0 & 0 & 0 & 0 \\ 0 & M_{3/2} & 0 & 0 & 0 & 0 \\ 0 & 0 & M_{3/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{3/2} & 0 & 0 \\ 0 & 0 & 0 & 0 & M_{1/2} & 0 \\ 0 & 0 & 0 & 0 & 0 & M_{1/2} \end{pmatrix} = \begin{pmatrix} M_{3/2}\mathbf{1}_{4\times 4} & 0 \\ 0 & M_{1/2}\mathbf{1}_{2\times 2} \end{pmatrix}. \quad (34)$$

To calculate the amplitude for particular processes, such as

$$
\begin{array}{ccl}
\pi^+ p & \to & \pi^+ p \\
\pi^- n & \to & \pi^- n \\
\pi^+ n & \to & \begin{cases} \pi^+ n \\ \pi^0 p \end{cases} \\
\pi^- p & \to & \begin{cases} \pi^0 n \\ \pi^- p. \end{cases} \\
\pi^0 p & \to & \begin{cases} \pi^+ n \\ \pi^0 p \end{cases} \\
\pi^0 n & \to & \begin{cases} \pi^- p \\ \pi^0 n. \end{cases} \quad ,
\end{array}
\tag{35}
$$

in terms of $M_{3/2}$ and $M_{1/2}$ we need to switch from the basis (33) to (32), which requires Clebsch-Gordon coefficients. Note the power of isospin symmetry, ten amplitudes (35) can be calculated in terms of only two unknown number $M_{3/2}$ and $M_{1/2}$.

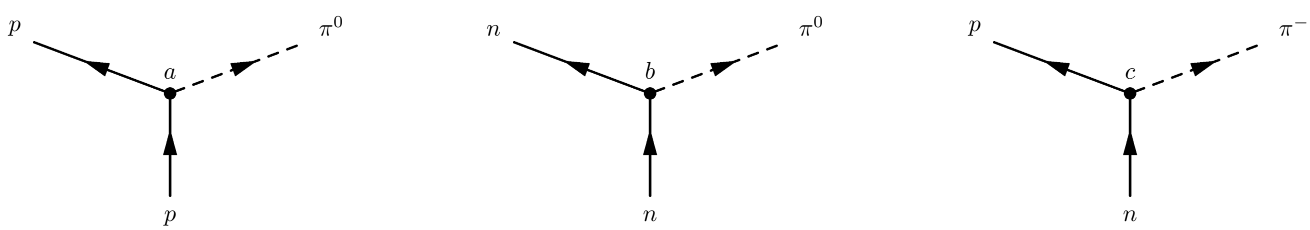
We shall now derive the relevant Clebsch-Gordon coefficients. First note that $\Big| 1, \frac{1}{2}; 1, \frac{1}{2} >$ has $I_3 = \frac{3}{2}$ and therefore must have total isospin $I = \frac{3}{2}$, so we can chose

$$\Big| 1, \frac{1}{2}; 1, \frac{1}{2} > = \Big| \ \frac{3}{2}; \frac{3}{2} >$$

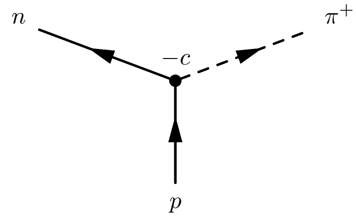
and similarly,

$$\Big| 1, \frac{1}{2}; 1, \frac{1}{2} > = \Big| \ \frac{3}{2}; \frac{3}{2} > .$$

Now we analyse $I_3 = \pm\frac{1}{2}$. There are three possible pion-nucleon couplings, which we shall denote by a, b and c

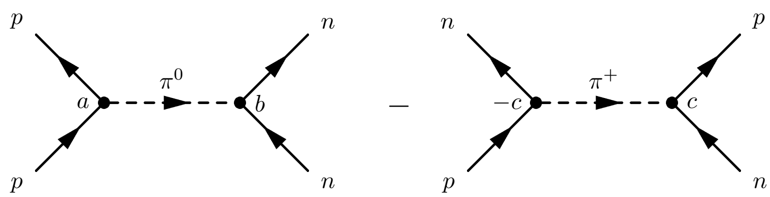


A fourth possibility



must have the opposite charge to $n \to p\pi^-$ because (n, p, π^-) have opposite third component of isospin to (p, n, π^+). The couplings a, b and c are real numbers, analogous to electric charge, but they are not independent — they are related by isospin symmetry as we shall now show.

From these couplings we can construct three amplitudes for $N-N$ scattering via single pion exchange:



where the last amplitude, for $pn \to pn$, is a linear combination of an interchange involving the product ab minus one involving $-c^2$ (the reason we subtract these two amplitudes to get the total $pn \to pn$ amplitude is that protons and neutrons are Fermions and Fermi statistics requires that there is an additional minus sign if the two final state particle are interchanged).

Isospin symmetry requires that all 3 amplitudes are actually the same (weak and electromagnetic interactions are being ignored here of course) which implies[42]

$$a^2 = b^2 = ab + c^2. \tag{36}$$

It is known that $c \neq 0$, since the charge exchange process is observed in experiments, so a solution requires

$$a = -b, \qquad c = \pm\sqrt{2}a = \mp\sqrt{2}b.$$

The above diagrams represent the fact that quantum mechanically a proton can dissociate into virtual $(\pi^0 p)$ or $(\pi^+ n)$ pairs, mathematically

$$\Big|\frac{1}{2};\frac{1}{2}>= a\Big|1,\frac{1}{2};0,\frac{1}{2}> -c\Big|1,\frac{1}{2};1,-\frac{1}{2}>, \tag{37}$$

and a neutron can dissociate into virtual $(\pi^0 n)$ or $(\pi^- p)$ pairs,

$$\Big|\frac{1}{2};-\frac{1}{2}>= b\Big|1,\frac{1}{2};0,-\frac{1}{2}> +c\Big|1,\frac{1}{2};-1,\frac{1}{2}> . \tag{38}$$

The states (37) and (38) must be orthonormal, as are the basis vectors on the right-hand sides, which implies, using (36),

$$1 = a^2 + c^2 = 3a^2 \qquad \Rightarrow \qquad a = \pm\frac{1}{\sqrt{3}}, \quad c = \pm\sqrt{\frac{2}{3}}$$

$$1 = b^2 + c^2 = 3b^2 \qquad \Rightarrow \qquad b = \pm\frac{1}{\sqrt{3}}, \quad c = \mp\sqrt{\frac{2}{3}}.$$

[42] The three amplitudes constitute the total isospin I=1 triplet.

The signs are a matter of convention and we shall use

$$b = \frac{1}{\sqrt{3}}, \quad a = -\frac{1}{\sqrt{3}} \qquad \Rightarrow \qquad c = -\frac{2}{\sqrt{3}},$$

so

$$\begin{aligned} \Big|\frac{1}{2};\frac{1}{2}> &= -\frac{1}{\sqrt{3}}\Big|1,\frac{1}{2};0,\frac{1}{2}> +\frac{2}{\sqrt{3}}\Big|1,\frac{1}{2};1,-\frac{1}{2}>, \\ \Big|\frac{1}{2};-\frac{1}{2}> &= \frac{1}{\sqrt{3}}\Big|1,\frac{1}{2};0,-\frac{1}{2}> -\frac{2}{\sqrt{3}}\Big|1,\frac{1}{2};-1,\frac{1}{2}> . \end{aligned} \tag{39}$$

The Clebsch-Gordon coefficients that we require are obtained by calculating A, B, C and D in

$$\begin{aligned} \Big|\frac{3}{2};\frac{1}{2}> &= A\Big|1,\frac{1}{2};0,\frac{1}{2}> +B\Big|1,\frac{1}{2};1,-\frac{1}{2}> \\ \Big|\frac{3}{2};-\frac{1}{2}> &= C\Big|1,\frac{1}{2};0,-\frac{1}{2}> +D\Big|1,\frac{1}{2};-1,\frac{1}{2}> \end{aligned}$$

(again we can choose phases so that A, B, C and D are real). Using orthonormality of the bases again and (39) gives

$$\begin{aligned} <\frac{3}{2};\frac{1}{2}\Big|\frac{3}{2};\frac{1}{2}> = 1 \qquad &\Rightarrow \qquad A^2+B^2 = 1 \\ <\frac{3}{2};-\frac{1}{2}\Big|\frac{3}{2};-\frac{1}{2}> = 1 \qquad &\Rightarrow \qquad C^2+D^2 = 1 \end{aligned}$$

and

$$\begin{aligned} <\frac{1}{2};\frac{1}{2}\Big|\frac{3}{2};\frac{1}{2}> = 0 \qquad &\Rightarrow \qquad -\frac{A}{\sqrt{3}}+\sqrt{\frac{2}{3}}B = 0 \qquad \Rightarrow \qquad A = \sqrt{2}B \\ <\frac{1}{2};-\frac{1}{2}\Big|\frac{3}{2};-\frac{1}{2}> = 0 \qquad &\Rightarrow \qquad \frac{C}{\sqrt{3}}-\sqrt{\frac{2}{3}}D = 0 \qquad \Rightarrow \qquad C = \sqrt{2}D, \end{aligned}$$

giving

$$A = \pm\sqrt{\frac{2}{3}}, \qquad B = \pm\sqrt{\frac{1}{3}}, \qquad C = \pm\sqrt{\frac{2}{3}}, \qquad D = \pm\sqrt{\frac{1}{3}}.$$

By convention we take the upper sign in each case,

$$A = \sqrt{\frac{2}{3}}, \qquad B = \sqrt{\frac{1}{3}}, \qquad C = \sqrt{\frac{2}{3}}, \qquad D = \sqrt{\frac{1}{3}}.$$

In summary

$$\begin{aligned} \Big|\frac{3}{2};\frac{3}{2}> &= \Big|\ 1,\frac{1}{2};1,\frac{1}{2}> \\ \Big|\frac{3}{2};\frac{1}{2}> &= \sqrt{\frac{2}{3}}\Big|1,\frac{1}{2};0,\frac{1}{2}> +\sqrt{\frac{1}{3}}\Big|1,\frac{1}{2};1,-\frac{1}{2}> \\ \Big|\frac{3}{2};-\frac{1}{2}> &= \sqrt{\frac{2}{3}}\Big|1,\frac{1}{2};0,-\frac{1}{2}> +\sqrt{\frac{1}{3}}\Big|1,\frac{1}{2};-1,\frac{1}{2}> \\ \Big|\frac{3}{2};-\frac{3}{2}> &= \Big|\ 1,\frac{1}{2};-1,-\frac{1}{2}>, \end{aligned}$$

$$\Big|\frac{1}{2};\frac{1}{2}>=-\sqrt{\frac{1}{3}}\Big|1,\frac{1}{2};0,\frac{1}{2}>+\sqrt{\frac{2}{3}}\Big|1,\frac{1}{2};1,-\frac{1}{2}>$$

$$\Big|\frac{1}{2};-\frac{1}{2}>=\sqrt{\frac{1}{3}}\Big|1,\frac{1}{2};0,-\frac{1}{2}>-\sqrt{\frac{2}{3}}\Big|1,\frac{1}{2};-1,\frac{1}{2}>$$

Inverting these gives

$$\Big|1,\frac{1}{2};1,\frac{1}{2}>=\Big|\frac{3}{2};\frac{3}{2}>$$

$$\Big|1,\frac{1}{2};1,-\frac{1}{2}>=\sqrt{\frac{1}{3}}\Big|\frac{3}{2};\frac{1}{2}>+\sqrt{\frac{2}{3}}\Big|\frac{1}{2};\frac{1}{2}>$$

$$\Big|1,\frac{1}{2};0,\frac{1}{2}>=\sqrt{\frac{2}{3}}\Big|\frac{3}{2};\frac{1}{2}>-\sqrt{\frac{1}{3}}\Big|\frac{1}{2};\frac{1}{2}>$$

$$\Big|1,\frac{1}{2};0,-\frac{1}{2}>=\sqrt{\frac{2}{3}}\Big|\frac{3}{2};-\frac{1}{2}>+\sqrt{\frac{1}{3}}\Big|\frac{1}{2};-\frac{1}{2}>$$

$$\Big|1,\frac{1}{2};-1,\frac{1}{2}>=\sqrt{\frac{1}{3}}\Big|\frac{3}{2};-\frac{1}{2}>-\sqrt{\frac{2}{3}}\Big|\frac{1}{2};-\frac{1}{2}>$$

$$\Big|1,\frac{1}{2};-1,-\frac{1}{2}>=\Big|\frac{3}{2};-\frac{3}{2}>,$$

from which we can read off the Clebsch-Gordon coefficients for the decomposition

$$\Big|1;I_3'>\otimes\Big|\frac{1}{2};I_3''>=\Big|1,\frac{1}{2};I_3',I_3''>=\sum_{I_3=-3/2}^{3/2}C^{I;I_3}_{I',I'';I_3',I_3''}\Big|\frac{3}{2};I_3>+\sum_{I_3=-1/2}^{1/2}C^{I;I_3}_{I',I'';I_3',I_3''}\Big|\frac{1}{2};I_3>.$$

In tabular form,

	$I=3/2$				$I=1/2$	
$I_3=$	$3/2$	$1/2$	$-1/2$	$-3/2$	$1/2$	$-1/2$
(π^+p) $\vert 1,\frac{1}{2};1,\frac{1}{2}>$	1	0	0	0	0	0
(π^+n) $\vert 1,\frac{1}{2};1,-\frac{1}{2}>$	0	$\sqrt{\frac{1}{3}}$	0	0	$\sqrt{\frac{2}{3}}$	0
(π^0p) $\vert 1,\frac{1}{2};0,\frac{1}{2}>$	0	$\sqrt{\frac{2}{3}}$	0	0	$-\sqrt{\frac{1}{3}}$	0
(π^0n) $\vert 1,\frac{1}{2};0,-\frac{1}{2}>$	0	0	$\sqrt{\frac{2}{3}}$	0	0	$\sqrt{\frac{1}{3}}$
(π^-p) $\vert 1,\frac{1}{2};-1,\frac{1}{2}>$	0	0	$\sqrt{\frac{1}{3}}$	0	0	$-\sqrt{\frac{2}{3}}$
(π^-n) $\vert 1,\frac{1}{2};-1,-\frac{1}{2}>$	0	0	0	1	0	0

We can now use this information to calculate ratios of cross-sections. Consider for example the following three reactions:

i) $\pi^+p \longrightarrow \pi^+p$

ii) $\pi^-p \longrightarrow \pi^-p$

iii) $\pi^-p \longrightarrow \pi^0n.$

In terms of isospin $i)$ is

$$\Big|1,\frac{1}{2};1,\frac{1}{2}> \quad\rightarrow\quad \Big|\,1,\frac{1}{2};1,\frac{1}{2}>$$

and can only proceed through the $I = 3/2$ channel, thus the cross-section

$$\sigma_{\pi^+p\rightarrow\pi^+p} = K\Big|<1,\frac{1}{2};1,\frac{1}{2}\Big|M\Big|1,\frac{1}{2};1,\frac{1}{2}>\Big|^{\,2} = K|M_{3/2}|^2,$$

where K is a kinematical factor, depending on the 4-momenta of the incoming and outgoing particles.

Reaction $ii)$ is

$$\Big|1,\frac{1}{2};-1,\frac{1}{2}> \quad\rightarrow\quad \Big|\,1,\frac{1}{2};-1,\frac{1}{2}>$$

and this is a combination of $I = 3/2$ and $I = 1/2$ and so involves both $M_{3/2}$ and $M_{1/2}$. Using the table above and the form of M in (34)

$$\begin{aligned}
&<1,\frac{1}{2};-1,\frac{1}{2}\Big|M\Big|1,\frac{1}{2};-1,\frac{1}{2}> \\
&\qquad = \left\{\sqrt{\frac{1}{3}}<\frac{3}{2};-\frac{1}{2}\Big| - \sqrt{\frac{2}{3}}<\frac{1}{2};-\frac{1}{2}\Big|\right\} M \left\{\sqrt{\frac{1}{3}}\Big|\frac{3}{2};-\frac{1}{2}> - \sqrt{\frac{2}{3}}\Big|\frac{1}{2};-\frac{1}{2}>\right\} \\
&\qquad = \frac{1}{3}M_{3/2} + \frac{2}{3}M_{1/2}.
\end{aligned}$$

Note that the matrix M in (34) is written in the $\Big|I;I_3>$ basis and not the $\Big|I',I'';I_3',I_3''>$ basis, which is why the table is essential in the derivation. We deduce that

$$\sigma_{\pi^-p\rightarrow\pi^-p} = \frac{1}{9}K\Big|M_{3/2} + 2M_{1/2}\Big|^2,$$

where K is the same kinematic factor as in $i)$ if the pions and the protons have the same energy and momentum.

Reaction $iii)$ is

$$\Big|1,\frac{1}{2};-1,\frac{1}{2}> \quad\rightarrow\quad \Big|\,1,\frac{1}{2};0,-\frac{1}{2}>$$

and

$$\begin{aligned}
&<1,\frac{1}{2};-1,\frac{1}{2}\Big|M\Big|1,\frac{1}{2};0,-\frac{1}{2}> \\
&\qquad = \left\{\sqrt{\frac{1}{3}}<\frac{3}{2};-\frac{1}{2}\Big| - \sqrt{\frac{2}{3}}<\frac{1}{2};-\frac{1}{2}\Big|\right\} M \left\{\sqrt{\frac{2}{3}}\Big|\frac{3}{2};-\frac{1}{2}> + \sqrt{\frac{1}{3}}\Big|\frac{1}{2};-\frac{1}{2}>\right\} \\
&\qquad = \frac{\sqrt{2}}{3}\big(M_{3/2} - M_{1/2}\big),
\end{aligned}$$

leading to

$$\sigma_{\pi^-p\rightarrow\pi^0n} = \frac{2}{9}K\Big|M_{3/2} - M_{1/2}\Big|^2.$$

K is the same kinematic factor as in i) and ii) if the initial π^- and the proton have the same energy and momentum in all the reactions and the final π^0 and neutron in iii) have the same energy and momentum as the final charged pion and proton in i) and ii) (the small mass differences between the neutral and the charge pions and between the proton and neutron are ignored since they would be zero if isospin were an exact symmetry).

The kinematical factor K can be eliminated by taking ratios

$$\sigma_{\pi^+p\to\pi^+p} : \sigma_{\pi^-p\to\pi^-p} : \sigma_{\pi^-p\to\pi^0n} = K|M_{3/2}|^2 : \frac{1}{9}K\left|M_{3/2}+2M_{1/2}\right|^2 : \frac{2}{9}K\left|M_{3/2}-M_{1/2}\right|^2.$$

If, for example, $|M_{3/2}| >> |M_{1/2}|$ we expect

$$\sigma_{\pi^+p\to\pi^+p} : \sigma_{\pi^-p\to\pi^-p} : \sigma_{\pi^-p\to\pi^0n} = 1 : \frac{1}{9} : \frac{2}{9} = 9 : 1 : 2$$

while, if $|M_{3/2}| << |M_{1/2}|$, we expect

$$\sigma_{\pi^+p\to\pi^+p} : \sigma_{\pi^-p\to\pi^-p} : \sigma_{\pi^-p\to\pi^0n} = 0 : \frac{4}{9} : \frac{2}{9} = 0 : 2 : 1.$$

The former case is nearer to the experimental situation than than the latter. If we look at the total cross-section for π^-p scattering

$$\sigma(\pi^-p) = \sigma_{\pi^-p\to\pi^-p} + \sigma_{\pi^-p\to\pi^0n}$$

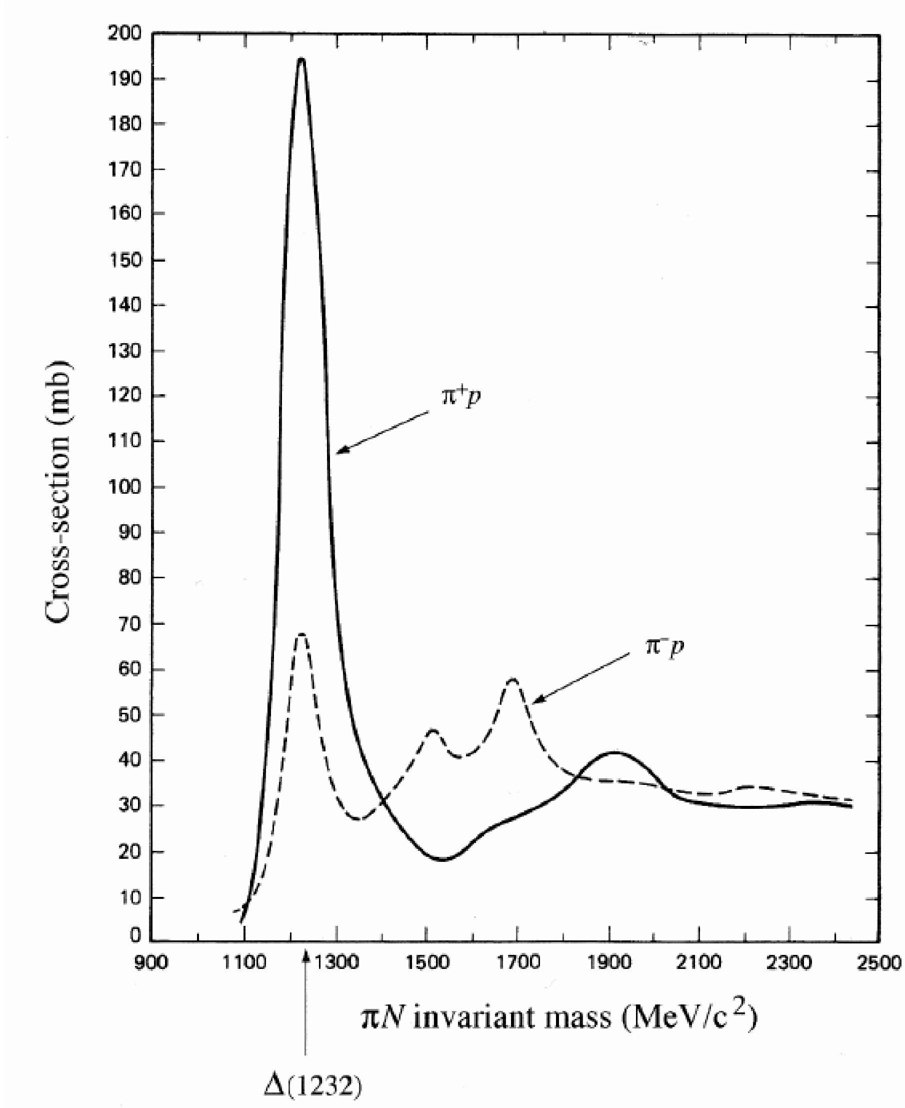
then, when the $I = 3/2$ channel dominates, this is

$$\sigma(\pi^-p) = \frac{1}{3}K|M_{3/2}|^2.$$

π^+p can only go to π^+p so $\sigma(\pi^+p) = \sigma(\pi^+p \to \pi^+p)$ is itself the total cross-section for π^+p scattering and $I = 3/2$ dominance implies that

$$\frac{\sigma(\pi^+p)}{\sigma(\pi^-p)} \approx 3.$$

These cross-sections are shown below and we see that the ratio is indeed very close to three at the total centre of mass energy equivalent to the Δ-hadron, which has $I = 3/2$.



Cross-section (mb)
200
190
180
170
160
150
140
130
120
110
100
90
80
70
60
50
40
30
20
10
0
900
1100
1300
1500
1700
1900
2100
2300
2500
$\pi^+ p$
$\pi^- p$
πN invariant mass (MeV/c^2)
$\Delta(1232)$

4. The Quark Model of Hadrons

Here is a list of the hadrons we have met so far:

$$\underbrace{\pi^{\pm}, \pi^0, K^0}_{mesons}, \quad \underbrace{p, n, \Lambda, \Delta^{++}, \Delta^{+}, \Delta^0, \Delta^-}_{baryons},$$

and there are many more. All observed hadrons can be built up from the six quarks

$$\begin{array}{cc} u\ (2.3\ MeV/c^2) & d\ (4.8\ MeV/c^2) \\ c\ (1.3\ GeV/c^2) & s\ (95\ MeV/c^2) \\ t\ (174\ GeV/c^2) & b\ (4.2\ GeV/c^2) \end{array}$$

where the masses are indicated in brackets. u, c and t have electric charge $Q = 2/3$ while d, s and b have charge $Q = -1/3$.

The three lightest quarks are u, d and s, so we shall first consider hadrons composed of these three quarks.

Strange Hadrons

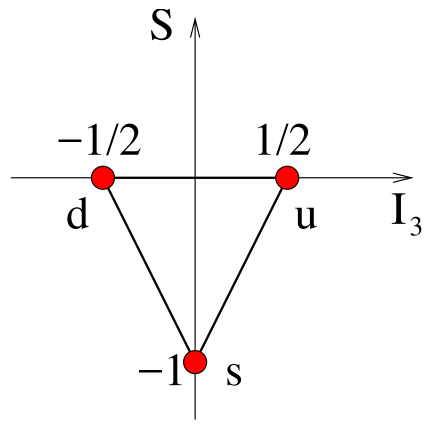
i) Mesons: the u, d and s quarks are characterised by quantum numbers which include, among others, the third component of isospin and strangeness:

$$(I_3, S) = (1/2, 0) \qquad \text{for the } u\text{-quark}$$
$$(I_3, S) = (-1/2, 0) \qquad \text{for the } d\text{-quark}$$
$$(I_3, S) = (0, -1) \qquad \text{for the } s\text{-quark.}$$

The s-quark is assigned $S = -1$ for compatibility with the Gell-Mann – Nishijima formula (all quarks have baryon number $B = 1/3$)

$$Q = I_3 + \frac{(B+S)}{2} = \begin{cases} \frac{1}{2} + \frac{1}{6} = \frac{2}{3} & u\text{-quark} \\ -\frac{1}{2} + \frac{1}{6} = -\frac{1}{3} & d\text{-quark} \\ 0 + \frac{(1/3)-1}{2} = -\frac{1}{3} & s\text{-quark.} \end{cases}$$

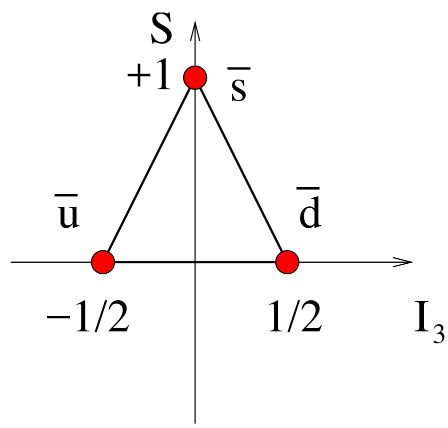
Using I_3 and S as Cartesian co-ordinates, the positions of the these three quarks can be plotted in a two dimensional plane. They sit at the corners of an isosceles triangle like this



The three corresponding anti-quarks have exactly opposite quantum numbers

$$(I_3, S) = (-1/2, 0) \qquad \text{for the } \bar{u}\text{-quark}$$
$$(I_3, S) = (1/2, 0) \qquad \text{for the } \bar{d}\text{-quark}$$
$$(I_3, S) = (0, 1) \qquad \text{for the } \bar{s}\text{-quark},$$

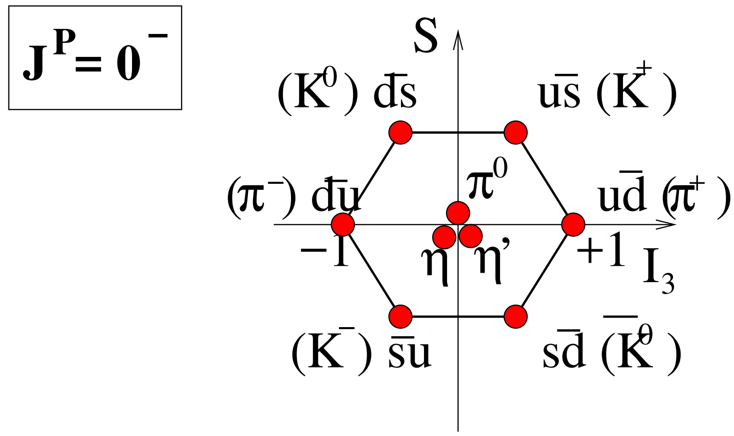
like this



The quantum numbers I_3 and S are additive, *i.e.* the third component of isospin I_3 and the strangeness S of a composite object, made up of components each of which has known values I_3 and S, are obtained simply by adding the values of I_3 and S of the components. So all the possible mesons, which are $q\bar{q}$ composites, that can be constructed from the u, d and s quarks can be found just by using vector addition of all possible pairs of u, d and s with $\bar{u}$, $\bar{d}$ and $\bar{s}$ in the (I_3, S) plane. There are nine possibilities:

$$u\bar{u}, u\bar{d}, u\bar{s}, d\bar{u}, d\bar{d}, d\bar{s}, s\bar{u}, s\bar{s} \text{ and } s\bar{s},$$

which can be represented graphically as a hexagon in the (I_3, S) plane together with three points at the origin.



The three states at the origin are linear combination of $u\bar{u}$, $d\bar{d}$ and $s\bar{s}$. There are five $S = 0$ states, four from the pairing of the $I = 1/2$ doublet (u, d) with the anti-doublet $(\bar{u}, \bar{d})$

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$$

and one from the $I = 0$ singlet s paired with $\bar{s}$

$$0 \otimes 0 = 0.$$

The first four we have already met: the $I = 1$ multiplet is the pion triplet which can be constructed from[43]

$$\mathbf{q} = \begin{pmatrix} u \\ d \end{pmatrix} \qquad \text{and} \qquad \bar{\mathbf{q}} = \begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix},$$

using the three Pauli matrices, as

$$(\,\bar{u}, \bar{d}\,) \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{(\sigma_1 - i\sigma_2)/2} \begin{pmatrix} u \\ d \end{pmatrix} = \bar{d}u = \pi^+, \qquad (\,\bar{u}, \bar{d}\,) \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{(\sigma_1 + i\sigma_2)/2} \begin{pmatrix} u \\ d \end{pmatrix} = \bar{u}d = \pi^-$$

and

$$(\,\bar{u}, \bar{d}\,) \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\sigma_3} \begin{pmatrix} u \\ d \end{pmatrix} = \bar{u}u - \bar{d}d = \pi^0.$$

Thus the π^0 is a linear combination of $\bar{u}u$ and $\bar{d}d$. The combination orthogonal to the π^0 is

$$(\,\bar{u}, \bar{d}\,) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix} = \bar{u}u + \bar{d}d$$

[43] The third component of isospin of an anti-particle is the eigenvalue of $-I_3$, rather than I_3. Hence $\bar{u}$ has $-I_3$=$+1/2$ while $\bar{d}$ has $-I_3$=$-1/2$.

which is related to the η meson mentioned in the previous section. We see from this discussion however that there is a *second* $I = 0$ singlet, made up of s and $\bar{s}$-quarks. This is related to a particle called the η' and it is much heavier than the η at $m_{\eta'} = 958\ MeV/c^2$. Actually the η and the η' are linear combinations of $\bar{u}u + \bar{d}d$ and $\bar{s}s$, a phenomenon known as *mixing*. In fact $\eta = u\bar{u} + d\bar{d} - 2s\bar{s}$ and $\eta' = u\bar{u} + d\bar{d} + s\bar{s}$.

There are four other states in the above hexagon diagram, the $I = 1/2$ doublets with $S = \pm 1$ which are labelled (K^0, K^+) and $(K^-, \overline{K^0})$ in the diagram. The K^0 meson has already been mentioned when the strangeness quantum number was first introduced in the last chapter. We see here in the quark model that the K^0 is only one component of an isospin doublet with $I_3 = -1/2$, the other component is a positively charged kaon, K^+ with $I_3 = +1/2$. The other doublet, $(K^-, \overline{K^0})$ with $S = -1$, are the anti-particles with $\overline{K^0}$ the anti-particle of the K^0 and K^- the anti-particle of the K^+.

These nine states are summarised in the table below where some of their physical properties, such as mass and lifetimes (or widths) are shown together with their dominant decay modes.

Particle	Mass (MeV/c^2)	τ (or Γ)
$\pi^\pm$	140 ($\to \mu^\pm \nu$)	$\tau = 2.6 \times 10^{-6}\ s$
π^0	135 ($\to 2\gamma$)	$\tau = 8.4 \times 10^{-17}\ s$
$K^\pm$	494 ($\to \mu^\pm \nu$)	$\tau = 1.2 \times 10^{-8}\ s$
$K^0, \bar{K}^0$	498 ($\to 2\pi$)	$\tau = 9 \times 10^{-11}\ s$
η	547 ($\to 2\gamma$)	$\Gamma = 1.3\ keV$
η'	958 ($\to \eta\pi\pi$)	$\Gamma = 0.2\ MeV$

Notice that all of the complete isospin multiplets, (π^+, π^0, π^-), (K^0, K^+), $(K^-, \bar{K}^0)$, η and η', have different masses but the particles within each multiplet have very similar masses.[44] This is due to isospin symmetry — it is assumed that, if the electromagnetic force could be turned off, all of the particles in each multiplet would have exactly the same mass. The mass difference between the multiplets can be thought of as being due to the fact that the strange quark has a much bigger mass than either the u or the d-quark.C [45]

A very important property that these nine states share is that they all have the same intrinsic spin and parity. Like the pion they all have $J^P = 0^-$, they are negative parity scalars (pseudo-scalars). They are scalars because the quark and the anti-quark in each particle have no orbital angular momentum relative to each other ($l = 0$) and the intrinsic spins of the quarks are in opposite directions ($\uparrow\downarrow$), so the particles have no intrinsic spin either, hence $J = 0$. The negative parity of the neutral pion, or the η for that matter, follows for the same reason as the parity of parapositronium is negative — they are bound states of a fermion with its anti-fermion in an s-wave ($l = 0$) state.

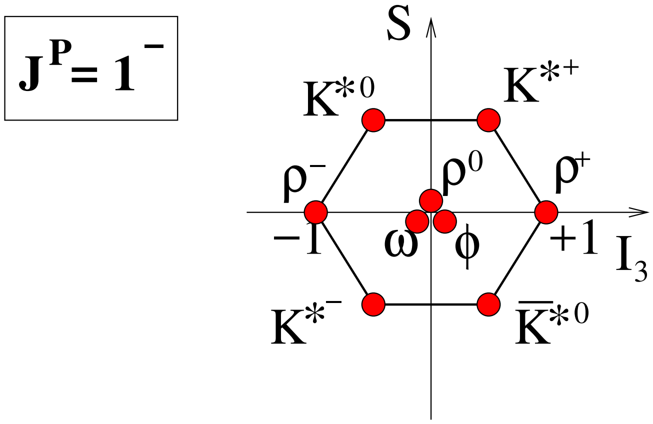
There are versions of these states in which the quark and the anti-quark have their spins parallel ($\uparrow\uparrow$), giving intrinsic spin one but still with zero orbital angular momentum (quark analogues of orthopositronium). These particles have intrinsic spin $J = 1$, but still have negative parity: they are $J^P = 1^-$ particles which are significantly heavier than the

---

[44] The K^+ and the K^- have the same mass, since they are particle and anti-particle.

[45] As mentioned previously, it is actually quite difficult to define what we mean by the mass of a quark, since no one has ever seen a free quark in isolation.

$J^P = 0^-$ mesons.



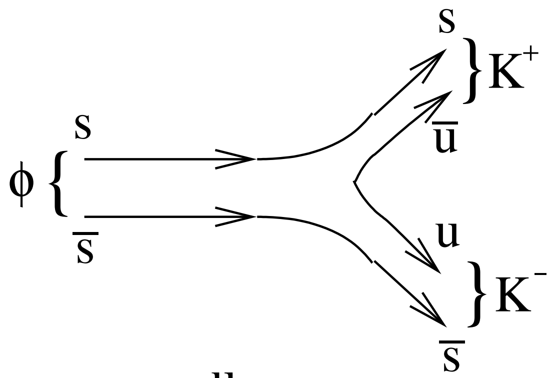
Here is a table with their properties

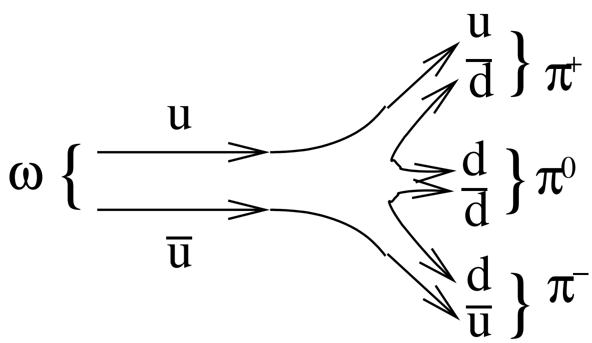
Particle	Mass (MeV/c^2)	Γ (MeV)
$\rho^{\pm,0}$	771 ($\to 2\pi$)	150
$K^{*\pm}$	892 ($\to \mu^\pm\nu$)	51
$K^{*,0}$	896 ($\to K\pi$)	51
ω	782 ($\to 3\pi$)	8.5
ϕ	1020 ($\to K\bar{K}$)	4.3

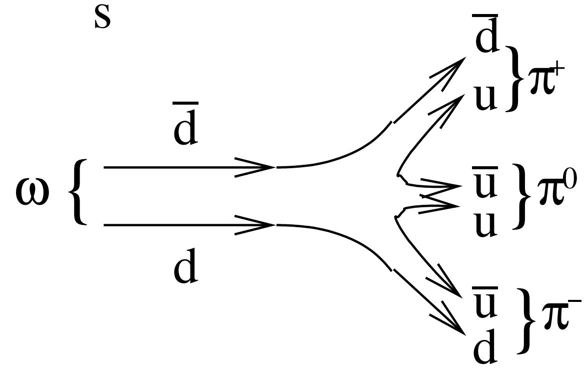
These particles are called *vector mesons* because their spin is $J = 1$, giving $2J + 1 = 3$ components, like a vector in three dimensions.[46]

The fact that the ϕ-meson decays predominantly to $K\bar{K}$ pairs is a hint that it is a bound state of mostly $s\bar{s}$ pairs, since $\bar{K}$'s contain an s-quark and K's contain an s-bar. The ω however is mostly a combination of $u\bar{u}$ and $d\bar{d}$ and decays predominantly to pions. These decays can be represented pictorially like this:

[46] Vector mesons are heavier than pseudo-scalar mesons because a state with the quark spins parallel has a higher energy than a state with the spins anti-parallel. Because of Einstein's famous formula $E{=}mc^2$ this energy difference manifests itself as a mass difference. There is an analogy here with the magnetic properties of matter. Some materials are made of atoms, or molecules, which have small magnetic moments parallel to their spin. In materials called ferromagnets the energy is lower when the spins of adjacent atoms are parallel and higher when they are anti-parallel (iron is an example). In materials called anti-ferromagnets the energy is lower when the spins of adjacent atoms are anti-parallel and higher when they are parallel (chromium is an example). In this language mesons behave like anti-ferromagnets.

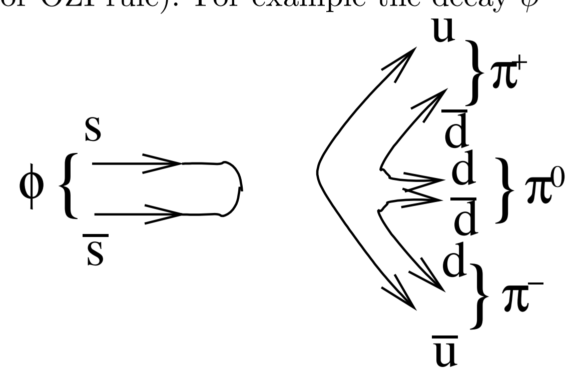






(the ω is a linear combination of $u\bar{u}$ and $d\bar{d}$, as is the π^0: the decay of the ω to $\pi^+\pi^-\pi^0$ is a linear combination of the lower two diagrams above).

There is a rule of thumb for hadron decays in the quark model which states that decays involving connected quarks lines between the initial particle and the decay products are favoured over decays which do not involve such connected quark lines. This is called the *Okubo-Zweig-Iizuka Rule* (or OZI rule). For example the decay $\phi \to 3\pi$



is suppressed by the OZI rule, because there are no continuous quark lines running through the figure from left to right, and indeed this decay is not observed despite the fact that there are no conservation laws forbidding it. Without the OZI rule we would expect $\phi \to 3\pi$ to dominate over $\phi \to K\bar{K}$ because the ϕ is much more massive ($m_\phi = 1020\ MeV/c^2$) than three pions ($m_{3\pi} = 415\ MeV/c^2$) but only slightly more massive than two kaons ($m_{2K} = 988\ Mev/c^2$) and this means that the number of final quantum states available for three pion decay is much greater than that for $K\bar{K}$ decay. This shortage of phase space for the final states is reflected in the width of the ϕ, 4.3 MeV, which is much less than that of a typical hadronic decay.

Strange quarks have isospin $I = 0$ and (u, d) quarks are an isospin doublet. One can however generalise the $SU(2)$ symmetry of isospin to a larger symmetry, a 3×3 unitary matrix[47] acting on a three component column vector $\begin{pmatrix} u \\ d \\ s \end{pmatrix}$,

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \to U \begin{pmatrix} u \\ d \\ s \end{pmatrix}.$$

Since U is unitary, $\det(U^\dagger) = (\det U^{-1})$, so $(\det U)^* = 1/(\det U)$ and $\det U$ is just a complex phase. An overall complex phase is not observable in quantum mechanics so we can always choose $\det U = 1$, in which case U is again called a 'special' unitary 3×3 matrix. Mathematically the set of all such matrices is denoted by $SU(3)$, and when applied to the three quark states u, d and s it is called *flavour* symmetry. Thus the $SU(3)$ of flavour is an extension of the $SU(2)$ of isospin. Unlike the $SU(2)$ of isospin, however, $SU(3)$ is not a very good symmetry of nature, nevertheless it can still be used as an organising principle for hadronic particles. For example, if we denote the flavour triplet $\begin{pmatrix} u \\ d \\ s \end{pmatrix}$ by **3** and the triplet of anti-quarks $\begin{pmatrix} \bar{u} \\ \bar{d} \\ \bar{s} \end{pmatrix}$ by $\bar{\mathbf{3}}$, then the nine mesons of the pion pseudo-scalar family, $J^P = 1^-$, come from[48]

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}$$

where there are two independent $SU(3)$ multiplets on the right-hand side.[49] These are: a singlet, denoted by **1** and multiple consisting of eight states, denoted by **8**, which are mixed up between each other by flavour symmetry (just as the three components of an ordinary vector in three dimensional space, in some given basis, are mixed up under the action of rotations in three dimensional space). The **1** is in fact the η' and the **8** consists of the remaining eight particles, $(\pi^+, \pi^0, \pi^-, K^0, K^+, \bar{K^0}, K^-, \eta)$ — often called the *pion octet.*

A measure of the violation, or breaking, of flavour symmetry is the mass difference between members of a single multiplet. For example, in the pion flavour octet, $m_K \neq m_\pi$

47 Remember a unitary matrix is one for which $U^{-1} = U^\dagger$, where † denotes the Hermitian conjugate, the complex conjugate of the transposed matrix

48 A similar notation is sometimes used for isospin, with a bold number denoting the number of components in a multiplet. Thus $I=0$ is denoted by **1**, $I=1/2$ by **2**, $I=1$ by **3**, and so on. In this notation $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$, for example, becomes $\mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{3}$.

49 The reason why the nine states in this equation arranged into **1** and **8** is beyond the scope of this course. Suffice it to say that there are well defined rules for combining flavour triplets into other flavour multiplets, just as there are well defined rules in quantum mechanics for combining two angular momenta to get a finite number of different angular momenta.

and the relative difference, a dimensionless number, is

$$2\left(\frac{m_K - m_\pi}{m_K + m_\pi}\right) = 2\left(\frac{494-140}{494+140}\right) = 1.12$$

(the factor of 2 is because we divide the mass difference by the average mass $(m_K+m_\pi)/2$). This is much larger than the typical mass difference between components of the same isospin multiplet, *e.g.* in the pion isospin triplet

$$2\left(\frac{m_{\pi^+} - m_{\pi^0}}{m_{\pi^+} + m_{\pi^0}}\right) = 2\left(\frac{140-135}{140+135}\right) = 0.036,$$

indicating that $SU(2)$ isospin symmetry is broken at the level of about 4% in the pion isospin triplet.

Empirically the masses in the pion octet are related by

$$m_{K^0}^2 = \frac{1}{4}\left(m_{\pi^0}^2 + 3m_\eta^2\right), \tag{40}$$

a formula which is called the *Gell-Mann – Okubo* mass formula. If flavour symmetry were an exact symmetry of nature then all particles in a given flavour multiplet would have exactly the same mass, so we would have[50]

$$m_{\pi^0} = m_{K^0} = m_\eta$$

and equation (40) would be automatic. Why equation (40) should still be satisfied, despite the fact that flavour symmetry is broken, is not well understood, even though the Gell-Mann – Okubo mass formula was first suggested in over 40 years ago in 1962.

The meson states described above all correspond to $q\bar{q}$ pairs which have zero orbital angular momentum. When orbital angular momentum is included there are more, higher mass, states which will not be described here: a full listing can be found on the Particle Data Group's home page `http://pdg.lbl.gov`.

ii) Baryons: now consider the possible ways of combining three quarks into baryons, which are qqq states. Write the wave-function for a baryon made up of three quarks as

$$\psi(x_1, x_2, x_3) = q_{ai}(x_1) q_{bj}(x_1) q_{ck}(x_3) \tag{41}$$

where $i, j, k = 1, 2$ are ordinary spin labels (*not* isospin, quarks have spin one-half, like electrons), 1 is spin up (↑) and 2 is spin down (↓) say, and $a, b, c = 1, 2, 3$ are flavour labels, 1 is the u-quark, 2 is the d-quark and 3 is the s-quark. We shall analyse the different possibilities by classifying the wave-functions according their symmetry properties under interchange of quarks. To simplify the notation in the following discussion the arguments

[50] Note that, since the η' is in a different multiplet to the pion octet, even exact flavour symmetry would not require the η' mass to be the same as that of the pion octet.

x_1, x_2 and x_3 will be omitted from the wave-function and will be understood from the order of the factors on the right-hand side of (41), so

$$\psi(x_1, x_2, x_3) = q_{ai} q_{bj} q_{ck}.$$

Suppose first that ψ is symmetric under interchange of any two quarks. Start with a proton uud, which has spin one-half. Assuming that the quarks have no orbital angular momentum this means that the three quark spins must be either $(u\uparrow)(u\uparrow)(d\downarrow)$) or $(u\uparrow)(u\downarrow)(d\uparrow)$), *i.e.* either both u-quarks spin up and the d-quark spin down or the d-quark spin up and the two u-quarks being one spin up and one spin down. Start first with the latter and suppose the spin down u-quark and the spin up d-quark are in a spin singlet, $(\uparrow\downarrow - \downarrow\uparrow)$ (the other combination, $(\uparrow\downarrow + \downarrow\uparrow)$, is not a spin singlet, it is part of a triplet). Now take the isospin singlet $(ud - du)$ and pair these to form

$$\begin{aligned}(ud - du)(\uparrow\downarrow - \downarrow\uparrow) &= \big\{(u\uparrow)(d\downarrow) - (u\downarrow)(d\uparrow) - (d\uparrow)(u\downarrow) + (d\downarrow)(u\uparrow)\big\} \\ &= \big\{(u\uparrow)(d\downarrow) + (d\downarrow)(u\uparrow)\big\} - \big\{(u\downarrow)(d\uparrow) + (d\uparrow)(u\downarrow)\big\} \qquad (42)\end{aligned}$$

which is clearly symmetric under interchange of the two quarks. Now add a spin-up u-quark

$$\begin{aligned}&(u\uparrow)\big\{(u\uparrow)(d\downarrow) + (d\downarrow)(u\uparrow) - (u\downarrow)(d\uparrow) - (d\uparrow)(u\downarrow) \\ &\quad = (u\uparrow)(u\uparrow)(d\downarrow) + (u\uparrow)(d\downarrow)(u\uparrow) - (u\uparrow)(u\downarrow)(d\uparrow) - (u\uparrow)(d\uparrow)(u\downarrow).\end{aligned}$$

This expression is a spin one-half wave-function (remember (42) is spin zero by construction) and it is symmetric under interchange of the second and third quarks. Now add all the terms necessary (another eight terms in all) to make it completely symmetric under interchange of *any* two quarks:

$$\begin{aligned}&(u\uparrow)(u\uparrow)(d\downarrow) + (u\uparrow)(d\downarrow)(u\uparrow) - (u\uparrow)(u\downarrow)(d\uparrow) - (u\uparrow)(d\uparrow)(u\downarrow) \\ &+(u\uparrow)(u\uparrow)(d\downarrow) + (d\downarrow)(u\uparrow)(u\uparrow) - (u\downarrow)(u\uparrow)(d\uparrow) - (d\uparrow)(u\uparrow)(u\downarrow) \\ &+(d\downarrow)(u\uparrow)(u\uparrow) + (u\uparrow)(d\downarrow)(u\uparrow) - (d\uparrow)(u\downarrow)(u\uparrow) - (u\downarrow)(d\uparrow)(u\uparrow) \\ &\quad = 2(u\uparrow)(u\uparrow)(d\downarrow) - (u\uparrow)(u\downarrow)(d\uparrow) - (u\downarrow)(u\uparrow)(d\uparrow) \\ &\qquad -(u\uparrow)(d\uparrow)(u\downarrow) + 2(u\uparrow)(d\downarrow)(u\uparrow) - (u\downarrow)(d\uparrow)(u\uparrow) \\ &\qquad -(d\uparrow)(u\uparrow)(u\downarrow) - (d\uparrow)(u\downarrow)(u\uparrow) + 2(d\downarrow)(u\uparrow)(u\uparrow).\end{aligned} \qquad (43)$$

This is the proton wave function, it has spin $J = 1/2$ and isospin $I = 1/2$ (remember (42) has isospin zero). It is more succcintly written, in what is meant to be an obvious matrix notation, as

$$|p\uparrow> = (\,|uud>\quad |udu>\quad |duu>\,)\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} |\uparrow\uparrow\downarrow> \\ |\uparrow\downarrow\uparrow> \\ |\downarrow\uparrow\uparrow> \end{pmatrix}.$$

The second member of this isospin doublet is the neutron, which has one of the u-quarks in (43) replaced with a d-quark — it does not matter which u-quark is replaced, since (43) is symmetric under interchange of any two quarks.[51]

$$|n\!\uparrow\,>= \left(\,|ddu> \quad |dud> \quad |udd>\,\right) \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} |\uparrow\uparrow\downarrow> \\ |\uparrow\downarrow\uparrow> \\ |\downarrow\uparrow\uparrow> \end{pmatrix}.$$

There is no symmetric $J = 1/2$ state with all three quarks u-quarks uuu (or all three quarks d-quarks ddd) since replacing the d in (43) with a u just makes the whole wave-function vanish, for example

$$\left(\,|uuu> \quad |uuu> \quad |uuu>\,\right) \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} |\uparrow\uparrow\downarrow> \\ |\uparrow\downarrow\uparrow> \\ |\downarrow\uparrow\uparrow> \end{pmatrix} = 0.$$

However, if we include the s-quark, there are at least five more possible states

$$uus, dds, uds, ssu, ssd.$$

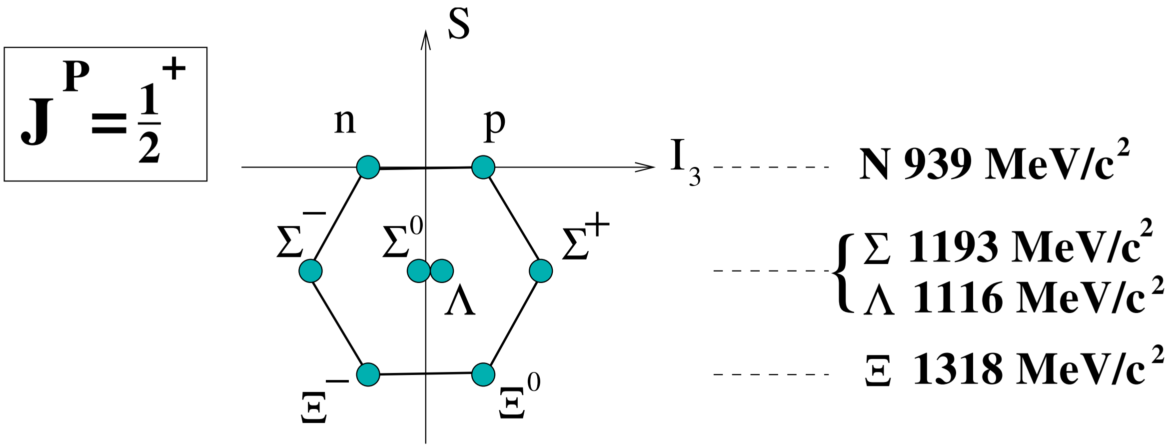
Actually there are two linearly independent uds states. One is the $I_3 = 0$ component of an isospin triplet, with the $I_3 = \pm 1$ components being the uus and the dds states — this is called the Σ triplet, $(\Sigma^{\pm}, \Sigma^0)$. The other is an $I = 0$ state which we have already met when strangeness was first introduced, it is the Λ-baryon.

The ssu and ssd states form an isospin doublet with strangeness $S = -2$, these are given the symbols Ξ^- and Ξ^0.[52]

So we have eight particles in all — called the baryon octet, which is an **8** of flavour — which are arranged in the I_3–S plane like this:

[51] The attentive reader may object to the fact that the proton wave-function constructed here is symmetric under interchange of any two quarks. Quarks are fermions and wave-functions should be anti-symmetric under interchange of fermions, not symmetric, because of the Pauli exclusion principle. This will be explained in the next chapter where the concept of the quark's 'strong charge', or *colour* as it is known, is introduced. Quarks must carry one of three different kinds of 'colour' and the three quarks in the proton wave-function all have different colour, so they are not identical. In fact, when colour is factored in, the proton wave-function is completely anti-symmetric under interchange of any two quarks, in agreement with the exclusion principle.

[52] They are sometimes called *cascade particles* because their decays look like a sequence of decays into a cascade of more and more particles.



The baryon octet: the average mass of the members of each isospin multiplet is indicated on the right in GeV/c^2 (the symbol N stands for nucleon, the familiar proton-neutron doublet).

Particle	Quark Content	Isospin	Strangeness	Mass
p	uud	$I = 1/2, \quad I_3 = 1/2$	$S = 0$	938.3 MeV/c^2
n	udd	$I = 1/2, \quad I_3 = -1/2$		939.6 MeV/c^2
Σ^+	uus	$I = 1, \quad I_3 = +1$	$S = -1$	1189.4 MeV/c^2
Σ^0	uds	$I = 1, \quad I_3 = 0$		1192.6 MeV/c^2
Σ^-	dds	$I = 1, \quad I_3 = -1$		1197.5 MeV/c^2
Λ	uds	$I = 0$	$S = -1$	1115.7 MeV/c^2
Ξ^0	uss	$I = 1/2, \quad I_3 = 1/2$	$S = -2$	1315 MeV/c^2
Ξ^-	dss	$I = 1/2, \quad I_3 = -1/2$		1321 MeV/c^2

The particles in this octet have parity $P = +1$, like the proton, so this is a $J^P = \frac{1}{2}^+$ octet. If the mass differences between the isospin multiplets were due solely to the mass of the strange quark we would expect that

$$\underbrace{m_\Lambda}_{1116\ GeV/c^2} = \underbrace{m_\Sigma}_{1193\ GeV/c^2} \qquad\qquad \underbrace{m_\Lambda - m_N}_{177\ GeV/c^2} = \underbrace{m_\Xi - m_\Lambda}_{202\ GeV/c^2}$$

whereas the actual masses are indicated underneath. The fact that we do not quite have equality here is due to spin effects. The small mass splittings between the individual members of an isospin multiplet are due to electromagnetic interactions.

With the single exception of the proton all of these particles are unstable. The dominant decay modes and lifetimes are:

Particle	Decay Mode	Lifetime
p	Stable	$\tau_p > 10^{31}$ $years$
n	$\to p + e^- + \bar{\nu}_e$	$\tau_n = 886$ s
Σ^+	$\to p\pi^0, n\pi^+$	$\tau_{\Sigma^+} = 8.0 \times 10^{-11}$ s
Σ^0	$\to \Lambda\gamma$	$\tau_{\Sigma^0} = 7 \times 10^{-20}$ s
Σ^-	$\to n\pi^-$	$\tau_{\Sigma^-} = 1.5 \times 10^{-10}$ s
Λ	$\to p\pi^-, n\pi^0$	$\tau_\Lambda = 2.6 \times 10^{-10}$ s
Ξ^0	$\to \Lambda\pi^0$	$\tau_{\Xi^0} = 2.9 \times 10^{-10}$ s
Ξ^-	$\to \Lambda\pi^-$	$\tau_{\Xi^-} = 1.6 \times 10^{-10}$ s

Now consider what happens when all quark spins are parallel, but still with zero orbital angular momentum, so we have spin $J = 3/2$ states. The number of symmetric states that can be made out three objects is $3(3+1)(3+2)/6 = 10$.[53] Explicitly the ten possibilities for the wave-function are

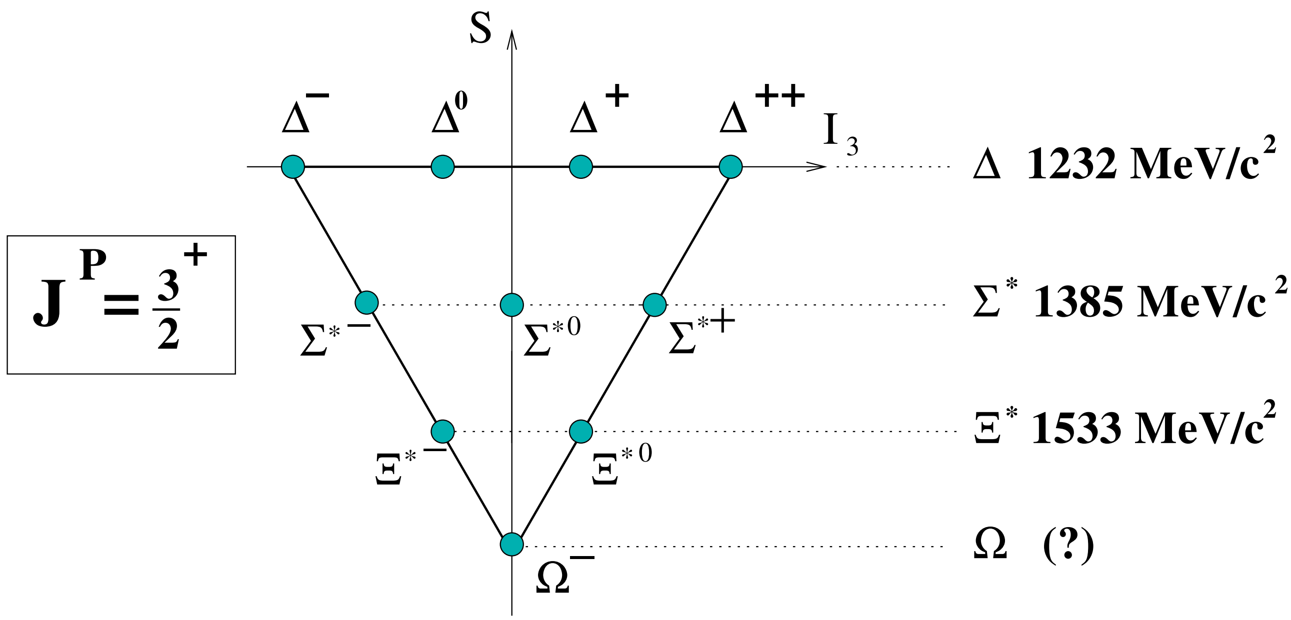
$$\begin{array}{ccccc} (u\uparrow)(u\uparrow)(u\uparrow) & (u\uparrow)(u\uparrow)(d\uparrow) & (u\uparrow)(d\uparrow)(d\uparrow) & (d\uparrow)(d\uparrow)(d\uparrow) & (u\uparrow)(u\uparrow)(s\uparrow) \\ (u\uparrow)(d\uparrow)(s\uparrow) & (d\uparrow)(d\uparrow)(s\uparrow) & (u\uparrow)(s\uparrow)(s\uparrow) & (d\uparrow)(s\uparrow)(s\uparrow) & (s\uparrow)(s\uparrow)(s\uparrow) \end{array}$$

all of which can be symmetrised in the obvious way,

$$\begin{gathered} (u\uparrow)(u\uparrow)(u\uparrow) \\ (d\uparrow)(d\uparrow)(d\uparrow) \\ (s\uparrow)(s\uparrow)(s\uparrow) \\ (u\uparrow)(u\uparrow)(d\uparrow) + (u\uparrow)(d\uparrow)(u\uparrow) + (d\uparrow)(u\uparrow)(u\uparrow) \\ (u\uparrow)(d\uparrow)(d\uparrow) + (d\uparrow)(u\uparrow)(d\uparrow) + (d\uparrow)(d\uparrow)(u\uparrow) \\ (u\uparrow)(u\uparrow)(s\uparrow) + (u\uparrow)(s\uparrow)(u\uparrow) + (s\uparrow)(u\uparrow)(u\uparrow) \\ (d\uparrow)(d\uparrow)(s\uparrow) + (d\uparrow)(s\uparrow)(d\uparrow) + (s\uparrow)(d\uparrow)(d\uparrow) \\ (u\uparrow)(s\uparrow)(s\uparrow) + (s\uparrow)(u\uparrow)(s\uparrow) + (s\uparrow)(s\uparrow)(u\uparrow) \\ (d\uparrow)(s\uparrow)(s\uparrow) + (s\uparrow)(d\uparrow)(s\uparrow) + (s\uparrow)(s\uparrow)(d\uparrow) \\ (u\uparrow)(d\uparrow)(s\uparrow) + (u\uparrow)(s\uparrow)(d\uparrow) + (d\uparrow)(u\uparrow)(s\uparrow) + (d\uparrow)(s\uparrow)(u\uparrow) + (s\uparrow)(u\uparrow)(d\uparrow) + (s\uparrow)(d\uparrow)(u\uparrow). \end{gathered}$$

This gives a baryon decuplet, denoted as a **10** of flavour (average masses of the isospin multiplets are indicated on the right):

[53] It we choose 3 objects out of n possibilities, without worrying about the order, there are $n(n+1)(n+2)/6$ possibilities: there are n in which all three objects are the same; $n(n-1)$ with two the same and the third different; and $n(n-1)(n-2)/6$ with all three different. These add to $n(n+1)(n+2)/6$.



Can you guess the mass of the Ω^-?

The average masses of the four isospin multiplets are indicated above, apart from the Ω^-. The mass differences are

$$m_{\Sigma^*} - m_{\Delta} = 153\ MeV/c^2, \qquad m_{\Xi^*} - m_{\Sigma^*} = 148\ MeV/c^2.$$

Here are the current experimental values of the masses (only one value is quoted for the Δ baryons as their electromagnetic splittings have not yet been clearly resolved):

Particle	Quark Content	Isospin	Strangeness	Mass
Δ^{++}	uuu	$I=3/2, \quad I_3=3/2$	$S=0$	$1232 \quad MeV/c^2$
Δ^{+}	uud	$I=3/2, \quad I_3=1/2$		
Δ^{0}	udd	$I=3/2, \quad I_3=-1/2$		
Δ^{-}	ddd	$I=3/2, \quad I_3=-3/2$		
Σ^{*+}	uus	$I=1, \quad I_3=+1$	$S=-1$	$1383\ MeV/c^2$
Σ^{*0}	uds	$I=1, \quad I_3=0$		$1384\ MeV/c^2$
Σ^{*-}	dds	$I=1, \quad I_3=-1$		$1387\ MeV/c^2$
Ξ^{*0}	uss	$I=1/2, \quad I_3=1/2$	$S=-2$	$1532\ MeV/c^2$
Ξ^{*-}	dss	$I=1/2, \quad I_3=-1/2$		$1535\ MeV/c^2$
Ω^{-}	sss	$I=0$	$S=-3$	$1672\ MeV/c^2$

In fact the Ω^- particle was predicted on the basis of the quark model three years before its discovery in 1964, and experimentalists even knew what the mass should be, to within a few GeV.

All particles in the baryon decuplet are unstable but we shall not go into the details of the decay modes. Again, as for the mesons, there are higher mass excited states of the above baryons, in which the quarks have orbital angular momentum and/or radial excitations but the details are many and varied and will not be described here.

Charming and Beautiful Hadrons

When the charmed quark is added to the list we have a quadruplet $\begin{pmatrix} u \\ d \\ s \\ c \end{pmatrix}$, suggesting an extension of flavour symmetry to 4×4 unitary matrices. However the mass of the charmed quark

$$m_c = 1600 \; MeV/c^2 >> m_s \approx 500 \; MeV/c^2 > m_u \approx m_d \approx 300 \; MeV/c^2$$

is so much greater than the mass of the s, d, and u-quarks that this would be a very badly broken symmetry.

Like the up quark, the charmed quark has electric charge $Q = 2/3$. In analogy with strangeness a new quantum number called *charm* is introduced and given the symbol C, not to be confused with charge parity $\mathcal{C}$. The c-quark has $C = 1$, the $\bar{c}$ has $C = -1$ and u, d and s have $C = 0$. C is an additive quantum number. The Gell-Mann – Nishijima formula must now be modified to allow for charmed particles

$$Q = I_3 + \frac{B + S + C}{2}.$$

i) Charmed hadrons: we can construct charmed mesons, $q\bar{q}$ states, by using a c-quark (or a $\bar{c}$). If the other quark is a u or a d these are called D-mesons. For example

$$\begin{array}{rclcrcl} D^+ & = & c\bar{d}, & \qquad & \bar{D^0} & = & u\bar{c} \\ D^0 & = & c\bar{u}, & \qquad & D^- & = & d\bar{c} \end{array}$$

are charmed analogues of strange pseudo-scalars, (K^+, K^0) and $(\bar{K^0}, K^-)$, in the pion octet. They are isospin doublets, $I = 1/2$.

Like strangeness, charm is conserved by strong interactions but not by weak interactions. Charmed mesons can be created in pairs by strong interactions involving collisions between hadrons with zero charm but the total charm of the final state must be zero so, if it contains particles with charm, both $C = +1$ and $C = -1$ must be present and these must decay to $C = 0$ particles via the weak interactions (charmed quarks decay predominantly to strange quarks). Their lifetimes are therefore comparatively long, at least compared to strong interaction decays. The physical properties of the D-mesons are

$$\begin{array}{rlcrl} m_{D^\pm} & = 1869 \; MeV/c^2 & \qquad & \tau_{D^\pm} & = 1.05 \times 10^{-12} \; s \\ m_{D^0} & = 1864 \; MeV/c^2 & \qquad & \tau_{D^0} & = 4.1 \times 10^{-13} \; s. \end{array}$$

The u or the d-quark in the D-mesons can also be replaced by a strange quark, giving a strange-charmed meson, D_s, which is an isospin singlet $I = 0$,

$$D_s^+ = c\bar{s} \qquad m_{D_s^+} = 1968 \; MeV/c^2 \qquad \tau_{D_s^+} = 4.9 \times 10^{-13} \; s,$$

together with its anti-particle $D_s^- = s\bar{c}$.

All of the D-mesons decay weakly, mostly into kaons plus other stuff, but the details of the decay modes are a complex and fascinating branch of modern particle physics.

There is also a $C = 0$ meson made up of a c-$\bar{c}$ pair, called *charmonium* in analogy with positronium, and denoted by η_c, because it is the analogue of the η in the pion octet. The η_c, like the η, is an isospin singlet but with mass and width

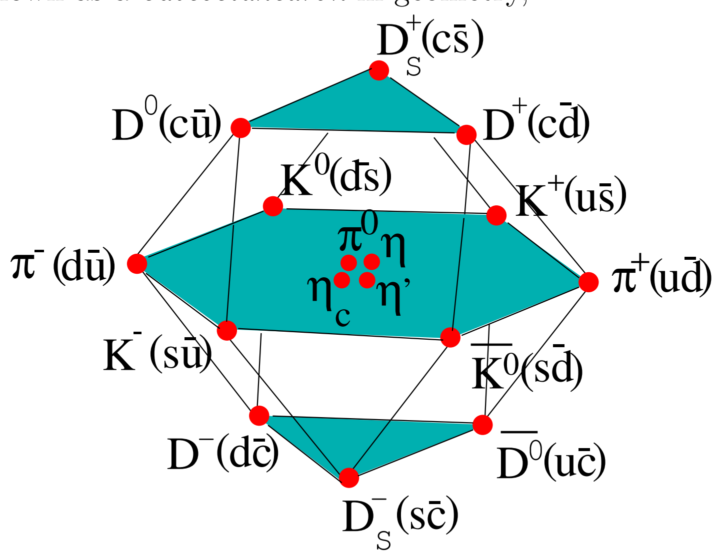
$$m_{\eta_c} = 2980\ MeV/c^2, \qquad \Gamma = 17 \pm 3\ MeV \Leftrightarrow \tau_{\eta_c} = 4 \times 10^{-23}\ s.$$

Since charmonium has $C = 0$ it can decay via strong interactions to other $C = 0$ states and this does not violate charm conservation, hence its short lifetime. The decay modes of charmonium are also very complicated and include

$$\eta_c \to \begin{cases} K\bar{K}\pi \\ \eta\pi\pi \\ \eta'\pi\pi \\ \rho\rho \\ \vdots \end{cases}$$

Being a bound state of a fermion and its anti-fermion, with zero orbital angular momentum, charmonium, like positronium, has parity $\mathcal{P} = -1$.

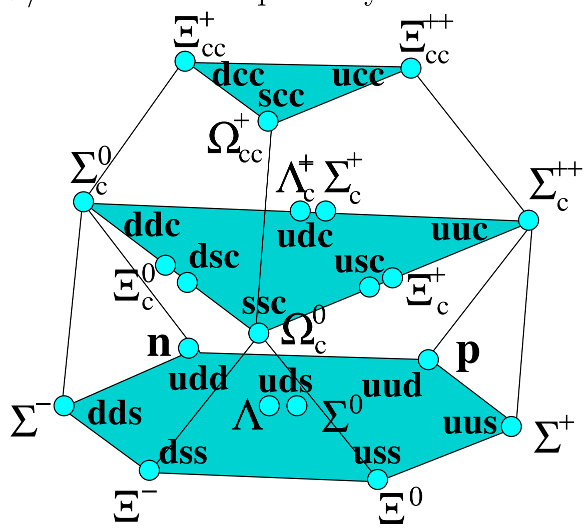
All of these mesons containing c-quarks (or $\bar{c}$ or both) can be represented pictorially by introducing C as a third co-ordinate and extending the (I_3, S)–plane into three dimensions, resulting in a shape known as a *cubeoctahedron* in geometry,

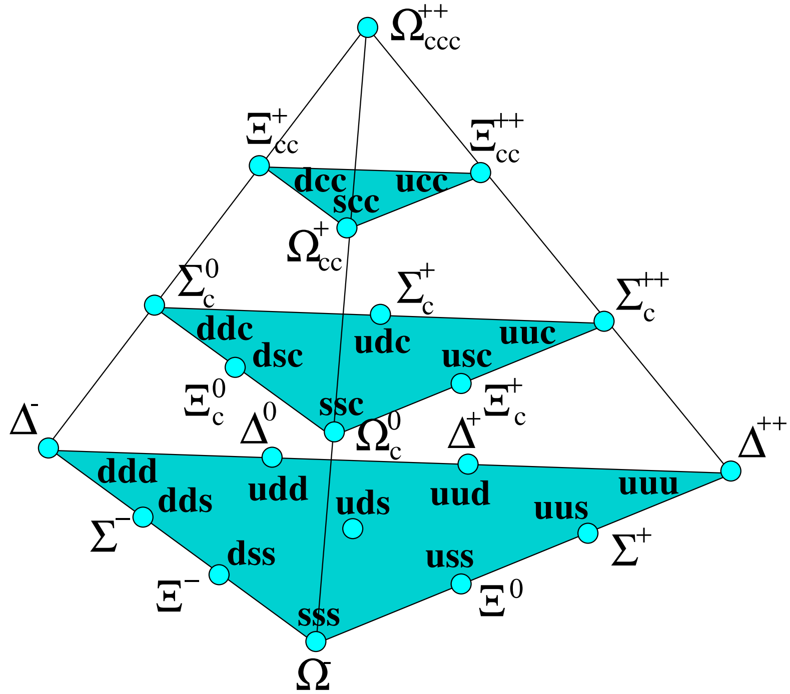


All of these particles are pseudo-scalars with $J^P = 0^-$ and charm extends the pion pseudo-scalar family from $9 = 3^2$ to $16 = 4^2$ particles.

There are also charmed analogues of the vector mesons with $J^P = 1^-$ and in fact the observation of the charmonium state with the quark spins parallel, $(c\uparrow)(\bar{c}\uparrow)$, marked the discovery of the charmed quark in 1974. It was discovered simultaneously by two groups in the United States working independently: one group called it the J-particle and gave it the same symbol J, the other group called it the Ψ-particle. It has since been known as the J/Ψ particle and is sometimes affectionately called the 'gypsy'. The J/Ψ is the charmed analogue of the ϕ-meson, with $J^P = 1^-$, but, unlike the ϕ which decays to $K\overline{K}$, the $J\Psi$ cannot avail of the analogous decay $D\overline{D}$ because it is below the mass threshold for $D\overline{D}$ production: it is exceptionally long lived for such a massive hadron and has a very high narrow resonance.

The J/Ψ decay modes are very complex and have been extensively studied. It weighs in at $m_{J/\Psi} = 3096.9\ MeV/c^2$ with an exceptionally narrow width of $91.0 \pm 3.2\ keV$.[54]





[54] The width is so narrow because the natural decay mode to D-mesons, $J/\Psi \to D\bar{D}$ (the charm analogue of $\phi \to K\bar{K}$), is forbidden by conservation of energy, since $m_{J/\Psi}$=3096 MeV/c^2<2m_D=3730 MeV/c^2. The complicated hadronic decay modes of the J/Ψ are all suppressed by the OZI rule making it relatively long-lived, and hence narrower, than other hadronic resonances.

There also charmed baryons extending both the baryon-octet and the baryon-decuplet. We shall not describe their properties in detail, but just represent them pictorially in the previous diagram. The previous picture shows the $J^P = \frac{1}{2}^+$ charmed baryons, in the shape of a truncated tetrahedron, and the the $J^P = \frac{3}{2}^+$, all $4(4+1)(4+2)/6 = 20$ of them, arranged in a tetrahedron.

ii) Bottom hadrons: including the b-quark makes the spectrum even richer and we now run out of dimensions for easy visualisation. The b-quark is some three times more massive than the c-quark with $m_b = 4.6\ GeV/c^2$ and, like the d and the s-quarks, has electric charge $Q = -1/3$. Like s and c-quarks, b-quarks are always produced in strong interactions in quark—anti-quark pairs, $b\bar{b}$, which can then separate in different particles. These particles subsequently do not decay by strong interactions, but only through weak interactions. b-quarks are therefore assigned an additive quantum number, called beauty $\mathcal{B}$ (not to be confused with baryon number B), which is conserved by strong interactions but not by weak interactions. $\mathcal{B} = +1$ for b-quarks and $\mathcal{B} = -1$ for $\bar{b}$-quarks.

There are b-mesons, analogues of the D-mesons, which enlarge our sixteen pseudo-scalar mesons even further. These are

$$B^+ = u\bar{b}, \qquad B^0 = d\bar{b}, \qquad m_B = 5279\ MeV/c^2$$
$$B^- = b\bar{u}, \qquad \overline{B}^0 = b\bar{d}, \qquad \tau_{B^\pm} = 1.67 \times 10^{-12}\ s \qquad \tau_{B^0} = 1.53 \times 10^{-12}\ s.$$

There is a $b\bar{b}$ bound state ('bottomonium') called the Upsilon particle[55]

$$\Upsilon = b\bar{b}, \qquad m_\Upsilon = 9460\ MeV/c^2, \qquad \Gamma = 53\ keV.$$

There are also bottom-charm mesons

$$B_c^+ = c\bar{b}, \qquad B_c^- = b\bar{c}, \qquad m_{B_c^\pm} = 6.4 \pm 0.4\ GeV/c^2, \qquad \tau_{B_c^\pm} = 0.46 \pm 0.17 \times 10^{-12}\ s$$

and bottom-strange mesons

$$B_s^0 = s\bar{b}, \qquad \bar{B}_s^0 = b\bar{s}, \qquad m_{B_s^0} = 5.370 \pm 0.002\ GeV/c^2, \qquad \tau_{B_s^0} = 1.46 \times 10^{-12}\ s.$$

With the exception of the Υ these are all pseudo-scalar mesons, $J^P = 0^-$.

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55 Actually the Υ is a vector meson J^P=1^- and not a pseudo-scalar meson, it is the bottom analogue of the J/Ψ. The J^P=0^- bottomonium state η_b, the bottom analogue of the η_c, has a slightly lower mass of 9398 MeV/c^2. The mass splittings between the J^P=1^- and the J^P=0^- in the $b\bar{b}$ and $c\bar{c}$ are due to spin effects (see footnote on page (69)). The splitting is expected to be proportional to the spin-spin coupling and inversely proportional to the square of the quark mass and the b-quark, m_b=4.2 GeV/c^2, is some three times heavier than the c-quark, m_c=1.3 GeV/c^2, so we might expect the mass splitting in the bottomonium system to be nine times less than in the charmonium system. Experimentally $m_{J/\psi}-m_{\eta_c}$=116 MeV/c^2 so we would expect $m_\Upsilon - m_{\eta_b} \approx 13\ MeV/c^2$ if the quark mass were the only relevant parameter. The fact that the Υ-η_b mass difference is some five times this implies that the spin-spin coupling between the b and the $\bar{b}$ in bottomium is some five times larger than the c-$\bar{c}$ spin coupling in charmonium.

Then there are bottom baryons, *e.g.*

$$\Lambda_b = udb, \qquad m_{\Lambda_b} = 5.624\ GeV/c^2, \qquad \tau_{\Lambda_b} = 1.2 \times 10^{-12}\ s.$$

There are many more possibilities that one can construct and not all have been seen experimentally, though many have. All spin-$\frac{1}{2}$ baryons with one b-quark have now been seen except for the electrically neutral Σ_b^0 with quark content udb (obtained by replacing the s-quark in the Σ^0 with a b-quark). The latest addition, the Ξ_b^0 with quark content usb (obtained by replacing one of the s-quarks in the Ξ^0 with a b-quark) was discovered in July 2011. The search goes on for the Σ_b^0!

The list is too long to describe here and the current status of known particle properties can be found on the particle data group's web page at `http://pdg.lbl.gov`.

The Top Quark

The top quark, which was only discovered in 1994 with the phenomenal mass of $174.3 \pm 5.3\ GeV/c^2$, decays so quickly, mostly to b-quarks and emitting W-bosons in the process, that bound states, either top-mesons or top-baryons, simply do not have time to form.

Exotic States

All of the hadronic states described above are either mesons, $q\bar{q}$ pairs, or baryons, qqq states. This is partially explained by the concept of *confinement* and the idea that no hadron can be a source for 'strong charge', or colour: all hadrons should be colourless (ideas that will be explained in more detail in the next chapter). However there are many possible states that would still be compatible with this idea and any hadronic state other than a meson or a baryon is termed 'exotic'.

In 2008 a hadronic state with quantum numbers that probably make it exotic was discovered. With a mass close to $4430\,MeV/c^2$ it is dubbed the $Z(4430)$. It appears to consist of two quarks and two anti-quarks, $\bar{c}c\bar{u}d$, and perhaps could be a strong 'molecule' consisting of a bound state of two mesons.

Glueballs

An important aspect of the underlying dynamics of the strong nuclear force, quantum-chromodynamics (described in the next section), is that the force carriers, the gluons, can interact strongly among themselves. This is in stark contrast to electromagnetism where photons only have very slight interactions among themselves. Classically electromagnetic waves have no interactions at all among themselves — two electromagnetic waves will pass through each other without any effect whatsoever — though quantum mechanically there is a slight, but very weak and in most situations neglible, interaction. It should be possible to form a bound state of gluons only, with no quarks, called a *glueball*.[56] There is a well established state near 1500 MeV/c^2 which is a good candidate for a glueball, called the $f_0(1500)$.

[56] Strictly speaking a glueball can contain virtual quarks, quark – anti-quark pairs that can fleetingly pop out of the vacuum only to annihilate each other almost immediately.

5. Chromodynamics

The baryon wave-functions for the octet $J^P = \frac{1}{2}^+$ and the decuplet $J^P = \frac{3}{2}^+$, described in the last chapter, were completely symmetric under the interchange of any two quarks. But quarks are fermions and fermion wave-functions should be totally *anti-symmetric* under interchange of any two fermions for compatibility with the exclusion principle.

The anti-symmetric nature of fermionic wave-functions can be accommodated in the quark model by postulating a new quantum number, taking three values, $\alpha = 1, 2, 3$ say. The baryon wave function (41) then becomes more complicated, it is a linear combination of

$$q_{\alpha ai}(x_1) q_{\beta bj}(x_1) q_{\gamma ck}(x_3)$$

with $\alpha, \beta, \gamma = 1, 2, 3$ labelling the new quantum number, $a, b, c = 1, 2, 3$ labelling flavour and $i, j, k = 1, 2$ labelling spin. The wave-function is then

$$\psi(x_1, x_2, x_3) = \sum_{\alpha,\beta,\gamma=1}^{3} \sum_{a,b,c=1}^{3} \sum_{i,j,k=1}^{2} C_{\alpha\beta\gamma;abc;ijk} \; q_{\alpha ai}(x_1) q_{\beta bj}(x_1) q_{\gamma ck}(x_3)$$

where $C_{\alpha\beta\gamma;abc;ijk}$ are constants, chosen so that $\psi(x_1, x_2, x_3)$ is anti-symmetric under interchange of any two quarks. Since all the wave-functions that we have constructed are symmetric under interchange of the flavour and spin indices, the C's must be anti-symmetric under interchange of colour indices, $\alpha\beta\gamma$, so $C \propto \epsilon_{\alpha\beta\gamma}$.

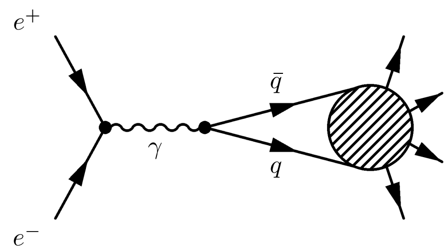
It is an experimental fact that the α, β, γ quantum numbers are not directly visible. They label a new kind of charge which, like electric charge, is conserved, but, unlike electric charge, it comes in three kinds together with three kinds of anti-charge. In electromagnetism there is only one type of charge (together with the anti-charge), for the strong force there are three distinct kinds of charge. Because this new strong charge has not been directly detected in any experiment it is assumed that these three 'charges' are like complex numbers, cube roots of unity 1, $e^{2i\pi/3}$ and $e^{-2i\pi/3}$ that sum to zero $1 + e^{2i\pi/3} + e^{-2i\pi/3} = 0$. The anti-charges, -1, $e^{-i\pi/3}$ and $e^{i\pi/3}$ also sum to zero. If each quark in a baryon carried a different 'strong' charge then they sum to zero and this would explain why the strong charge has not been observed. A rather more picturesque language is to use an analogy with colour, where the three primary colours, red, green and blue, combine to give white, which is colourless. This charge is therefore called 'colour': quarks carry one of three colours, red green or blue and baryons consist of three quarks each with a different colour which combine so that baryons are white, or 'colourless'. There are also anti-colours: the opposite of red is cyan; the opposite of green is magenta and the opposite of blue is yellow.[57] Mesons are also colourless, but this time because they consist of a quark and an anti-quark with the anti-colour, such as a red quark and a cyan anti-quark for example. The strong force that binds quarks inside hadrons is therefore usually called the colour force and the dynamics of the colour force is called *chromodynamics*. The quantum theory of chromodynamics, quantum chromodynamics, is usually abbreviated to QCD. QCD is

[57] In fact printers talk of cyan, magenta and yellow as the three 'primary' colours.

the fundamental theory of strong interactions, but the details are too advanced for this course.

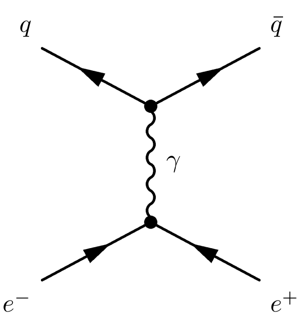
Evidence for colour

Colour is not just a theoretical idea, it is a real charge and has physical consequences albeit indirect ones. Evidence for colour comes from $e^+ - e^-$ annihilation experiments, where electrons and positrons collide together and we watch what comes out. In particular, if there is enough energy, hadrons can be produced in the collision via an intermediate (virtual) photon which decays into a quark – anti-quark pair:



(in this diagram time can be thought of as running from left to right and four hadrons emerge on the right). Since a photon carries no colour charge, it is colourless, conservation of colour charge implies that the quark and the anti-quark produced here must have opposite colour charge, *i.e.* whatever the colour charge of the quark the anti-quark must have the corresponding anti-colour. The blob here indicates that the $q\bar{q}$ pair must combine into white hadrons in some unspecified way, a poorly understood process that is called *hadronisation*, and four hadrons are depicted emerging from the collision in the diagram above for example.

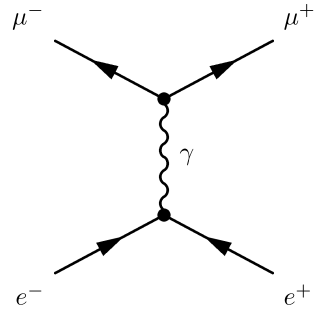
Let us focus on the quark production process for the moment,



(in this diagram time runs from the bottom to the top).

This is simpler because it does not involve QCD, it is a purely electromagnetic process, QCD only enters at the hadronisation stage. Diagrams like this are very important in particle physics. They have a precise meaning and there is a well defined set of rules, which we shall not go into in detail here, that associate a quantum mechanical amplitude with the process depicted above. A quantum mechanical amplitude is a complex number which tells us the probability of an incoming electron and a positron, with specific momentum, annihilating into a virtual photon which subsequently disintegrates into a $q\bar{q}$ pair. Diagrams like this are called *Feynman diagrams*. Feynman diagrams are *not* pictures of processes in real space, they are graphical representations of quantum mechanical amplitudes and as such they can be added, subtracted and multiplied using the usual rules of complex addition and multiplication.

To understand the physics here, first consider the simpler process of $e^+e^- \to \mu^+\mu^-$,



Ignoring spin, the cross-section for this process can be derived from equation (14),

$$\frac{d\sigma}{d\Omega} = \frac{|M_{fi}|^2}{4\pi^2\hbar^4}\frac{p_f^2}{v_f v_i} \tag{14}$$

where M_{fi} is the quantum mechanical amplitude, the complex number associated with the Feynman diagram.

The calculation proceeds in a very similar manner to the calculation of the cross-section for α-particle scattering off an atomic nucleus, calculated earlier, equation (15). The amplitude can be calculated using the Coulomb potential between the initial electron and the final muon

$$U(r) = \frac{e^2}{4\pi\epsilon_0 r}$$

and it is convenient to Fourier transform this to

$$\widetilde{U}(\mathbf{k}) := \int d^3x \, \mathrm{e}^{i\mathbf{k}.\mathbf{r}} U(r) = \frac{e^2}{\epsilon_0 k^2}$$

(the derivation of this Fourier transform is left as an exercise) and to express it in terms of the fine structure constant $\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c}$,

$$\widetilde{U}(\mathbf{k}) = \frac{e^2\hbar^2}{\epsilon_0 q^2} = \frac{4\pi\hbar^3 c\alpha}{q^2} \tag{44}$$

where $q = \hbar k$ is the momentum of the photon that is exchanged in the process above. This photon is time-like, if its 4-momentum is $\underset{\sim}{Q}$ then

$$\underset{\sim}{Q}.\underset{\sim}{Q} = -q^2 = -s/c^2 < 0,$$

where s is the square of the total energy of the incoming electron and positron in the centre of mass frame. For large energies we can ignore the particle masses m_e and m_μ, provided $s >> m_\mu^2 c^4$, and set

$$v_i = 2c, \qquad v_f = 2c \tag{45}$$

(remember that v_i in (14) is the relative speed of the two incoming particles in the centre of mass frame and v_f is the relative speed of the two outgoing particles). Conservation of energy requires that the total energy of the outgoing muons must also be $\sqrt{s} = qc$ so, in the centre of mass frame, each has energy $E_\mu = \sqrt{s}/2 = qc/2$, so the momentum of either muon is

$$p_f = \frac{E_\mu}{c} = q/2. \tag{46}$$

The amplitude calculated earlier for α-scattering was

$$M_{fi} = \widetilde{U}(q) \tag{47}$$

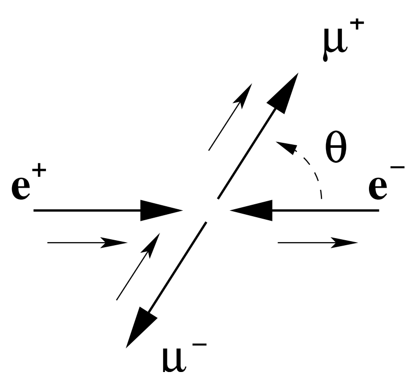
and we can use the same amplitude here, combining (14) (44) (45) (46) and (47) to get

$$\frac{d\sigma}{d\Omega} = \frac{\hbar^2\alpha^2}{4q^2}. \tag{48}$$

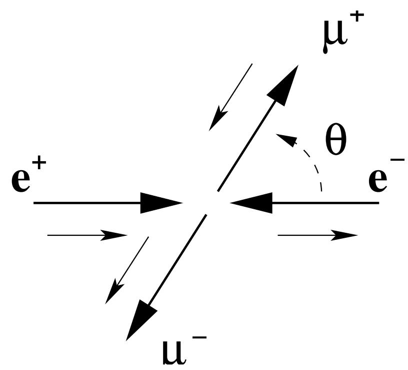
Integrating over all possible directions of the final state muons requires multiplying this by 4π to get

$$\sigma = \frac{\pi\hbar^2\alpha^2}{q^2} = \frac{e^4}{16\pi\epsilon_0^2 c^2 q^2}.$$

This calculation has ignored the spin of the initial electron and positron and the final muons. If we include spin the differential cross-section is modified, because the amplitude actually depends on the spin of the incoming and outgoing particles. For example suppose the incoming electron has positive helicity and the positron has negative helicity (they must have opposite helicity, if their helicities were the same the total incoming angular momentum would be zero and a spin one photon could not be created in the collision). There are then two possible orientations for the outgoing μ^- and μ^+ spins: the μ^- could have negative helicity and the μ^+ positive helicity, like this



or vice versa, like this



Now the incoming e^+ and the outgoing μ^+ prefer to have their spins parallel and the amplitude for the first case carries a factor $(1+\cos\theta)$, which is zero when the e^+ and μ^+ spins are anti-parallel ($\theta = \pi$). The amplitude for the second case carries a factor $(1-\cos\theta)$, which is again zero when the e^+ and μ^+ spins are anti-parallel ($\theta = 0$ in this case). The quantum probability is calculated from the square of the amplitudes and, according to (17) we should sum over final spins states. Squaring and adding the amplitudes for these two cases gives a factor

$$(1+\cos\theta)^2 + (1-\cos\theta)^2 = 2(1+\cos^2\theta).$$

There is also the possibility that the incoming electron has negative helicity and the positron has positive helicity — the analysis in this case is the same, again giving $2(1+\cos^2\theta)$ which must be added to the first case to give an overall factor of $4(1+\cos^2\theta)$. Another two possibilities are that the electron and the positron have the same helicity, either positive or negative, but these states have spin zero and so cannot produce a photon — giving amplitude zero. Averaging over the four initial spin states then requires dividing by 4 and the final effect of including the spin is to multiply (48) by $(1+\cos^2\theta)$ to give

$$\frac{d\sigma}{d\Omega} = \frac{\hbar^2\alpha^2}{4q^2}(1+\cos^2\theta) = \frac{e^4}{64\pi^2\epsilon_0^2c^2}\frac{(1+\cos^2\theta)}{q^2}. \quad (49)$$

Integrating this over all possible directions of the final muons gives

$$\int \frac{d\sigma}{d\Omega}d\Omega = \frac{\hbar^2\alpha^2}{4q^2}\int_0^{\pi}\int_0^{2\pi}(1+\cos^2\theta)\sin\theta d\theta d\phi = \frac{4\pi\hbar^2\alpha^2}{3q^2},$$

so the cross-section for the process $e^+e^- \to \mu^+\mu^-$, with total centre of mass energy $\sqrt{s} = qc$, is finally

$$\sigma(e^+e^- \to \mu^+\mu^-) = \frac{4\pi\hbar^2\alpha^2}{3q^2}. \quad (50)$$

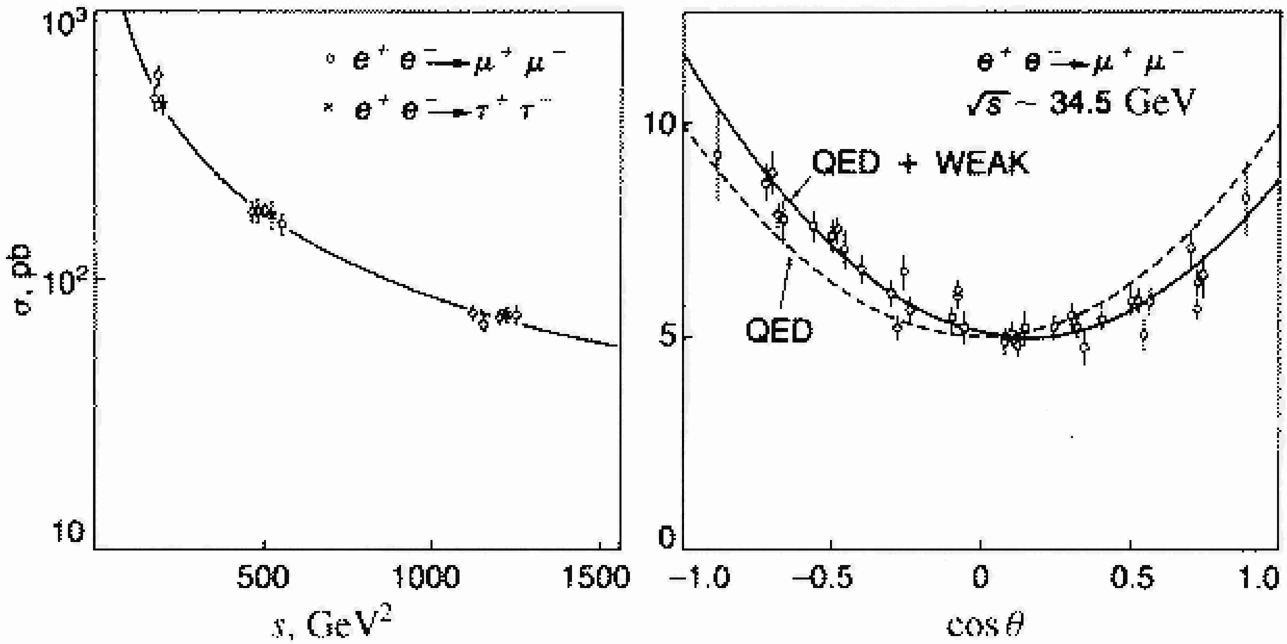


Fig. 2: Experimental $e^+e^- \to \mu^+\mu^-$ *cross-sections.* The figure on the left is the total cross-section, measured in picobarns, and shows the $1/s$ behaviour predicted by (50) (both $e^+e^- \to \mu^+\mu^-$ and $e^+e^- \to \tau^+\tau^-$ are shown). The dashed line in the figure on the right shows the $1 + \cos^2\theta$ dependence of (49) — the solid line includes corrections to (49) due to the weak force.

It is important that the total cross-section decreases as the centre of mass energy $\sqrt{s} = qc$ increases. Cross-sections represent probabilities and probabilities must always be less than one, a cross-section that grows indefinitely would be physically unacceptable.

Having derived the cross section for e^+e^- annihilation to $\mu^+\mu^-$ the answer for $e^+e^- \to q\bar{q}$ is easy: the only difference, when masses are ignored, is that the electric charge on the final state quarks is different to that of a muon final state, it can be either $2/3$ for u, c or t-quarks or $-1/3$ for d, s or b-quarks. This means that the factor of α in (44) must be replaced by $Q_u\alpha = \frac{2}{3}\alpha$, if a $u\bar{u}$ pair is created for example, or $Q_d\alpha = -\frac{1}{3}\alpha$ if a $d\bar{d}$ pair is created and so (50) is multiplied by Q_i^2, with $i = u$ or d. If k different types of quark with charges Q_i, $i = 1, \ldots, k$ can be created then the probabilities for their production must be added and

$$\sigma(e^+e^- \to q\bar{q}) = \left(\sum_{i=1}^{3} Q_i^2\right) \frac{4\pi\hbar^2\alpha^2}{3q^2}.$$

If we measure the cross-sections and take the ratio

$$R := \frac{\sigma(e^+e^- \to q\bar{q})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \sum_{i=1}^{3} Q_i^2$$

we get direct experimental information about the quark charges. (Experimentally it is the cross-section $\sigma(e^+e^- \to \mathit{hadrons})$ rather than $\sigma(e^+e^- \to q\bar{q})$ that is measured, but these are equal because the quarks create hadrons with probability one .)

Now c, b and t quarks are significantly heavier than the others so suppose that the centre of mass energy is not high enough to create these quarks and only u, d or s-quarks can be created. Then summing over these three possibilities gives the ratio

$$R = \left(\frac{4}{9} + \frac{1}{9} + \frac{1}{9}\right) = \frac{2}{3}.$$

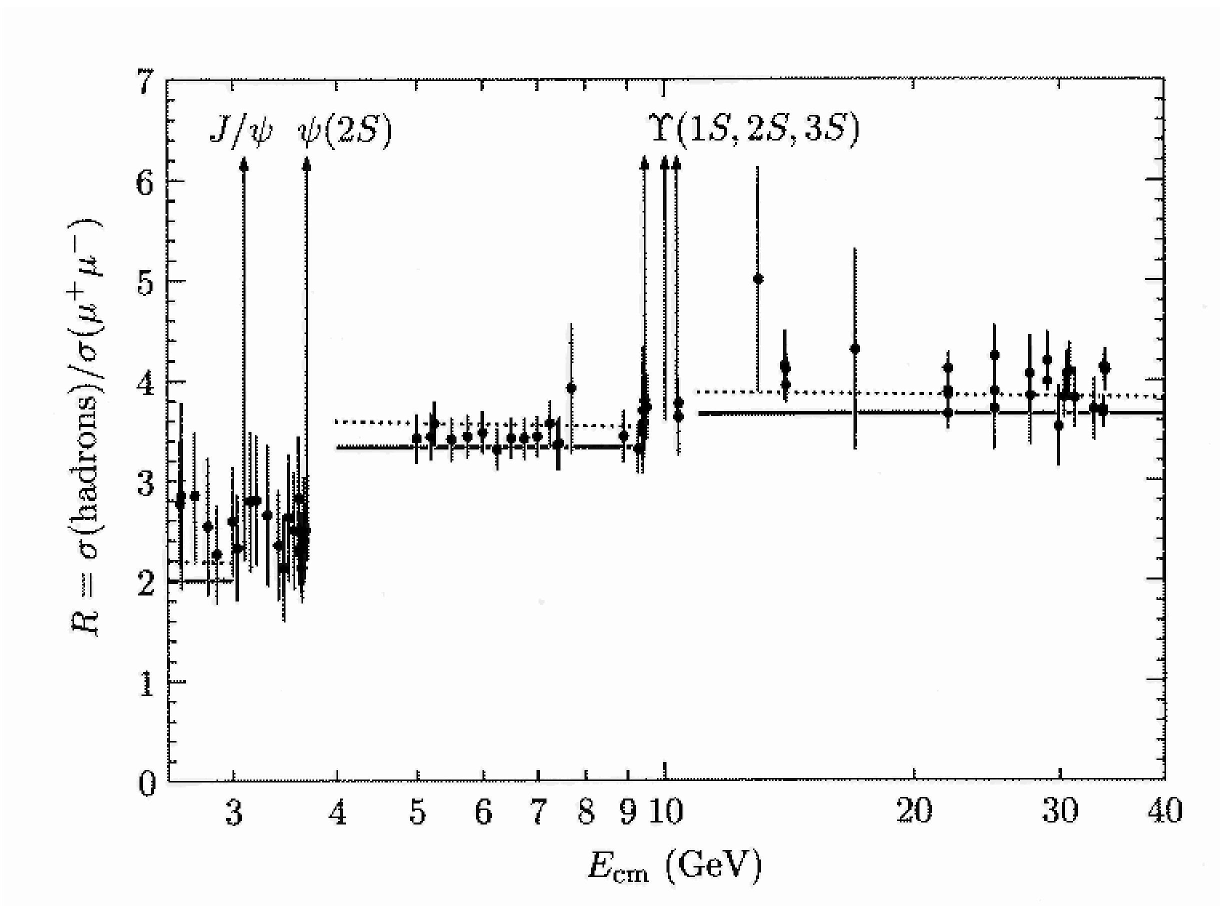
If the centre of mass energy is increased so that $c\bar{c}$ quarks can be created, but b and t are still forbidden by conservation of energy, the ratio is increased to

$$R = \left(\frac{4}{9} + \frac{1}{9} + \frac{1}{9} + \frac{4}{9}\right) = \frac{10}{9},$$

and if b-quarks are created as well

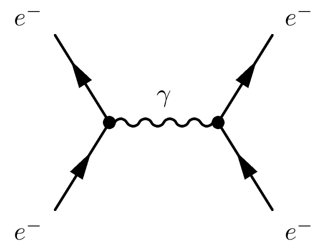
$$R = \left(\frac{4}{9} + \frac{1}{9} + \frac{1}{9} + \frac{4}{9} + \frac{1}{9}\right) = \frac{11}{9}.$$

The experimental measurement of R, shown below, gives $R = 2$ below $c\bar{c}$ threshold and $R = \frac{10}{3}$ above, when J/Ψ particles can be created, and another increase to $R = \frac{11}{3}$ above $b\bar{b}$ threshold, when Υ particles can be created. In each case the ratio R is 3 times larger than our naive result. The explanation of this is that each quark comes in three colours and all three colour possibilities should be added together in the final $q\bar{q}$ state.

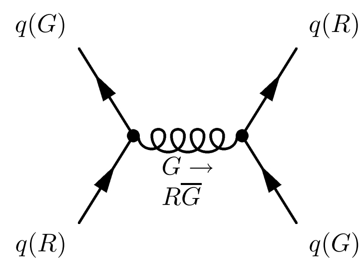


Gluons and Jets*

The relativistic quantum theory of electrodynamics is called *quantum electrodynamics*, or QED, and the relativistic quantum theory of chromodynamics is called *quantum chromodynamics*, or QCD. The details of QED and QCD are too advanced for this course but these two theories have many features in common, although there are also some crucial differences. In QED there is only one kind of charge, which has opposite sign for particles and anti-particles, whereas in QCD there are three kinds of colour charge (red, green and blue) and three kinds of anti-colour (cyan, magenta and yellow). In QED the electromagnetic force is mediated by photons which are massless electrically neutral particles with spin one and negative parity, $J^P = 1^-$.



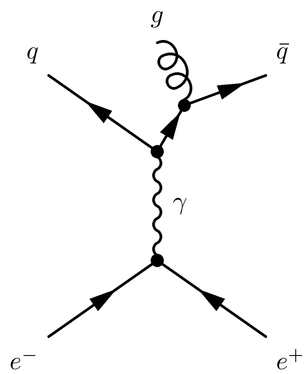
In QCD the colour force is also mediated by massless particles with spin one and negative parity, called *gluons*, but with the difference that gluons carry colour charge. For example a red (r) quark can emit a gluon and turn into a green (g) quark and conservation of colour charge dictates that the gluon must carry away a unit of red charge and a unit of anti-green, or magenta, charge ($r\bar{g}$). All gluons carry one unit of colour and one unit of anti-colour, giving nine kinds of gluon, except there is no colourless gluon in the symmetric combination ($r\bar{r} + g\bar{g} + b\bar{b}$), with b standing for "blue", so there are actually only eight independent types of gluon.



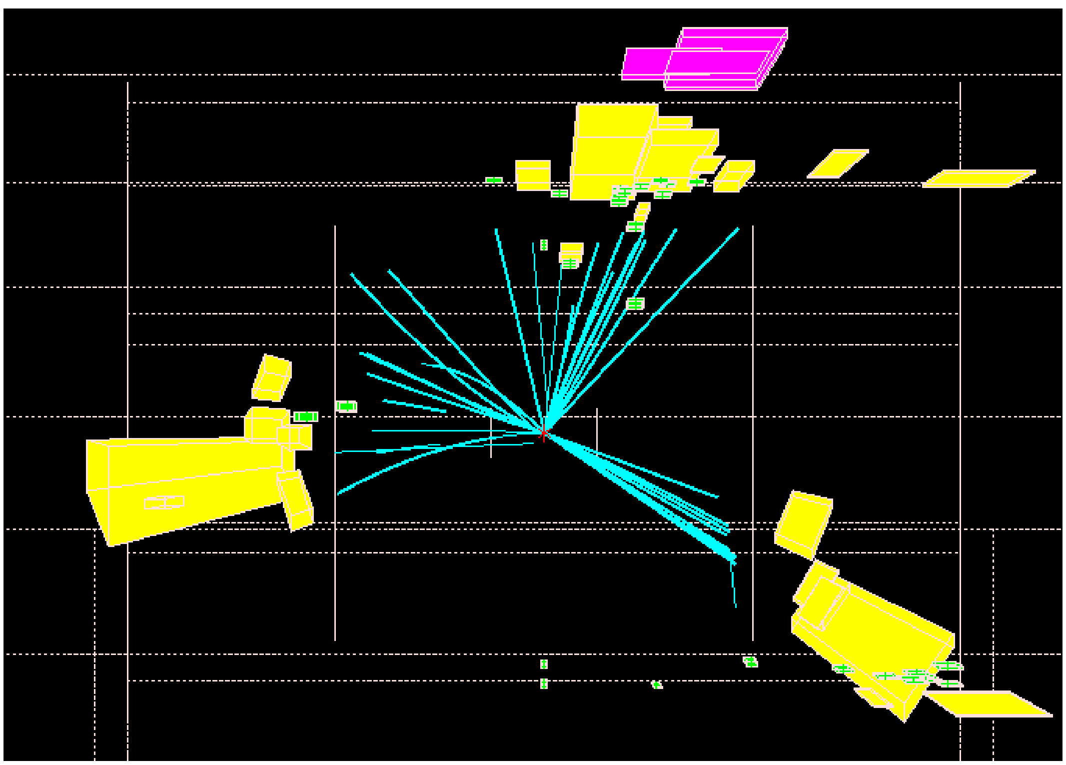
Despite the fact that gluons are believed to be massless, the colour force is not long range — its effect does not extend much beyond the size of a proton (except in so far as pions, colourless particles, mediate the strong force between two baryons, but pions are not fundamental). Neither quarks nor gluons have ever been seen directly, they appear to be confined inside baryons by the colour force. Experimental evidence for gluons can be found in the angular distribution of the hadrons produced in electron-positron annihilation via $e^+e^- \to q\bar{q}$. At high enough centre of mass energies one of the emerging quarks can emit a gluon. Although the quarks and the gluon are not directly observed, they hadronise as they exit giving only hadrons as the observed particles, the outgoing hadrons do carry a memory of the directions of the original quarks and the gluon — the hadrons emerge are

* Sections marked with an asterisk were not covered on the lectures and are include for general interest only.

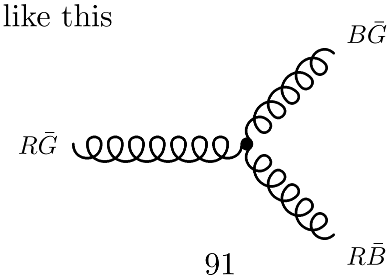
three well collimated 'jets' of particles



Below is a picture of an event produced in a particle accelerator at CERN (the European particle physics centre, Centre Europeén de Recherche Nucléaire) in Geneva by colliding a positron and an electron together at high energy. Three 'jets' of particles radiating away from the interaction point are clearly visible (the rectangular blobs indicate the amount of energy deposited in the detector by particles as they leave the interaction point)

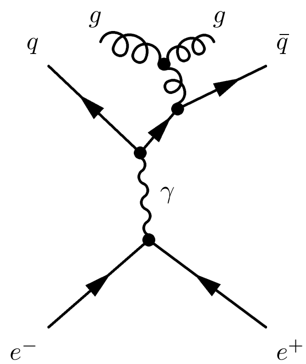


Unlike photons, gluons carry colour charge, which has the interesting consequence that, unlike photons, gluons can interact directly with other gluons and a single gluon can disintegrate into two gluons, like this

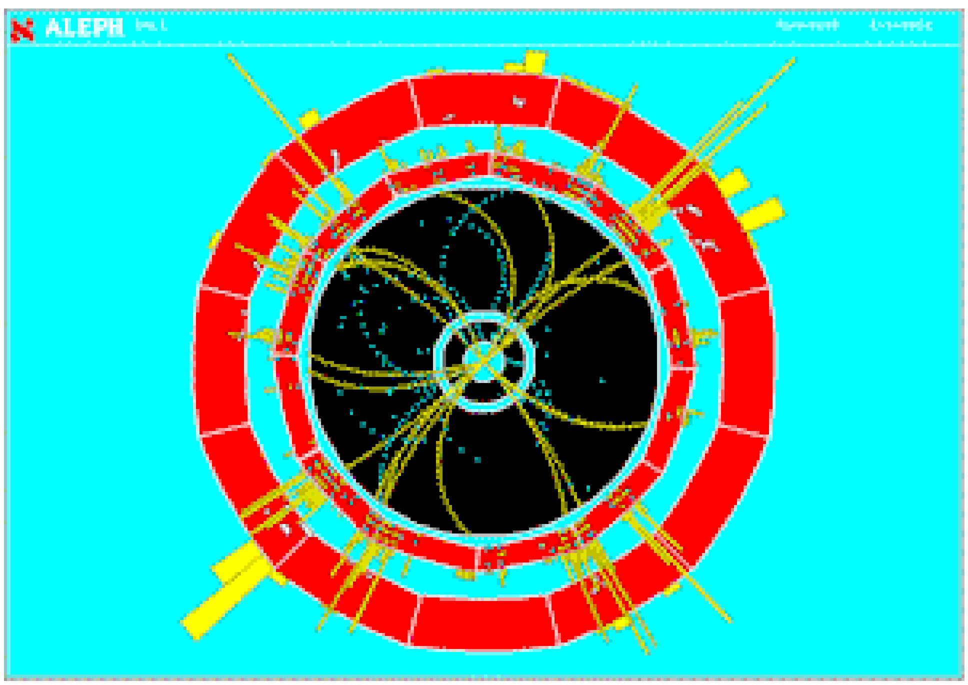


(a photon cannot disintegrate directly into two photons, at least not without creating intermediate electrons and positrons, because photons have zero charge).

This possibility allows for processes like this



which produce four jets of well collimated hadrons emerging from the interaction point. Here is a picture of a four-jet event produced in an electron-positron collision at CERN



By analysing pictures like those above, in which jets of hadrons are seen emanating from a point where an electron and a positron collide at high energy, physicists are able to deduce the physical characteristics of quarks and gluons, such as their spin and electric charge. The experimental results agree well with the predictions of QCD.

Strong decays of particles can be understood in terms of gluons, for example the decay of a J/Ψ-meson (a $c\bar{c}$ bound state) to a ρ^- $(d\bar{u})$ and a π^+ $(u\bar{d})$, must involve a minimum of three gluons, though it can involve more,[58]

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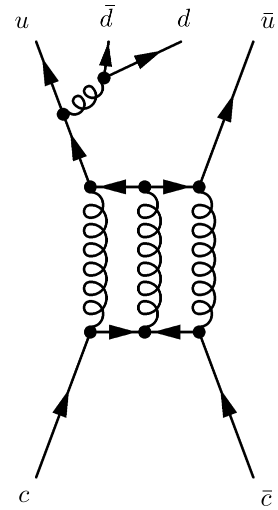
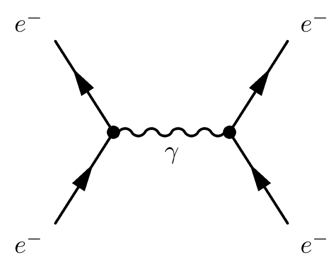
[58] There must be a minimum of three gluons exchanged between the $c\bar{c}$ pair and the $u\bar{u}$ pairs because of conservation of colour and parity in the strong interactions. The incoming J/Ψ particle is colourless and a single gluon has colour charge so conservation of colour implies that single gluon exchange is forbidden. Two gluon exchange is forbidden by



Asymptotic Freedom*

There is a subtle quantum mechanical effect associated with gluon exchange which has the surprising consequence that the closer two quarks get to one another the *weaker* the colour force gets. As the distance between the quarks goes to zero the force between them in fact vanishes and the quarks essentially become free particles (or at least they would if it were not for the Coulomb force due to their electric charges). This phenomenon is called *asymptotic freedom*, because the force vanishes asymptotically as the quarks approach each other. The effect relies once again on virtual particles borrowing energy to appear out of nothing for a fleeting existence before disappearing again. Consider the simpler process of a photon being exchanged between two electrons:

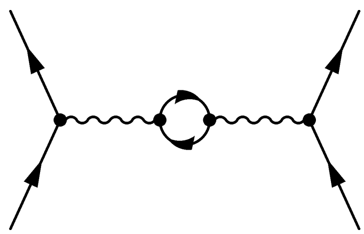


The photon exchanged between the two electrons in this process has a space-like four momentum, $\underset{\sim}{Q}.\underset{\sim}{Q} = -s/c^2 > 0$ (the proof of this is left as an exercise). The amplitude for a similar process, $e^+e^- \rightarrow \mu^+\mu^-$ for which the virtual photon was time-like, was calculated earlier, equation (44), at least when masses and spins were ignored. For our purposes here we can use the same expression (44) and take the amplitude to be proportional to α/s

parity conservation, the J/Ψ is a $J^P{=}1^-$ state with negative parity and gluons, like photons, also have negative parity. Two gluons would have parity $(-1)^2{=}{+}1$ and parity conservation requires that an odd number of gluons must be exchanged. The minimum number that can satisfy both these criteria is three. While any odd number greater than three can also be exchanged, and this does happen, the probability of it happening is less than that for three gluon exchange.

where $\alpha = \frac{e^2}{4\pi\epsilon_0}$ is the fine structure constant (there is an unimportant overall factor of $4\pi\hbar^3 c$ in (44) which we shall ignore here).

If $|s| \gtrsim 4m_e^2c^4$ then the virtual photon can create a virtual electron-positron pair that can exist for a very short time before annihilating each other leaving behind another virtual photon (remember that an electron with space-like four momentum $\underset{\sim}{P}$, with $\underset{\sim}{P}.\underset{\sim}{P} > 0$, can only be virtual because a real electron must have time-like four momentum with $\underset{\sim}{P}.\underset{\sim}{P} = -m_e^2c^2 < 0$).



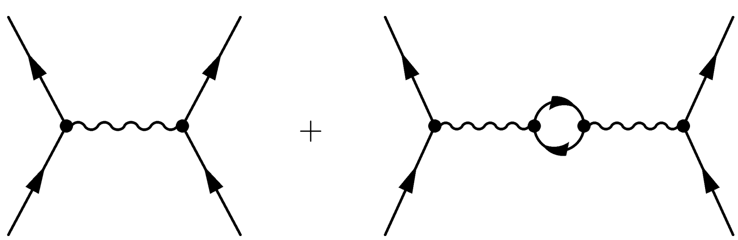
Now Feynman diagrams are a pictorial representation of quantum mechanical amplitudes and there are rules in QED that allow the complex numbers corresponding to each amplitude to be calculated. To obtain the full amplitude we must add the amplitude for each process.[59] Although it will not be derived here, that is beyond the scope of this course, we quote the amplitude for the above process. It depends on the four momentum of the exchanged photon: it vanishes for $\sqrt{|s|} \lesssim 2m_ec^2$ and, for $\sqrt{|s|} \gtrsim 2m_ec^2$, it is (again ignoring an overall constant factor of $-4\pi\hbar^3 c$)

$$\frac{\alpha}{s} X(s)$$

where

$$X(s) = \frac{\alpha}{3\pi} \ln\left(\frac{|s|}{4m_e^2c^4}\right).$$

Adding this to the first diagram we get

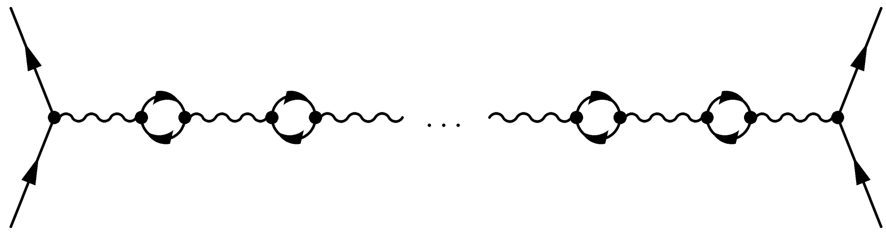


with numerical value

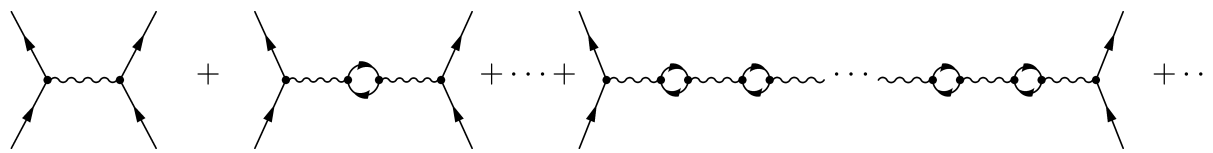
$$\frac{\alpha}{s}\left(1 + X(s)\right).$$

In fact any number of virtual electron-pairs can be created in series

[59] An apparent problem showed up here when people first tried to calculate the amplitude associated with the electron-positron pair diagram in that the answer was infinite! This was a serious problem for the quantum theory of electrodynamics for many years until it was realised that these infinities are harmless and can be removed in a clever process called *renormalisation*. The details of the renormalisation process are too advanced for this course.



and, according to the rules of quantum mechanics, these should all be added to get the total amplitude:



Algebraically the total amplitude, up to an overall constant, can be written as

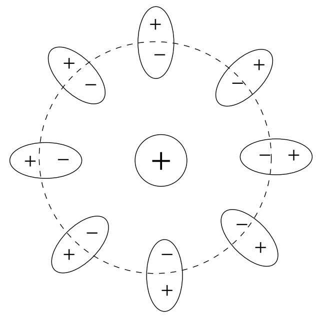
$$\widetilde{U} = \frac{\alpha}{s}\sum_{n=0}^{\infty} X^n(s) = \frac{\alpha}{s}\left(\frac{1}{1-X(s)}\right) = \frac{\alpha}{s}\left(\frac{1}{1-\frac{\alpha}{3\pi}\ln(|s|/4m_e^2c^4)}\right), \tag{51}$$

for $\sqrt{|s|} \gtrsim 2m_ec^2$. Finally the physical effect of all these virtual electron-positron pairs can be accounted for by making the fine structure constant a function of the momentum transferred during the collision and writing

$$\widetilde{U} = \frac{\alpha(s)}{s} \qquad \text{with} \qquad \alpha(s) = \begin{cases} \frac{\alpha}{1-\frac{\alpha}{3\pi}\ln(|s|/4m_e^2c^2)} & \sqrt{|s|} \gtrsim 2m_ec^2 \\ \alpha & \sqrt{|s|} \lesssim 2m_ec^2 \end{cases} .$$

The net result of all this is that the value of the fine structure constant effectively depends on the momentum transferred and grows as $|s|$ grows. Large momentum is equivalent to short distances and the charge on the electron gets larger as we get closer to the electron, leading to a modification of the Coulomb potential. In fact, if this result is used for arbitrarily high $|s|$, we would conclude that $\alpha(s)$ becomes infinite when $\sqrt{|s|} = 2m_ec^2\exp\left(\frac{3\pi}{2\alpha}\right)$, but this corresponds to a colossal energy, $\sqrt{|s|} \approx 2m_ec^2\exp\left(\frac{3\pi}{2\alpha}\right) \approx 2m_ec^2 \times 10^{280}$, equivalent to many times the mass of the universe, so this conclusion is unwarranted — new physical processes will certainly become important before this energy scale is reached and even within the theory of QED itself equation (51) is only valid when $\alpha(s)$ is small.

An intuitive picture of what is happening here is that short-lived electron pairs can pop out of the vacuum and a free charge, like a real physical electron, will preferentially attract the opposite charge of the pair during the pair's brief existence. This effect is more pronounced the closer we get to the free charge and the vacuum itself behaves like a dielectric medium. In a dielectric, such as water or wax, the electric field produced by a charge is less than it would be in a vacuum, because the molecules in the medium respond to the field produced by the charge by becoming polarised in a manner that reduces the field. For example a sphere with a positive charge at the centre actually contains less charge than one would expect because the molecules of the medium behave like little electric dipoles and line up with the radial electric field, like this



This effect is called *screening* because the medium has the effect of reducing the charge, in a sense it tries to hide the charge reducing its effectiveness. Mathematically the effect is encoded in the dielectric constant of the medium through the electric permittivity $\epsilon > \epsilon_0$, so

$$\nabla.\mathbf{E} = \rho/\epsilon < \rho/\epsilon_0$$

where ϵ_0 is the short distance (vacuum) permittivity. The closer we get to the charge the less effective the medium is at screening it and the 'effective' charge increases as we approach.

A similar effect occurs in a vacuum, except it is not real charges in a polarisable medium that cause the effect, but virtual dipoles that are electron-positron pairs. The increase in $\alpha(s)$ as $|s|$ increases can be interpreted as a decrease in the electric permittivity of the vacuum, that is the vacuum behaves as though it has a dielectric constant $\epsilon(s)$ that decreases with $|s|$ and gets smaller as we approach the free charge, increasing as we recede from it,

$$\alpha(s) = \frac{e^2}{4\pi\epsilon(s)\hbar c}$$

and approaching ϵ_0 only for large distances (small $|s|$). This affects Maxwell's equation so that

$$\nabla.\mathbf{E} = \frac{\rho}{\epsilon(s)} > \frac{\rho}{\epsilon_0}$$

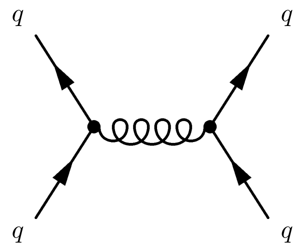
The usual permittivity of the vacuum, ϵ_0, is only the large distance (small $|s|$) permittivity. Close to a physical charge (large $|s|$) $\epsilon(s)$ is smaller and the vacuum becomes more and more dielectric as $|s|$ decreases. There is strong experimental evidence that $\alpha(s)$ really does grow with energy and this phenomenon is known as *vacuum polarisation.*

Alternatively, since the speed of light in the vacuum is invariant, $c^2 = 1/(\epsilon\mu) = 1/(\epsilon_0\mu_0)$, we can write

$$\alpha(s) = \frac{e^2}{4\pi\epsilon(s)\hbar c} = \frac{e^2\mu(s)c}{4\pi\hbar}$$

and we say that the effective magnetic permeability of the vacuum $\mu(s)$ decreases as $|s|$ decreases, approaching the vacuum value μ_o for large distances: the vacuum behaves like a diamagnet, a material in which the magnetic permeability is less than μ_0, so that currents generate smaller magnetic fields that they would if the magnetic permeability were μ_0.

Now consider the exchange of a single gluon between two quarks in QCD:



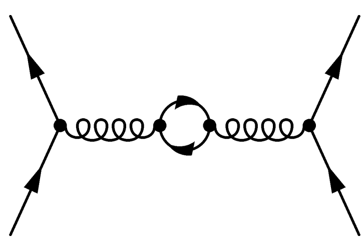
The amplitude of this process is

$$\widetilde{U} = \frac{\alpha_{QCD}}{s}$$

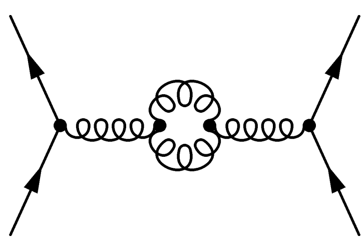
where $\underset{\sim}{Q}$ is the 4-momentum of the gluon transferred between the quarks ($s = -(\underset{\sim}{Q}.\underset{\sim}{Q})c^2$) and

$$\alpha_{QCD} = \frac{g_{QCD}}{4\pi\hbar c}$$

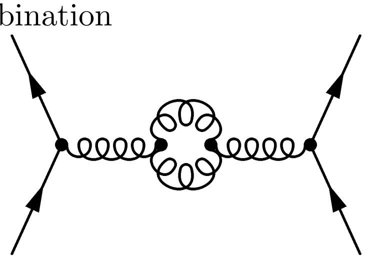
is the QCD analogue of the fine structure constant of electromagnetism, depending on the magnitude of the colour charge g_{QCD} (the QCD analogue of ϵ_0 is set equal to one by convention — it is essentially absorbed into the definition of g_{QCD}). Just as in QED, the exchanged gluon can produce a $q\bar{q}$ pair if $|s|$ is large enough:



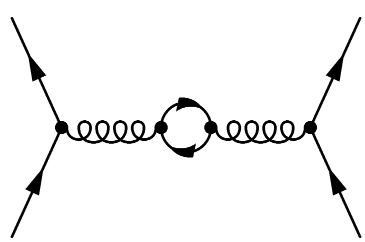
But now in QCD there is a second process which is allowed — the exchanged gluon can also produce a gluon–anti-gluon pair like this



The combination



$+$



should be added as quantum mechanical amplitudes. The numerical value requires a calculation that is too advanced for this course and we shall just quote the result as being proportional to

$$\frac{\alpha_{QCD}}{s}Y(s)$$

where

$$Y(s) = \frac{(2N-33)\alpha_{QCD}}{6\pi}\ln(|s|/4m^2c^4)$$

where N is the number of quarks with $2mc^2 < \sqrt{|s|}$.[60] The -33 comes from the gluon diagram above.

Again an infinite series of quark and gluon diagrams, like those above, can be connected together to give an s-dependent strong coupling constant

$$\frac{\alpha_{QCD}(s)}{s} = \frac{\alpha_{QCD}}{s}\sum_{n=0}^{\infty}Y^n(s) = \frac{\alpha_{QCD}}{s}\left(\frac{1}{1-Y(s)}\right).$$

The net result is that the QCD fine structure constant $\alpha_{QCD}(s)$ depends on $|s|$ and is

$$\alpha_{QCD}(s) = \frac{\alpha_{QCD}}{1+\frac{(33-2N)\alpha_{QCD}}{6\pi}\ln(|s|/4m^2c^2)},$$

The crucial point about this formula is that, provided $N < 33/2$, $\alpha_{QCD}(s)$ decreases as $|s|$ increases, *i.e.* QCD becomes weaker at higher energies and shorter distances. In the language of magnetism, the quarks give a diamagnetic contribution to the magnetic permeability of the vacuum (the $-2N$ in the formula above) while the gluons give a paramagnetic contribution (the $+33$ in the formula above). As long as there are not too many quarks, the gluons win and the vacuum behaves more and more like a paramagnetic as the quarks get further away from one another.

At very short distances quarks behave as though they are free particles as far as QCD is concerned — this phenomenon is called asymptotic freedom. By the same token the colour force gets stronger as we recede from a quark. Suppose we try to pull a quark out of a proton, for example. As it recedes from the other two quarks the colour force between them gets *stronger* and it gets more and more difficult to pull the quark further out. This is believed to be the reason why quarks are confined inside hadrons, though the details of the process are still poorly understood.

The Noble prize for physics was awarded in 2004 to the three physicists who first published the calculations proving asymptotic freedom.

[60] For simplicity the formula is written assuming all the quark masses are the same, but in fact N changes every time s is increased to the point where a new type of quark can contribute.

5. Weak Interactions

Weak interactions occur between all types of leptons and quarks, indeed they are the only interactions in which neutrinos participate. They are mediated by W and Z-bosons, in a manner very similar to the way that photons mediate the electromagnetic force and gluons mediate the colour force. A crucial difference however is that W and Z-bosons are massive $m_W = 80.4\ GeV/c^2$, $m_Z = 91.2\ GeV/c^2$, so the weak force is very short range, and also W-bosons carry electric charge ± 1 (there are two kinds of W-bosons W^+ and W^- which are anti-particles of one another, the Z-boson is electrically neutral and this often exhibited explicitly as Z^0).

Interactions among particles can be classified as electromagnetic, weak or strong.

An example of a purely electromagnetic interaction is the scattering of an electron off a quark (inside a proton, for example) by exchanging a photon,

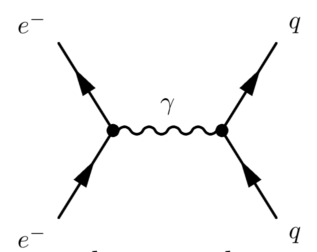


Fig. 3: Electron-quark scattering via photon exchange.

But electrons and quarks can also scatter off one another by exchanging a W-boson, a purely weak interaction,

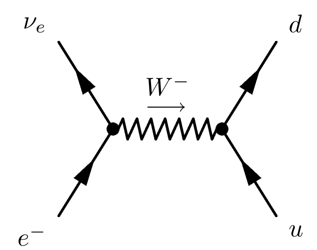


Fig. 4: Electron-quark scattering via W-exchange.

This is called a *charged current interaction* because the W^+ that is exchanged can be thought of as carrying a small electric current with it, due to its electric charge.

Neutrinos however do not interact with photons, since they have no electric charge, but they do interact with W and Z-bosons. For example an electron and a neutrino can exchange a Z^0:

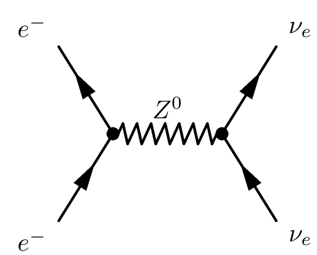


Fig. 5: Electron-neutrino scattering via Z^0-exchange.

This is called a *neutral current interaction*, to distinguish it from the charged current interaction carried by the W-boson above.

More complicated processes can involve two or more different kinds of interactions. For example the decay $\Lambda \to p + \pi^-$ of a Λ-baryon (usd) to a proton (uud) and a π^- ($d\bar{u}$) necessarily involves both the weak and the strong interactions:

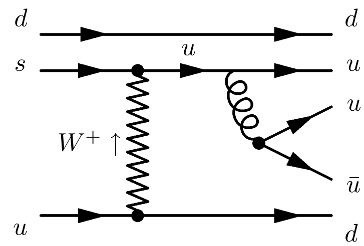


Fig. 6: Λ-decay. A weak decay, proceeding via W-exchange, but with strong interactions dictating the final decay products. Note that, by absorbing a W^+-boson, an s-quark turns into a u-quark in this decay — a phenomenon known as *quark mixing.* (In this diagram the particles enter from the left and exit on the right.)

Weak interactions differ from electromagnetic and colour interactions in a number of ways. They occur between all types of matter particles, both leptons and quarks. They are classified as *leptonic*, *semi-leptonic* and *non-leptonic*, depending on whether or not they involve leptons, leptons and quarks or quarks only. For example, in figures 5 and 6 above we have

$$e^- + \nu_e \to e^- + \nu_e \qquad \text{(leptonic)}$$
$$\Lambda \to p + \pi^- \qquad (s \to u + \bar{u} + d) \qquad \text{(non-leptonic)}$$

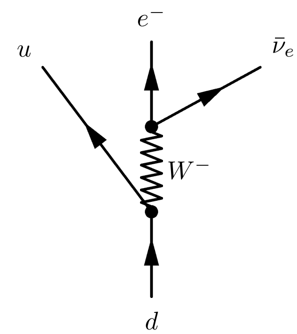
while figure 4 could represent what happens if a high energy electron penetrates a proton and turns one of the u-quarks into a d-quark,

$$p + e^- \to n + \nu_e \qquad (e^- + u \to \nu_e + d) \qquad \text{(semi-leptonic)}$$

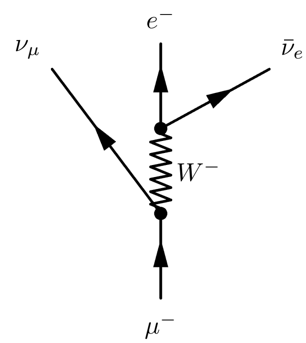
A related process is β-decay, when a neutron decays to a proton and emits an electron (β-particle),

$$n \to p + e^- + \bar{\nu}_e \qquad (d \to u + e^- + \bar{\nu}_e) \qquad \text{(semi-leptonic)}$$

At the quark level a d-quark in the neutron emits a W^- and turns into a u-quark, like this



An example of a purely leptonic decay is $\mu^- \to \mathrm{e}^- + \bar{\nu}_e + \nu_\mu$,



In terms of the above classification these decays are:

$$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu \qquad \text{(leptonic)}$$
$$n \to p + e^- + \bar{\nu}_e \qquad \text{(semi-leptonic)}$$
$$\Lambda \to p + \pi^- \qquad \text{(non-leptonic).}$$

Weak interactions violate parity. This originates in the fact that W-bosons do not couple to left-handed and right-handed fermions in the same way. Right-handed fermions *do not feel the weak force at all*! Only left-handed fermions couple to the W and Z-bosons.[61]

There is a potential associated with the weak interactions which, because the W and Z-bosons are massive, takes the same form as the Yukawa potential discussed earlier for pion

$$U(r) = \frac{g_W^2}{4\pi r}\mathrm{e}^{-\kappa r}$$

where $\kappa = \frac{Mc}{\hbar}$ is the inverse Compton wavenumber of the exchanged particle (the Compton wavelength is $\lambda = \frac{h}{Mc}$ where M is either the W or the Z mass) and g_W is a weak 'charge' analogous to the electric charge in the Coulomb potential, $\frac{e^2}{4\pi\epsilon_0 r}$. The Fourier transform of the weak potential is

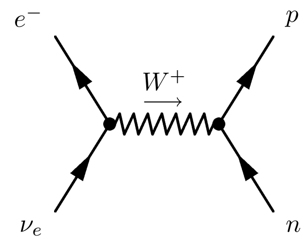
$$\widetilde{U}(\mathbf{k}) = \int U(r)\mathrm{e}^{i\mathbf{k}.\mathbf{x}} d^3x = \frac{g_W^2}{\kappa^2 + k^2}, \tag{52}$$

[61] Strictly speaking it is not helicity that dictates the type of particle that couples to W and Z-bosons, but a related property called *chirality*. The technical definition of chirality is beyond the scope of these lectures, suffice it to say that chirality, unlike helicity, is always Lorentz invariant. For massless particles with positive energy helicity and chirality are the same thing.

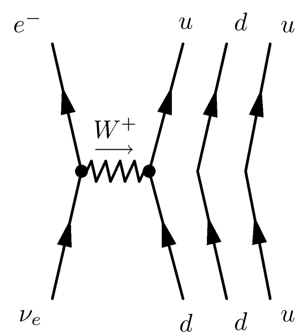
where $\kappa = \frac{2\pi}{\lambda}$. Using this potential we can calculate the cross-sections for some weak processes. Consider for example inverse β-decay

$$n + \nu_e \rightarrow p + e^-$$

which can happen if a high energy neutrino strikes a neutron, [62]



Inside the neutron the W^+-boson is absorbed by a d-quark, turning it into a u-quark



Let the 4-momenta of the incoming neutron and neutrino be $\underset{\sim}{P}_n$ and $\underset{\sim}{P}_\nu$ respectively and for the outgoing proton and electron let the 4-momenta be $\underset{\sim}{P}_p$ and $\underset{\sim}{P}_e$ respectively. Conservation of 4-momentum demands that the total 4-momentum $\underset{\sim}{P}$ remains unchanged during the collision

$$\underset{\sim}{P} = \underset{\sim}{P}_n + \underset{\sim}{P}_\nu = \underset{\sim}{P}_p + \underset{\sim}{P}_e.$$

In the centre of mass frame the total 4-momentum is

$$\underset{\sim}{P} = (E/c, \mathbf{0})$$

where E is the total energy,

$$E = E_n + E_\nu = E_p + E_e.$$

Let $\underset{\sim}{Q}$ represent the 4-momentum of the exchanged W^+ particle, then conservation of 4-momentum at each vertex in the diagram above for $n + \nu_e \rightarrow p + e^-$ requires that

$$\underset{\sim}{P}_n + \underset{\sim}{Q} = \underset{\sim}{P}_p \qquad \text{and} \qquad \underset{\sim}{P}_\nu = \underset{\sim}{Q} + \underset{\sim}{P}_e.$$

[62] The time reversed process, $p+e^- \rightarrow n+\nu_e$, happens at the end of a massive star's life. When a star runs out of nuclear fuel it can no longer sustain itself against gravitational collapse and it contracts, forcing the protons and electrons into a smaller and smaller volume until they fuse into neutrons, releasing a huge number of neutrinos in a massive nuclear explosion — a supernova.

If we decompose the individual 4-momenta into their time-like and space-like components in the centre of mass frame

$$\underset{\sim}{P}_n = (E_n/c, \underline{P}), \qquad \underset{\sim}{P}_\nu = (E_\nu/c, -\underline{P}), \qquad \underset{\sim}{P}_e = (E_e/c, -\underline{P}'), \qquad \underset{\sim}{P}_p = (E_p/c, \underline{P}'),$$

and

$$\underset{\sim}{Q} = (Q^0, \underline{Q}) = \Big((E_p - E_n)/c, \underline{P}' - \underline{P}\Big) = \Big((E_\nu - E_e)/c, -\underline{P} + \underline{P}'\Big).$$

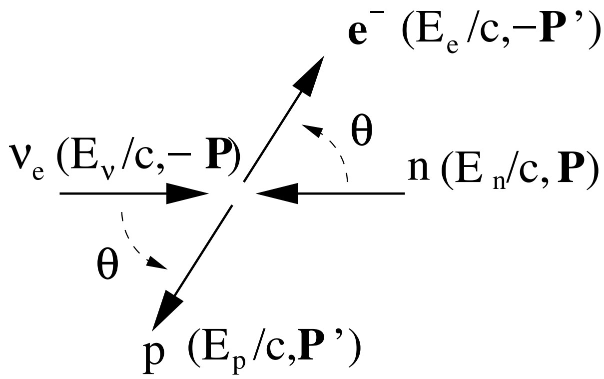
If the neutrino is very energetic, $E_\nu >> m_e c^2$, then we can ignore m_e and so, assuming also that the neutrino is massless,

$$\begin{array}{rclcl} \underset{\sim}{P}_\nu . \underset{\sim}{P}_\nu = 0 & \Rightarrow & P^2 := \underline{P}.\underline{P} = E_\nu^2/c^2 & \Rightarrow & P = E_\nu/c \\ \underset{\sim}{P}_e . \underset{\sim}{P}_e = -m_e^2 c^2 \approx 0 & \Rightarrow & P'^2 := \underline{P}'.\underline{P}' = E_e^2/c^2 & \Rightarrow & P' = E_e/c \end{array} \qquad (53)$$

So

$$\begin{aligned} \underset{\sim}{Q}.\underset{\sim}{Q} &= -(Q^0)^2 + \underline{Q}.\underline{Q} = -(E_\nu - E_e)^2/c^2 + (\underline{P}' - \underline{P}).(\underline{P}' - \underline{P}) \\ &= 2\Big(E_\nu E_e/c^2 - \underline{P}.\underline{P}'\Big) = 2\frac{E_\nu E_e}{c^2}(1 - \cos\theta) = \frac{4E_\nu E_e}{c^2}\sin^2(\theta/2), \end{aligned}$$

where the θ is the angle between the incoming neutron and the outgoing proton, $\underline{P}.\underline{P}' = PP'\cos\theta$,



Note that

$$q^2 := \underset{\sim}{Q}.\underset{\sim}{Q} = 2\frac{E_\nu E_e}{c^2}(1 - \cos\theta) \geq 0$$

is space-like. The electron and proton energies can be expressed in terms of the neutrino and neutron energies as follows: since

$$E_e = E - E_p \quad \Rightarrow \quad E_e^2 = E^2 - 2EE_p + E_p^2$$

and

$$E_p^2/c^2 - P'^2 = m_p^2 c^2 \quad \Rightarrow \quad c^2 P'^2 = E_e^2 = E_p^2 - m_p^2 c^4,$$

using (53), from which

$$E_p = \frac{E^2 + m_p^2 c^4}{2E} \qquad \text{and} \qquad E_e = \frac{E^2 - m_p^2 c^4}{2E}.$$

Using this in q^2 above gives

$$q^2 = \frac{E_\nu(E^2 - m_p^2c^4)}{Ec^2}(1 - \cos\theta).$$

In fact we can even eliminate E and express everything in terms of the incoming neutrino energy and known constants using

$$E = \sqrt{E_\nu^2 + m_n^2c^4} + E_\nu$$

(we leave the proof of this as an exercise in relativistic kinematics).

Ignoring spin the cross-section can now be calculated using $M_{fi} = \widetilde{U}(k)$, with $\widetilde{U}(k)$ given in (52) and $q = \hbar k$,

$$\frac{d\sigma}{d\Omega} = \frac{1}{4\pi^2\hbar^4}\frac{|M_{fi}|^2p_f^2}{v_fv_i} = \frac{g_W^4}{4\pi^2(M_W^2c^2+q^2)^2}\frac{p_f^2}{v_fv_i}.$$

Including spin is a little subtle because of the way the W-boson interacts with quarks and leptons, we need to know something about the amplitude $M_{fi}^{h_eh_u;h_\nu h_d}$ where h_e, h_u, h_ν and h_d are the helicities of the electron, the u-quark, the neutrino and the d-quark respectively. We will not go into too many details here, but the final answer is that spin effects reduce the cross-section by a factor of 2. Only left-handed neutrinos have ever been observed directly in any experiment so far and this implies that the W-boson only couples to left-handed neutrinos: even if right-handed neutrinos exist they are never produced by weak interactions. The incoming neutrino beam is therefore 100% polarised, it is purely left-handed, and we therefore do not average over the incoming helicity states of the neutrino. The incoming d-quark can be either left or right-handed but the parity violation of the weak interactions has been established to be such that the W-boson only couples to left-handed quarks,[63] so $M_{fi}^{h_eh_u;h_\nu h_d}$ is zero if the incoming quark is right-handed. Averaging over the incoming d-quark helicity (or, equivalently, averaging over the incoming neutron helicities) therefore introduces a factor of a half — basically the cross-section is reduced by a factor of 2 because the W-boson only interacts with half of the incoming neutrons. The net result is that spin effects reduce the cross-section by a factor of 2 giving

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{g_W^4}{8\pi^2(M_W^2c^2+q^2)^2}\frac{p_f^2}{v_fv_i}.}$$

In this expression p_f is the relativistic 3-momentum of one of the outgoing particles, in the centre of mass frame, and q is the momentum transferred between the two colliding particles,

$$p_f = P' = E_e/c = \frac{E^2 - m_p^2c^4}{2Ec} \qquad \text{and} \qquad q^2 = 2\frac{E_\nu p_f}{c}(1 - \cos\theta).$$

[63] Strictly speaking this is only true when the particles are relativistic, but a more detailed analysis yields the same factor of one-half.

v_i is the relative speed of the two initial state incoming particles and v_f is the relative speed of the two outgoing final state particles.

To understand the implications of this cross-section consider some limiting cases:
i) $m_e c^2 << E_\nu << m_n c^2$. This implies that n and p are slow, so we can set $E_n \approx m_n c^2$ and $v_n/c \approx 0$ together with $E_p \approx m_p c^2$ and $v_p/c \approx 0$, and that the electron is relativistic with $v_e \approx c \Rightarrow v_f = v_p + v_e \approx c$ (remember v_f is the relative speed of the outgoing particles) together with $v_\nu = c \Rightarrow v_i = v_n + v_\nu \approx c$. Also

$$E^2 - m_p^2 c^4 \approx m_n^2 c^4 + 2E_\nu m_n c^2 - m_p^2 c^4 = (m_n - m_p)(m_n + m_p)c^4 + 2E_\nu m_n c^2 \approx 2E_\nu m_n c^2$$

since $E_\nu >> m_e c^2 \approx (1/2)(m_n - m_p)c^2$. So

$$q^2 = \frac{E_\nu (E^2 - m_p^2 c^4)}{Ec^2}(1 - \cos\theta) \approx \frac{2E_\nu^2 m_n c^2}{(m_n c^2 + E_\nu)c^2}(1 - \cos\theta) \approx \frac{2E_\nu^2}{c^2}(1 - \cos\theta) << M_W^2 c^2$$

and

$$p_f = \frac{E^2 - m_p^2 c^4}{2Ec} \approx \frac{2E_\nu m_n c^2}{2(E_n + E_\nu)c} \approx \frac{2E_\nu m_n c^2}{2(m_n c^2 + E_\nu)c} \approx \frac{E_\nu}{c}.$$

Substituting these into the differential cross-section gives

$$\frac{d\sigma}{d\Omega} \approx \frac{g_W^4}{8\pi^2 M_W^4 c^4}\frac{E_\nu^2}{c^2}\frac{1}{c^2} = \frac{g_W^4 E_\nu^2}{8\pi^2 M_W^4 c^8}$$

which is independent of θ, so the total cross-section

$$\sigma \approx \frac{g_W^4 E_\nu^2}{2\pi M_W^4 c^8}$$

grows quadratically with energy, for $E_\nu << m_n c^2$.

ii) $E_\nu >> m_n c^2$. In this extreme relativistic limit we can ignore the proton and neutron masses and set $m_n \approx 0$, $m_p \approx 0$, $E_n \approx E_\nu \approx E/2$, $E_p \approx E_e \approx E/2$, $v_i \approx v_f \approx 2c$. This gives

$$p_f \approx \frac{E}{2c} \approx \frac{E_\nu}{c}, \qquad q^2 \approx \frac{2E_\nu^2}{c^2}(1 - \cos\theta)$$

so

$$\frac{d\sigma}{d\Omega} \approx \frac{g_W^4}{8\pi^2}\frac{1}{(M_W^2 c^2 + q^2)^2}\left(\frac{E_\nu}{c}\right)^2 \frac{1}{4c^2} = \frac{g_W^4 E_\nu^2}{32\pi^2}\left\{\frac{1}{M_W^2 c^4 + 2E_\nu^2(1 - \cos\theta)}\right\}^2$$

which goes like $\sim 1/E_\nu^2$ for $E_\nu >> M_W c^2$ and θ is not too small.

The total cross-section is now

$$\begin{aligned}\sigma &= \int \frac{d\sigma}{d\Omega} d\Omega = 2\pi \int_0^\pi \frac{d\sigma}{d\Omega}\sin\theta d\theta \\ &\approx \frac{g_W^4 E_\nu^2}{16\pi}\int_0^\pi \left\{\frac{1}{M_W^2 c^4 + 2E_\nu^2(1 - \cos\theta)}\right\}^2 \sin\theta d\theta = \frac{g_W^4 E_\nu^2}{8\pi M_W^2 c^4}\left(\frac{1}{M_W^2 c^4 + 4E_\nu^2}\right).\end{aligned}$$

If $m_n c^2 << E_\nu << M_W c^2$ (which means E_ν is somewhere in the region of 10 GeV) the denominator is independent of E_ν, to a good approximation,

$$\sigma \approx \frac{g_W^4 E_\nu^2}{8\pi M_W^4 c^8}$$

which again grows quadratically, but with a different pre-factor to that of case i). Experimentally

$$\sigma(E_\nu \approx 10\ GeV) \approx 10^{-40}\ m^2 = 10^{-12}\ bn = 1\ pb,$$

where 1 $barn = 10^{-28}\ m^2$. This is a small cross-section: compare it, for example, with the typical cross-section for $e^- e^+ \to \mu^- \mu^+$ at $s = q^2 c^2 = (10\ GeV)^2$ of around $8 \times 10^{-38}\ m^2 = 8 \times 10^{-10}\ bn = 0.8\ nb$, as calculated earlier in (50). This is why the weak interactions are called "weak". But using the experimental value for the W-mass, $M_W = 80\ GeV/c^2$, the weak analogue of the fine structure constant at $E_\nu \approx 10\ GeV$, is

$$\alpha_W := \frac{g_W^2}{4\pi\hbar c} \approx \frac{1}{30}$$

which is significantly greater than the electromagnetic fine structure constant $\alpha = \frac{e^2}{4\pi\epsilon_o \hbar c} \approx \frac{1}{137}$. The weak force is not intrinsically weak, in absolute terms it is in fact stronger than the electromagnetic force, it only appears to be weak because the W and Z-bosons are so massive.

Kaon decay and CP violation

We have already come across an example of a weak decay in the β-decay of the neutron

$$n \to p + e^- + \bar{\nu}_e$$

and charged pions also decay by weak interactions

$$\pi^+ \to \mu^+ + \nu_\mu$$
$$\pi^- \to \mu^- + \bar{\nu}_\mu.$$

Kaons decay by weak interactions too. The physics of kaon decay is a fascinating subject, exhibiting some rather subtle quantum effects. Kaons are hadronic particles and are produced by strong interactions, but they have strangeness $S = \pm 1$ and decay mostly to pions, violating conservation of strangeness, $\Delta S = \pm 1$, so they can only decay by weak interactions. Kaons are part of the pion octet, $J^P = 0^-$, and so are eigenstates of parity with $\mathcal{P} = -1$. In particular K^0 and $\bar{K}^0$ are neutral particles and charge conjugation C sends particles to anti-particles, so it sends K^0 to $\bar{K}^0$ and *vice versa*. We can choose the phases of K^0 and $\bar{K}^0$ so that

$$\mathbf{CP}|K^0> = |\bar{K}^0> \qquad \text{and} \qquad \mathbf{CP}|\bar{K}^0> = |K^0> .$$

Although $|K^0>$ and $|\bar{K}^0>$ are not themselves eigenstates of **CP** the linear combinations

$$|K_S> := \frac{1}{\sqrt{2}}(|K^0> + |\bar{K}^0>), \qquad \mathbf{CP} = +1$$
$$|K_L> := \frac{1}{\sqrt{2}}(|K^0> - |\bar{K}^0>), \qquad \mathbf{CP} = -1$$

are, and weak interactions conserve the combination **CP** although they do not conserve **C** and **P** separately.

$|K^0>$ and $|\bar{K}^0>$ are defined by their mode of production (*e.g.* $\pi^- + p \to K^0 + \Lambda$) while $|K_S>$ and $|\bar{K}_L>$ are defined by their mode of decay:

$$K_S \to \begin{cases} \pi^0\pi^0 & 31\% \\ \pi^+\pi^- & 69\% \end{cases} \qquad \tau = 9.0 \times 10^{-11}\ s$$
$$K_L \to \begin{cases} \pi^\pm e^\mp \nu_e(\bar{\nu}_e) & 39\% \\ \pi^\pm \mu^\mp \nu_\mu(\bar{\nu}_\mu) & 27\% \\ \pi^0\pi^0\pi^0 & 21\% \\ \pi^+\pi^-\pi^0 & 13\%. \end{cases} \qquad \tau = 5.2 \times 10^{-8}\ s$$

The lifetime of the K_S is $\tau_S = 9.0 \times 10^{-11}\ s$ while that of the K_L is $\tau_L = 5.2 \times 10^{-8}\ s$ and this is the reason for the notation K_S and K_L: K_S (K-short) has a much shorter life-time than K_L (K-long).

These decays are all weak decays, but they can be further classified by the decay products:

$$\text{Hadronic decays} \quad \begin{cases} K_S \to 2\pi \\ K_L \to 3\pi \end{cases} \qquad \text{Semi-leptonic decays} \quad \begin{cases} K_L \to \pi e \nu_e \\ K_L \to \pi \mu \nu_\mu \end{cases}$$

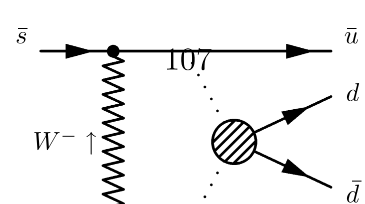
Now look at the **CP** content of some the final states in kaon decay.

i) $\pi^0\pi^0$, $\pi^+\pi^-$: kaons and pions have $J^P = 0^-$. If the initial and final states both have intrinsic spin zero then conservation of angular momentum dictates that the pions are produced with orbital angular momentum zero, $l = 0$ *i.e.* s-wave. Pions are bosons so the final state wave-function should be symmetric under interchange of the two pions — this is true both for $\pi^0\pi^0$, where the final state bosons are identical, and for $\pi^+\pi^-$, where the π^+ and the π^- are particle anti-particles. Now

$$\mathbf{C}\left(\overset{\pi^+}{\longleftarrow} \bullet \overset{\pi^-}{\longrightarrow}\right) = \left(\overset{\pi^-}{\longleftarrow} \bullet \overset{\pi^+}{\longrightarrow}\right) \qquad \Rightarrow$$

$$\mathbf{PC}\left(\overset{\pi^+}{\longleftarrow} \bullet \overset{\pi^-}{\longrightarrow}\right) = \mathbf{P}\left(\overset{\pi^-}{\longleftarrow} \bullet \overset{\pi^+}{\longrightarrow}\right) = \left(\overset{\pi^+}{\longleftarrow} \bullet \overset{\pi^-}{\longrightarrow}\right)\mathcal{P}_\pi^2 = \left(\overset{\pi^+}{\longleftarrow} \bullet \overset{\pi^-}{\longrightarrow}\right)$$

so $\mathbf{PC}|\pi^+\pi^- >= +|\pi^+\pi^- >$ and the two pion final state has $\mathbf{CP} = +1$, the same as K_S. In terms of the quark model the K^0 component of the $K_S \to \pi^+\pi^-$ decay looks like this, reading from left to right,



where the dots denote the hadronisation process.

ii) $\pi^0\pi^0\pi^0$ and $\pi^+\pi^-\pi^0$: the argument here requires an intermediate step. Denote the orbital angular momentum of two of the three pions (either $\pi^0\pi^0$ or $\pi^+\pi^-$) about their centre of mass by $L_{12} = l_{12}\hbar$. Let the angular momentum of the third pion (π^0) about the centre of mass of the first two be $L_3 = l_3\hbar$. Then, by the rules of quantum mechanical angular momentum addition, the total angular momentum must be given by one of $L/\hbar = l_{12} + l_3, l_{12} + l_3 - 1, l_{12} + l_3 - 2, \cdots, |l_{12} - l_3|$. Since the initial total angular momentum is $L = 0$ conservation of angular momentum requires that the final angular momentum must also be $L = 0$, and this is only possible if $l_{12} = l_3$. The parity of the final state is then

$$\mathcal{P} = (-1)^{l_{12}}(-1)^{l_3}\mathcal{P}_\pi^3 = -1,$$

since $l_{12} = l_3$ and the intrinsic parity of the pion is $\mathcal{P}_\pi = -1$.

The π^0 is even under charge conjugation $\mathcal{C}_{\pi^0} = +1$ so

$$\mathbf{C}|\pi^0\pi^0\pi^0 >= (+1)^3|\pi^0\pi^0\pi^0 >= |\pi^0\pi^0\pi^0 >,$$

hence

$$\mathbf{CP}|\pi^0\pi^0\pi^0 >= -|\pi^0\pi^0\pi^0 >$$

and the $\pi^0\pi^0\pi^0$ final state has $\mathbf{CP} = -1$.

As in *i)* above $\mathbf{CP}|\pi^+\pi^- >= +|\pi^+\pi^- >$. Also $\mathbf{CP}|\pi^0 >= -|\pi^0 > \pi^0$, since the π^0 has $\mathcal{C} = +1$ and $\mathcal{P} = -1$, also $\mathbf{P}$ gives a factor of $(-1)^{l_3}$, due to orbital angular momentum between the $\pi^+\pi^-$ pair and the π^0, so

$$\mathbf{CP}|\pi^+\pi^-\pi^0 >= -(-1)^{l_3}|\pi^+\pi^-\pi^0 > .$$

However the decay is predominantly to $l_3 = 0$, because higher angular momentum requires more energy and the mass difference between the kaon and three pions is only 83 MeV/c^2, in relative terms 83 $MeV/M_K c^2 = 0.17$, and there is very little energy available to excite any non-zero orbital angular momentum. $l_3 > 0$ is highly suppressed and $\mathcal{CP} = -1$ to a high level of accuracy.
In either case the three pion final state is $\mathcal{CP} = -1$.

Now K_L has $\mathcal{CP} = -1$, K_S has $\mathcal{CP} = +1$ and these particles decay via the weak interaction, which conserves $\mathbf{CP}$, so we expect that

$$K_S \to 2\pi \qquad \text{and} \qquad K_L \to 3\pi,$$

which is indeed the pattern in the decay modes quoted above. However experiments on very large numbers of K_L decays show that

$$\frac{\sigma(K_L \to 2\pi)}{\sigma(K_L \to \text{anything})} = (2.12 \pm 0.09) \times 10^{-3} > 0$$

which is small but non-zero. The fact that this is non-zero implies that **CP** is being violated in kaon decays. The **CPT** theorem would then force us to conclude that **T** is also violated — the fundamental laws of nature are not symmetric under time reversal! **CP** violation was first observed in kaon decay in 1964, but direct experimental evidence for **T** violation was not found until 1998, though the **CPT** theorem led physicists to believe, even before it was seen, that it had to be there. **CP**-violation has also been observed recently in B-decays, in 2001. and in D-meson decays in 2011.

CP violation, though a tiny effect, is crucial to our existence because without it there would be no matter in the Universe! Huge numbers of electrons and positrons (and quarks and anti-quarks) were produced in the Big Bang 13.7 billion years. As the Universe expanded and cooled most of these annihilated with each other to produce photons but there was a tiny excess of electrons and quarks over positrons and anti-quarks which was left over, and this tiny excess accounts for all the Hydrogen and other elements that we see in stars today. There must have been CP violating process in the early Universe for this excess to have been produced. It turns out that the CP violation in the Kaon system is not large enough to account for all the matter in the present day Universe, there must be some other source of CP-violation that is not yet understood!

Strangeness Oscillation and Kaon Regeneration

K_S and K_L have different lifetimes and decay modes, unlike K^0 and $\bar{K}^0$ they are not particle and anti-particles of one another and their masses do not have to be equal. Indeed a tiny mass difference can be detected via a subtle effect known as kaon regeneration. Because the lifetimes are different relativistic K_L's and K_S's will travel different distances, on average, before decaying,

$$
\begin{aligned}
\tau_S &= 9 \times 10^{-11}\ s \qquad \Rightarrow \qquad c\tau_S = 2.7 \times 10^{-2}\ m = 2.7\ cm \\
\tau_L &= 5 \times 10^{-8}\ s \qquad \Rightarrow \qquad c\tau_L = 15\ m.
\end{aligned}
$$

The actual distance travelled will be increased by the Lorentz γ-factor, due to time dilation, but this cancels in the ratio and we expect K_L to travel about 500 times further than K_S.

Assuming that the K_S and K_L wave-functions are energy eigenstates (which is reasonable because they have definite lifetimes) in the kaon rest-frame we can write

$$
\begin{aligned}
|K_S(t)> &= e^{-\left(\frac{\Gamma_S}{2}+im_S c^2\right)t/\hbar}|K_S(0)> \\
|K_L(t)> &= e^{-\left(\frac{\Gamma_L}{2}+im_L c^2\right)t/\hbar}|K_L(0)>,
\end{aligned}
$$

where the complex phases are due to the usual Schrödinger time evolution in quantum mechanics $\psi(t) = e^{-iHt/\hbar}\psi(0)$, with $H = mc^2$ the energy in the particle's rest frame, and the exponential decay represents the fact that the particles are unstable and decay to other things, so the amplitude of their wave-functions decreases exponentially with time.

Now suppose we produce a K^0 by a strong interaction process at time $t = 0$, [64]

$$\psi(0) = |K^0 >= \frac{1}{\sqrt{2}} \left(|K_S(0)> + |K_L(0)> \right)$$
$$\Rightarrow \psi(t) = \frac{1}{\sqrt{2}} \left(e^{-\left(\frac{\Gamma_S}{2} + i m_S c^2\right) t/\hbar} |K_S(0)> + e^{-\left(\frac{\Gamma_L}{2} + i m_L c^2\right) t/\hbar} |K_L(0)> \right).$$

Choosing $\{|K_S(0)>, |K_L(0)>\}$ to be an orthonormal basis, in particular $< K_S(0)|K_L(0)>= 0$, then after a time t the quantum mechanical amplitude for finding a K^0 in $\psi(t)$ is

$$\psi^*(0)\psi(t) = \frac{1}{2} \left(e^{-\left(\frac{\Gamma_S}{2} + i m_S c^2\right) t/\hbar} + e^{-\left(\frac{\Gamma_L}{2} + i m_L c^2\right) t/\hbar} \right)$$
$$= \frac{1}{2} e^{-i m_L c^2 t/\hbar} \left(e^{-\left(\frac{\Gamma_S}{2} + i \Delta m c^2\right) t/\hbar} + e^{-\frac{\Gamma_L}{2\hbar} t} \right)$$

where $\Delta m = m_S - m_L$. The probability is

$$|\psi^*(0)\psi(t)|^2 = \frac{1}{4} \left| e^{-\frac{\Gamma_S}{2\hbar} t} \cos(\Delta m c^2 t/\hbar) + e^{-\frac{\Gamma_L}{2\hbar} t} - i e^{-\frac{\Gamma_S}{2\hbar} t} \sin(\Delta m c^2 t/\hbar) \right|^2$$
$$= \frac{1}{4} \left\{ \left(e^{-\frac{\Gamma_S}{2\hbar} t} \cos(\Delta m c^2 t/\hbar) + e^{-\frac{\Gamma_L}{2\hbar} t} \right)^2 + e^{-\Gamma_S t/\hbar} \sin^2(\Delta m c^2 t/\hbar) \right\}$$
$$= \frac{1}{4} \left\{ e^{-\Gamma_S t/\hbar} + e^{-\Gamma_L t/\hbar} + 2 e^{-(\Gamma_S + \Gamma_L) t/2\hbar} \cos(\Delta m c^2 t/\hbar) \right\}.$$

Repeating the calculation for $\overline{K}^0$, the quantum mechanical amplitude for finding a $|\overline{K}^0 >= \frac{1}{\sqrt{2}}(|K_S(0)> - |K_L(0)>)$ in $\psi(t)$ after a time t is

$$\frac{1}{4} \left| e^{-\frac{\Gamma_S}{2\hbar} t} \cos(\Delta m c^2 t/\hbar) - e^{-\frac{\Gamma_L}{2\hbar} t} - i e^{-\frac{\Gamma_S}{2\hbar} t} \sin(\Delta m c^2 t/\hbar) \right|^2$$
$$= \frac{1}{4} \left\{ e^{-\Gamma_S t/\hbar} + e^{-\Gamma_L t/\hbar} - 2 e^{-(\Gamma_S + \Gamma_L) t/2\hbar} \cos(\Delta m c^2 t/\hbar) \right\}.$$

This calculation has the surprising implication that, if we create a beam of pure K^0's ($S = +1$) then after a time it becomes a mixture of K^0 and $\overline{K}^0$ with both $S = +1$ and $S = -1$, that is some of the K^0's turn into their anti-particles $\overline{K}^0$'s before they decay. An experimental graph of the number of K^0's (upper curve) and the number of $\overline{K}^0$'s (lower curve), in time units of τ_S, is shown on the next page.

What is measured here is the number of K^0's and $\overline{K}^0$'s. A beam which is initially pure K^0 eventually becomes pure K_L as all the K_S decay. If the beam then hits a target, such as a slab of material containing protons and neutrons, strong interactions will pick

[64] It is relatively easy to produce Kaons in strong reactions like $p + \pi^- \to K^0 + \Lambda$ or $n + \pi^0 \to K^0 + \Lambda$, remember the Λ is a baryon with $S = -1$. It is not so easy to produce $\overline{K^0}$ using protons or neutrons as it would have to be accompanied by a baryon with strangeness $S = +1$ and there is no feasible candidate.

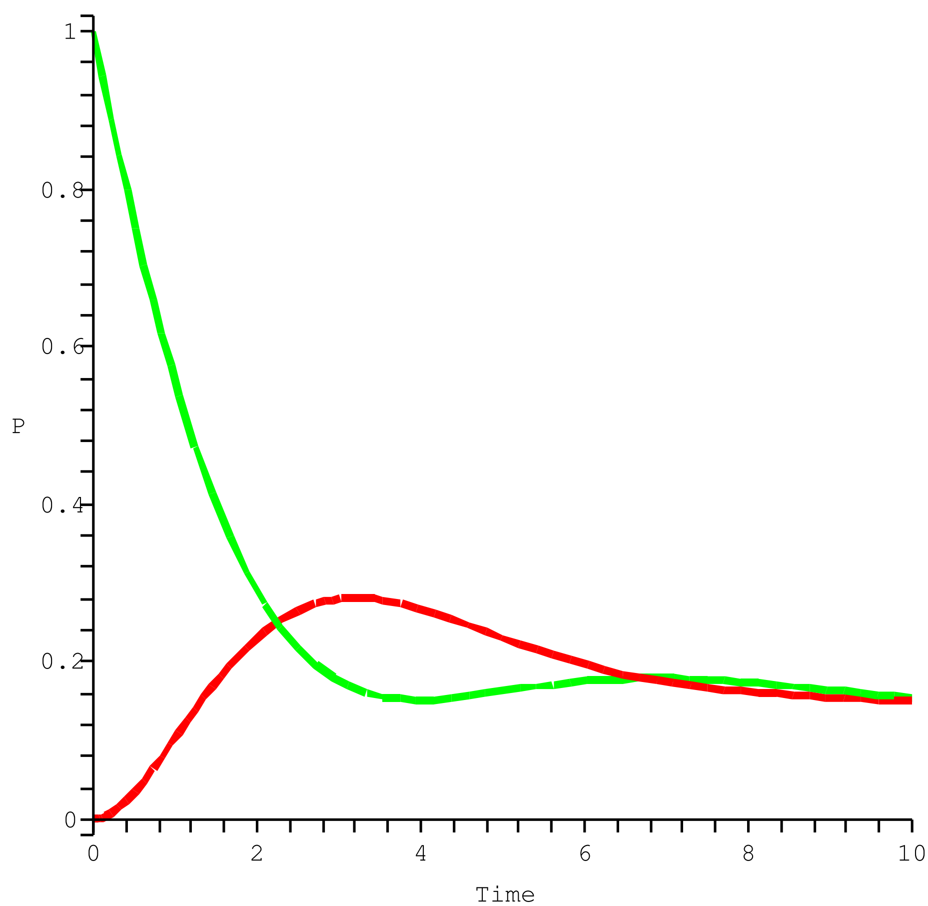
out the $S = +1$ (K^0) and $S = -1$ ($\overline{K}^0$) components of the beam. An initially pure K^0 beam can produce $S = -1$ particles in a target a few metres away from the source. $\overline{K}^0$ are detected via Λ-production in reactions like

$$\overline{K}^0 + p \to \Lambda + \pi^+.$$

From strangeness oscillation experiments like this we deduce the tiny K_L-K_S mass difference of

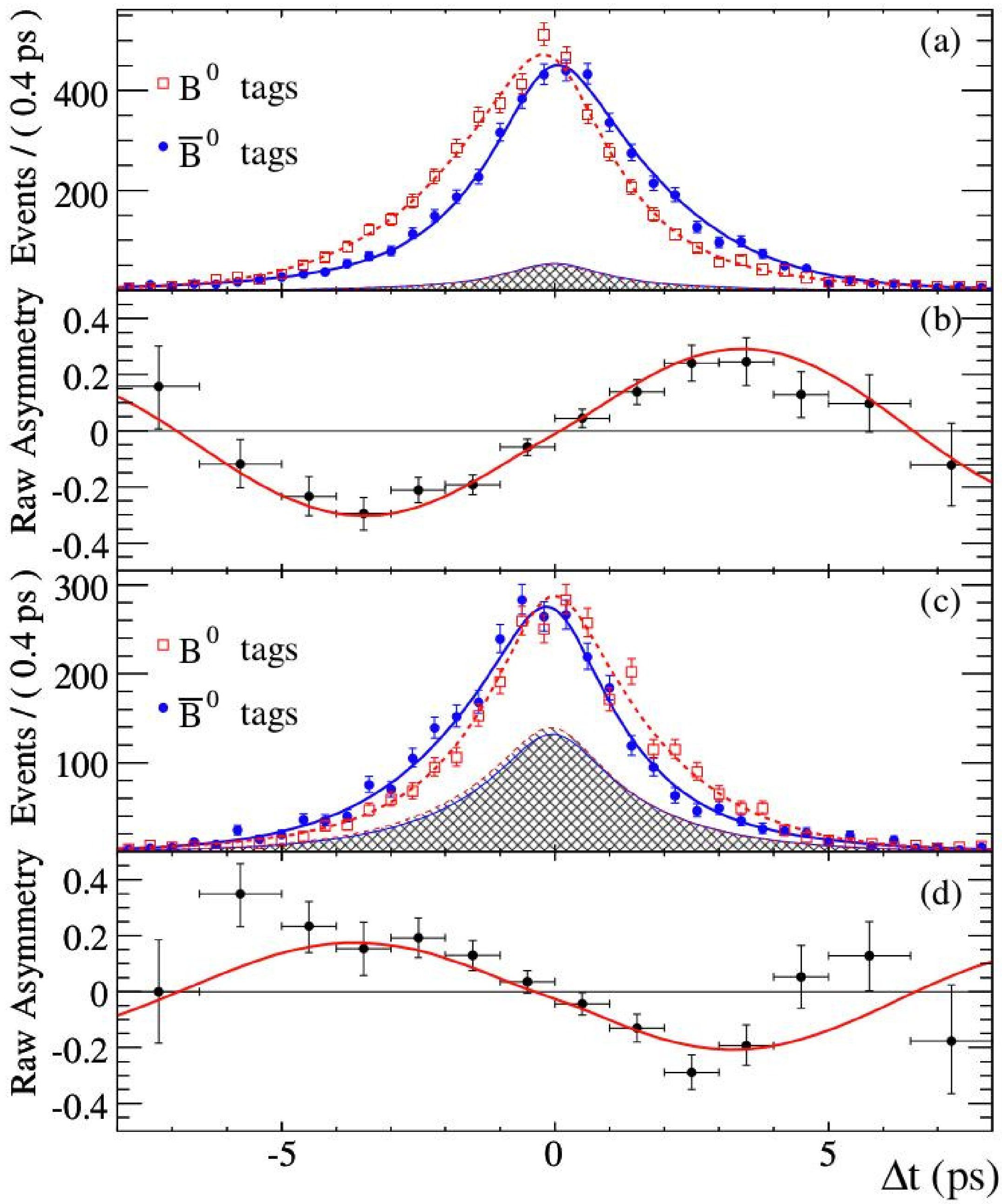
$$\Delta m = 3.5 \times 10^{-6} \ eV/c^2.$$

Oscillations have also recently (2006) been observed in the $B_s^0 - \overline{B}_s^0$ system. Remember the quark content of the B_s^0 meson is $s\bar{b}$, so $\overline{B}_s^0$ is $b\bar{s}$ and an oscillation changes both S and B quantum numbers (see http://www.fnal.gov/pub/presspass/press_releases/CDF_04-11-06.html). Charm oscillations in the $D^0 - \overline{D}^0$ system were first observed at CERN in November 2011 (http://cerncourier.com/cws/article/cern/48323).



CP-violation in neutral B-mesons

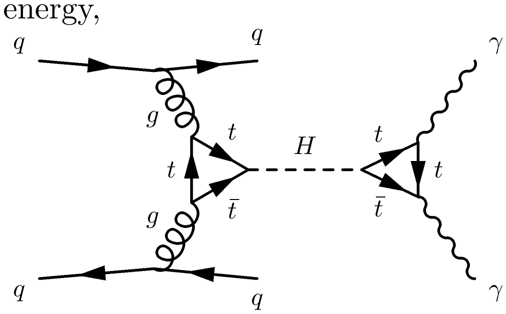
CP violation has also been seen in the B^0-$\overline{B}^0$ system, first observed in 2009. Although it is a much larger effect there than in kaons, of order 30% compared to 0.1%, it was not observed till more than 50 years after its discovery in the K^0-$\overline{K}^0$ system, because B-mesons are harder to produce and to work on. We shall not enter into the details but show here a graph of the time difference between B^0 and $\overline{B}^0$ decaying into CP-even and CP-odd states (note the timescale, 1 ps (picosecond) is 10^{-12} s, as compared to $\approx 10^{-8}$ s for kaons — this is one of the reasons that experiments on neutral B-mesons are harder than for kaons). The CP violation here is 30% (raw symmetry 0.3 in the figure).



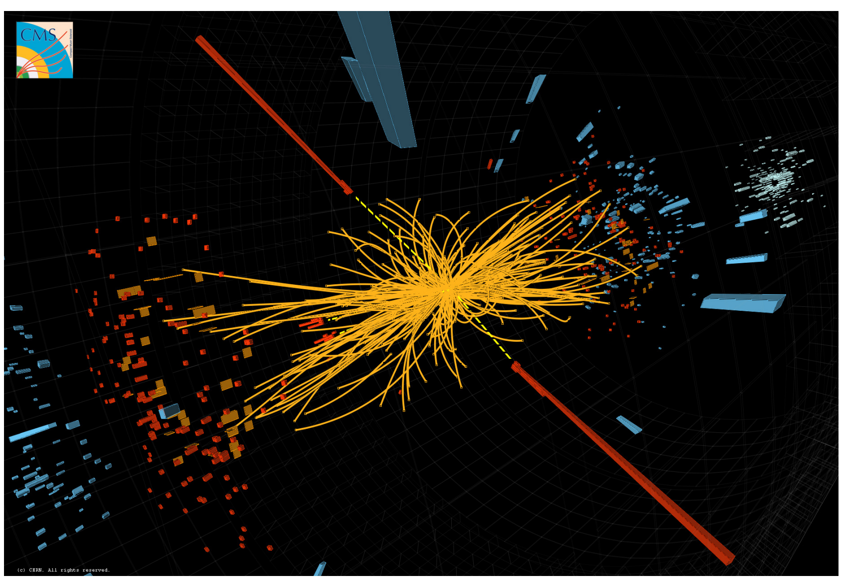
Higgs bosons

The standard model of particle physics consists of the fermions in (14) together with the force particles: the photon, the $W^{\pm}$ and Z^0 bosons and eight gluons, but there is one other piece that we have not yet described. There is one other particle which is needed to understand a fundamental difference between photons and gluons on the one hand and $W^{\pm}$ and Z^0 bosons in the other: namely photons and gluons are massless while $W^{\pm}$ and Z^0 bosons are massive. This particle goes by the name of the Higgs boson.[65] Its existence was predicted in 1964 and it was finally discovered 48 years later in 2012, in high energy proton-proton at the Large Hadron Collider at CERN.

It was first found by its decay into two photons, when a quark in each incoming proton emits a gluon and the two gluons form a t-$\bar{t}$ quark pair which immediately decay to form a Higgs boson. The Higgs does not live long enough to be detected directly, rather it subsequently decays into another t-$\bar{t}$ pair which then annihilate into two photons whose combined energy give the Higgs mass of 125 MeV/c^2 — the Higgs particle itself is not seen directly, it is the photons that are detected, but a resonance at 125 MeV/c^2 is a solid signal of a particle at that energy,



Below is an image of an actual Higgs decay, from the CMS detector at the LHC. The red tracks are a reconstruction of the photon trajectories, all the other stuff is a jungle of lower energy QCD bi-products.



[65] Named after Peter Higgs at the University of Edinburgh in Scotland who first proposed that such a particle might exist and described a mechanism through which it could make force particles massive.

6. Neutrino Masses and Oscillations

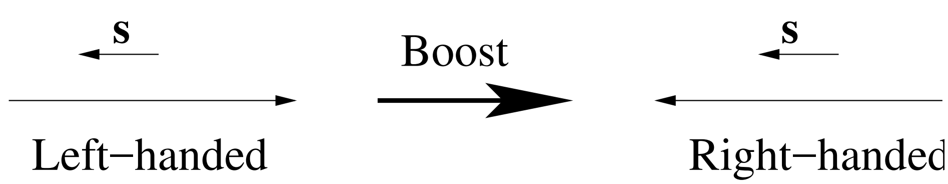
Direct measurements have never been able to detect a mass for neutrinos, to date they have only produced upper bounds on possible neutrino masses

$$m_{\nu_e} \lesssim 3\ eV/c^2 \approx 6 \times 10^{-6} m_e, \qquad m_{\nu_\mu} \lesssim 0.2\ MeV/c^2, \qquad m_{\nu_\tau} \lesssim 18\ MeV/c^2.$$

There is also an indirect constraint coming from cosmology — the Universe is bathed in a thermal background of photons (the microwave background) at a temperature of $2.7K$, corresponding to a few hundred photons per cubic centimetre. The Big Bang model predicts that there should be a similar background of neutrinos, but if neutrinos have mass this would contribute significantly to the energy density of the Universe and slow up the cosmological expansion, because of the extra gravitational attraction caused by neutrino mass. Current cosmological observations indicate that, if neutrinos do have a mass then the *total* mass of all three types must be less than $\sum_{i=1}^{3} m_{\nu_i} \lesssim 0.7\ eV/c^2$.

However, over the last 30 years, there has been mounting indirect evidence that neutrinos do have a non-zero, but very small, mass and in the last five years the evidence for this has become very strong. If neutrinos have a mass then a number of things that have been said previously must be qualified.

i) Positive helicity neutrinos have never been directly observed, only negative helicity neutrinos have been directly detected in weak interaction processes, but if neutrinos have a mass positive helicity neutrinos must exist. To see this suppose that a negative helicity neutrino is moving in the positive x-direction, with helicity $h = -1$ so the spin is in the negative x-direction. If the neutrino has a mass then it must necessarily be travelling at less than the speed of light so we can Lorentz boost to an inertial reference frame moving in the positive x-direction at a speed greater than that of the neutrino (but still less than the the speed of light relative to the first frame, of course). In this new frame the neutrino is now moving in the negative x-direction, but its spin has not changed so its helicity is now $h = +1$ *i.e.* we can turn a negative helicity massive particle into a positive helicity particle just by a Lorentz boost.



This cannot be done for truly massless particles because they must travel at the speed of light and we can never overtake such particles. So if neutrinos have a mass, right-handed neutrinos must exist even thought they have never been directly detected.

The fact that right-handed neutrinos have never been seen directly means that, if they exist, then they are not produced in weak interactions, which in turn means that they do not participate in the weak force, that is they carry no ‘weak’ charge. In this regard they are the same as other fermions — only left-handed fermions see the weak force, right handed fermions do not (see discussion on page 101).

ii) If neutrinos have mass it is possible for different generations to mix as they evolve in time giving rise to *neutrino oscillations*. This would mean that the individual lepton numbers

L_e, L_μ and L_τ are not absolutely conserved, though experimentally they do appear to be conserved by weak interactions. If individual lepton number conservation is violated, the possibility still remains that the total lepton number $L_T = L_e + L_\mu + L_\tau$ is conserved, even allowing for neutrino oscillations.

The mechanism for neutrino oscillations is similar in principle, but different in detail, to that of kaon regeneration in that particles produced by specific interactions, weak interactions in this case, are not necessarily mass, or equivalently energy, eigenstates and so can mix under time evolution. To illustrate the process of neutrino oscillations consider just two generations for simplicity, for example the electron and muon neutrinos. Write a neutrino quantum state as a linear combination of the two flavours of neutrino

$$|\nu>= \alpha|\nu_e> + \beta|\nu_\mu>$$

where α and β are complex numbers. $|\nu_e>$ and $|\nu_\mu>$ represent the electron and muon neutrinos that are produced in weak interactions, such as in the decay of the positive pion for example

$$\begin{array}{lll} \pi^+ & \rightarrow & \mu^+ + \nu_\mu \qquad\qquad , \\ & & \hookrightarrow e^+ + \nu_e + \bar{\nu}_\mu . \end{array}$$

or the neutrino produced in proton-proton collisions

$$p + p \quad \rightarrow \quad p + n + e^+ + \nu_e \tag{54}$$

(the latter is the main source of neutrinos coming from the Sun). A general neutrino state $|\nu>$ is a complex vector in a 2-dimensional Hilbert space and we can chose $|\nu_e>$ and $|\nu_\mu>$ as an orthonormal basis, $<\nu_e|\nu_\mu>= 0$. Since L_e and L_μ are conserved by weak interactions the electron and muon neutrinos cannot mix at the time of creation — it is the subsequent time evolution that allows them to mix. $|\nu_e>$ and $|\nu_\mu>$ are called *weak eigenstates*. The important point is that mass, being a physical observable, corresponds to an Hermitian operator on the Hilbert space but it does not have to be diagonal in the $(|\nu_e>, |\nu_\mu>)$ basis. Represent the mass by a 2×2 Hermitian matrix M and denote its (real) eigenvalues by m_1 and m_2 with corresponding orthogonal eigenvectors $|\nu_1>$ and $|\nu_2>$ with $<\nu_i|\nu_j>= \delta_{ij}$,

$$\begin{aligned} M|\nu_1> &= m_1|\nu_1> \\ M|\nu_2> &= m_2|\nu_2> . \end{aligned}$$

$|\nu_1>$ and $|\nu_2>$ are called *mass eigenstates*, but it is $|\nu_e>$ and $|\nu_\mu>$ that are produced in weak interactions.

Now the basis vectors $(|\nu_e>, |\nu_\mu>)$ can be written in terms of the alternative basis $(|\nu_1>, |\nu_2>)$

$$\begin{aligned} |\nu_e> &= \cos\theta|\nu_1> + \sin\theta|\nu_2> \\ |\nu_\mu> &= -\sin\theta|\nu_1> + \cos\theta|\nu_2> \end{aligned}$$

(we can always choose the complex phases of $|\nu_e>$, $|\nu_\mu>$, $|\nu_1>$ and $|\nu_2>$ so that the co-efficients on the right-hand side of this relation are real). In matrix notation this reads

$$\begin{pmatrix} |\nu_e> \\ |\nu_\mu> \end{pmatrix} = \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} \\ -\sin\theta_{12} & \cos\theta_{12} \end{pmatrix} \begin{pmatrix} |\nu_1> \\ |\nu_2> \end{pmatrix}.$$

Suppose weak eigenstates $|\nu_e(0)>$ and/or $|\nu_\mu(0)>$ with energy E are produced by some weak interaction process at time $t=0$. The quantum states satisfy a wave-like equation[66] and we take them to be plane waves of the form

$$\begin{aligned} |\nu_1(t)> &= e^{i(\mathbf{P}_1.\mathbf{x}-Et)/\hbar}|\nu_1(0)> \\ |\nu_2(t)> &= e^{i(\mathbf{P}_2.\mathbf{x}-Et)/\hbar}|\nu_2(0)>, \end{aligned}$$

where $\mathbf{P}_i$ are and relativistic 3-momentum respectively of $|\nu_i>$, with $i=1,2$. It is the mass eigenstates, $|\nu_1>$ and $|\nu_2>$, that are eigenstates of energy and momentum, not $|\nu_e>$ or $|\nu_\mu>$.

Suppose and electron neutrino, ν_e, is produced at $t=0$, for example in an electron-proton collision $e^- + p = \nu_e + n$. Then the wave-function is initially

$$\psi(0) = |\nu_e> = \cos\theta_{12}\,|\nu_1(0)> + \sin\theta_{12}\,|\nu_2(0)>$$

and evolves to

$$\begin{aligned} \psi(t) &= \cos\theta_{12}|\nu_1(t)> + \sin\theta_{12}|\nu_2(t)> \\ &= \cos\theta_{12}\,e^{i(\mathbf{P}_1.\mathbf{x}-Et)/\hbar}|\nu_1(0)> + \sin\theta_{12}\,e^{i(\mathbf{P}_2.\mathbf{x}-Et)/\hbar}|\nu_2(0)> \end{aligned}$$

at time t.

We can always assume that the motion is in the z-direction, $\mathbf{P}_i.\mathbf{x} = P_i z$. The neutrinos, being almost massless, will be highly relativistic and move almost at the speed of light, so they will travel a distance $z=L$ in a time $t=L/c$, and hence

$$\mathbf{P}_i.\mathbf{x} - Et = \left(P_i - \frac{E}{c}\right)L.$$

Of course P_i can be expressed in terms of the energy,

$$\mathbf{P}_i.\mathbf{P}_i = P_i^2 = \frac{E^2}{c^2} - m_i^2c^2 = \frac{1}{c^2}\left(E^2 - m_i^2c^4\right) \qquad \Rightarrow \qquad P_i = \frac{1}{c}\sqrt{E^2 - m_i^2c^4}.$$

So, for $m_i << \frac{E}{c^2}$,

$$P_i = \frac{E}{c}\left(1 - \frac{m_i^2c^4}{E^2}\right)^{\frac{1}{2}} \approx \frac{E}{c}\left(1 - \frac{m_i^2c^4}{2E^2}\right),$$

[66] The relevant equation, for spin-$\frac{1}{2}$ fermions, is called the Dirac equation — it is a relativistic version of the Schrödinger equation.

and

$$\left(P_i - \frac{E}{c}\right) L \approx -\left(\frac{m_i^2 c^3}{2E}\right) L.$$

Thus we can write

$$\psi(t) = \cos\theta_{12}\, e^{-i\phi_1} |\nu_1(0) > + e^{-i\phi_2} |\nu_2(0) >,$$

where $\phi_i = \frac{m_i^2 c^3 L}{2E\hbar}$.

We can now calculate the quantum mechanical amplitude for finding a muon neutrino, $\nu_\mu = -\sin\theta_{12}\,\nu_1 + \cos\theta_{12}\,\nu_2$, at time t in an initially pure beam of electron neutrinos,

$$\begin{aligned} M_{\nu_\mu \nu_e} &=< -\sin\theta_{12}\,\nu_1(0) + \cos\theta_{12}\,\nu_2(0) | \cos\theta_{12}\, e^{-i\phi_1} \nu_1(0) + \sin\theta_{12}\, e^{-i\phi_2} \nu_2(0) > \\ &= -\sin\theta_{12}\cos\theta_{12}\, e^{-i\phi_1} + \cos\theta_{12}\sin\theta_{12}\, e^{-i\phi_2} = -\sin\theta_{12}\cos\theta_{12}\left(e^{-i\phi_1} + e^{-i\phi_2}\right). \end{aligned}$$

From which follows the probability

$$\begin{aligned} P_{\nu_e \to \nu_\mu} &= \sin^2\theta_{12}\cos^2\theta_{12} \left| -e^{-i\phi_1} + e^{-i\phi_2} \right|^2 = \sin^2\theta_{12}\cos^2\theta_{12}\left(1 + 1 - e^{i(\phi_1 - \phi_2)} - e^{-i(\phi_1 - \phi_2)}\right) \\ &= 2\sin^2\theta_{12}\cos^2\theta_{12}\left(1 - \cos(\phi_1 - \phi_2)\right) = \sin^2(2\theta_{12})\sin^2\left(\frac{\phi_1 - \phi_2}{2}\right). \end{aligned}$$

Now

$$\phi_1 - \phi_2 = \frac{c^3 L}{2\hbar}\left(\frac{m_1^2}{E} - \frac{m_2^2}{E}\right) \approx \frac{c^3 L}{2\hbar}\frac{\left(m_1^2 - m_2^2\right)}{E} = \frac{c^3 L}{2\hbar}\frac{\left(\Delta m^2\right)}{E}, \tag{55}$$

where $\Delta m^2 = m_1^2 - m_2^2$.

Suppose therefore that we initially have a beam of purely electron neutrinos with energy E, produced for example by nuclear reactions in the core of the Sun, then after travelling a distance L some of the electron neutrinos will have turned into muon neutrinos with a probability

$$P_{\nu_e \to \nu_\mu} = \sin^2(2\theta_{12})\sin^2\left(\frac{\Delta m^2 c^3 L}{4E\hbar}\right).$$

Similarly the probability of an electron neutrino being found a distance L in an initially pure beam of electron neutrinos is

$$P_{\nu_e \to \nu_e} = 1 - \sin^2(2\theta_{12})\sin^2\left(\frac{\Delta m^2 c^3 L}{4E\hbar}\right),$$

since

$$P_{\nu_e \to \nu_\mu} + P_{\nu_e \to \nu_e} = 1,$$

unlike kaon oscillations the neutrinos are not actually decaying, they are just oscillating, and there is no leakage of probability into creation of other particles.

For neutrinos of a given energy the *oscillation length*, L_{osc}, is defined as the distance over which $P(\nu_\mu \to \nu_e; L)$ achieves its first maximum, *i.e* at

$$L_{osc} = \frac{2\pi E\hbar}{\Delta m^2 c^3}.$$

Such oscillations are observed in attempts to detect electron neutrinos produced by nuclear reactions in the Sun. An astrophysical analysis of reactions like (54) in the Sun leads on to expect a flux of about $10^{15}\ m^2 s^{-1}$ electron neutrinos at the Earth's surface. The neutrinos are produced in the core of the Sun with energy E and then travel a distance of $L = 1.5 \times 10^8\ m$ to Earth, in about eight and a half minutes. The number of neutrinos detected is about half of that expected and this can be interpreted as being due to electron neutrinos oscillating to muon neutrinos with

$$P_{\nu_e \to \nu_\mu} = \frac{1}{2}\sin^2(2\theta_{12})\left\{1 - \cos\left(\frac{\Delta m^2 c^3 L}{2E\hbar}\right)\right\}.$$

Because the Earth-Sun distance is so large the cosine is rapidly oscillating as a function of L and it averages to zero with small variations in L, such as the Earth's rotation ($\Delta L = 12,000\ km$ in 12 hours) and the eccentricity of its orbit ($\Delta L = 8 \times 10^6\ km$ in 6 months), so

$$P_{\nu_e \to \nu_\mu} = \frac{1}{2}\sin^2(2\theta_{12})$$

and the observation that $P_{\nu_e \to \nu_\mu} \approx 1/2$ implies that $\theta_{12} \approx \pi/4$, which is a large mixing angle. A detailed analysis of the experiments reveals that

$$\Delta m^2 = 7.5 \pm 0.2 \times 10^{-5}\ eV^2/c^4$$

and

$$\theta_{12} = 32.4^\circ \pm 0.8^\circ \quad \Rightarrow \quad \sin^2(2\theta_{12}) \approx 0.82. \tag{56}$$

Another place where neutrinos appear to oscillate is in measurements of fluxes coming from cosmic rays. Cosmic rays are highly energetic particles produced in astrophysical sources, often from outside the solar system. When an energetic charged particle, such as a proton for example, slams into the Earth's atmosphere pions are produced. There is a continuous flux of pions raining down on the Earth's surface from cosmic rays (indeed the pion was first detected in these cosmic rays). On their journey from the upper atmosphere to the Earth's surface some of these pions decay to muons (or anti-muons), producing a muon neutrino, and the muon subsequently decays to an electron (or a positron), producing an electron neutrino in processes like this

$$\begin{array}{lll} \pi^+ & \to & \mu^+ + \nu_\mu \\ & & \hookrightarrow e^+ + \nu_e + \bar{\nu}_\mu \end{array}$$

for example. If both muon neutrinos and anti-neutrinos are detected at the Earth's surface, without distinguishing which is which, one expects to find twice as many muon neutrinos as electron neutrinos because each decaying pion produces two muon neutrinos for every electron neutrino. Experiments indicate that the total number of muon neutrinos and electron neutrinos arriving at the Earth's surface is the same. This can be interpreted as being due to muon neutrinos oscillating and the experiments indicate that they are not turning into electron neutrinos but into τ-neutrinos.

These muon-tau neutrino oscillations involve a different mixing angle θ_{23} and a different mass difference Δm^2 than the electron-muon neutrino oscillations observed from the Sun. For atmospheric neutrino oscillations the experimental parameters are

$$\Delta m^2 = 2.45 \pm 0.07 \times 10^{-3}\ eV^2/c^4$$

and

$$\theta_{23} = 40.4^\circ \pm 1.3^\circ \quad \Rightarrow \quad \sin^2(2\theta_{23}) \approx 0.97. \tag{57}$$

A full analysis of neutrino oscillations must include three angles for rotations, R, between three different types of neutrinos,

$$\begin{pmatrix} |\nu_e > \\ |\nu_\mu > \\ |\nu_\tau > \end{pmatrix} = R \begin{pmatrix} |\nu_1 > \\ |\nu_2 > \\ |\nu_3 > \end{pmatrix},$$

where $|\nu_e >$, $|\nu_\mu >$ and $|\nu_\tau >$ are produced in weak interactions and $|\nu_1 >$, $|\nu_2 >$ and $|\nu_3 >$ are mass eigenstates. The angle measured in Solar neutrino experiments (56) is usually denoted θ_{12}, as it rotates between $|\nu_1 >$ and $|\nu_2 >$, and the one measured in atmospheric neutrino oscillations (57) is denoted θ_{23}. The third angle, θ_{13} was first measured in 2011, and was found to be smaller than the other two at $\theta_{13} = 8.7^\circ \pm 0.45^\circ$ giving mixing at the level of 10%, $\sin^2 2\theta_{13} \approx 0.09$. Thus the current experimental status on neutrino mixing angles is

$$\theta_{12} = 32.4^\circ \pm 0.8^\circ, \qquad \theta_{23} = 40.4^\circ \pm 1.3^\circ \qquad \theta_{13} = 8.7^\circ \pm 0.45^\circ.$$