Lorentz transformations

In relativity time and 3-dimensional space are in a sense combined into 4-dimensional space-time. Ordinary 3-dimensional vectors become vectors in a 4-dimensional "space", with time as the extra dimension. Physical quantities are described relative to an inertial reference frame with a specified inertial co-ordinate system related to a time axis and three spatial axes. All 4-vectors can then be decomposed in this co-ordinate system into a time-like component (the component along the chosen time-axis) and a relativistic 3-vector (consisting of the 3 spatial components in the chosen frame). For example time, t, becomes a co-ordinate in space-time, along with the usual three Cartesian co-ordinates, x, y and z, and all 4 are combined into a space-time position vector $\underset{\sim}{X} = (ct, x, y, z)$. The components of a 4-vector are often written using indices, $a, b, c, \ldots = 0, 1, 2, 3$. For example $\underset{\sim}{X}$ has components X^a with the super-script ^{0} representing the time-like component, thus the components of the 4-vector $\underset{\sim}{z}$ are $X^0 = ct$, $X^1 = x$, $X^2 = y$ and $X^3 = z$. For a massive particle moving with speed v the energy E and the non-relativistic 3-momentum $\mathbf{p}$ are combined into the 4-momentum, $\underset{\sim}{P} = (\frac{E}{c}, \mathbf{P})$ where $cP^0 = E = \gamma(v)mc^2$ is the relativistic energy and $\mathbf{P} = \gamma(v)m\mathbf{v} = \gamma(v)\mathbf{p}$ is the relativistic 3-momentum.

Transformations between inertial co-ordinate systems are implemented by rotations and/or Lorentz transformations — the latter corresponding to switching between inertial reference frames that are moving with constant velocity relative to one another. For example boosting[1] from an inertial reference frame with 4-dimensional Cartesian co-ordinates X^a to another inertial reference frame, moving with constant speed v in the $x = X^1$ direction relative to the first, we are free to choose Cartesian co-ordinates $X^{a'}$ in the latter frame such that $y' = y$ and $z' = z$. The Lorentz transformation between (ct', x') and (ct, x) in this case is

$$t' = \gamma(V)\left(t - \frac{V}{c^2}x\right)$$
$$x' = \gamma(V)\,(x - Vt)$$

where $\gamma(V) = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}$ is the Lorentz γ-factor. A more succinct way of writing this is to use a matrix notation

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \gamma(V) \begin{pmatrix} 1 & -\frac{V}{c} \\ -\frac{V}{c} & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}. \qquad (1)$$

A neater formula is obtained in terms of the *rapidity*, $\alpha(V)$, defined by

$$\frac{V}{c} = \tanh\alpha = \frac{e^\alpha - e^{-\alpha}}{e^\alpha + e^{-\alpha}},$$

with $-\infty < \alpha < \infty$ for $-1 < \frac{V}{c} < 1$. Then $\gamma(V) = \cosh\alpha = \frac{1}{2}\left(e^\alpha + e^{-\alpha}\right)$ and $\gamma(V)\frac{V}{c} = \sinh\alpha = \frac{1}{2}\left(e^\alpha - e^{-\alpha}\right)$ so, using the X^a and $X^{a'}$ notation

$$\begin{pmatrix} X^{0'} \\ X^{1'} \end{pmatrix} = \begin{pmatrix} \cosh\alpha & -\sinh\alpha \\ -\sinh\alpha & \cosh\alpha \end{pmatrix} \begin{pmatrix} X^0 \\ X^1 \end{pmatrix} \qquad (2).$$

[1] An unfortunate name, as no actual acceleration is involved. Neither of the two reference frames is accelerating, they are moving with constant velocity relative to one another

We can choose $\alpha < 0$, and hence $V < 0$, to describe boosting to a reference frame moving in the negative x-direction relative to the original frame.

Note the similarity, and the obvious differences, between equation (2) for a *boost* in the x-direction and the formula describing the change in co-ordinates brought about by a *rotation* about the x-axis through an angle ϕ, which only affects the y and z co-ordinates leaving t and z unchanged,

$$\begin{pmatrix} y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}.$$

Just as for electron spin we can simplify the description of rotations by using a (necessarily complex)[2] variable $y_\pm := y \pm iz$ with $y_- = \overline{y}_+$, so that, under a rotation,

$$y_\pm \longrightarrow e^{\pm i\phi} y_\pm.$$

We can also simplify the description of Lorentz transformations by defining $X_\pm = X^1 \mp X^0$ so that, under a boost,

$$X_\pm \longrightarrow e^{\pm\alpha} X_\pm.$$

The same transformations apply to the components of any 4-vector. For the 4-momentum $\underset{\sim}{P}$, for example, a boost in the z-direction gives

$$\begin{pmatrix} P^{0'} \\ P^{1'} \\ P^{2'} \\ P^{3'} \end{pmatrix} = \begin{pmatrix} \cosh\alpha & -\sinh\alpha & 0 & 0 \\ -\sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P^0 \\ P^1 \\ P^2 \\ P^3 \end{pmatrix} = \begin{pmatrix} \gamma(V) & -\frac{V}{c}\gamma(V) & 0 & 0 \\ -\frac{V}{c}\gamma(V) & \gamma(V) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P^0 \\ P^1 \\ P^2 \\ P^3 \end{pmatrix}$$

A particle at rest in the X^a frame has rest energy $cP^0 = mc^2$, while $P^1 = P^2 = P^3 = 0$. In the boosted reference frame $cP^{0'} = \gamma(V)mc^2$ is the relativistic energy for a particle with rest mass m moving with speed V, while $P^{1'} = -\gamma(V)mV$ is the only non-zero component of the relativistic 3-momentum in the moving frame.[3]

A very important aspect of Lorentz transformations is that, just like rotations, they leave the length of a vector invariant, provided the "length" of a 4-vector is properly defined. In relativity, just like in ordinary 3-dimensional Euclidean space, the length of a vector is related to the dot product, with the twist that the dot product of two 4-vectors, or of a 4-vector with itself, involves a minus sign for the time-like component, for example

$$\underset{\sim}{X}.\underset{\sim}{X} = -(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2$$

or, in terms of $X^\pm$,

$$\underset{\sim}{X}.\underset{\sim}{X} = X^+X^- + (X^1)^2 + (X^2)^2 + (X^3)^2$$

[2] This is nothing more than a notational convenience however, the vector $\underset{\sim}{X}$ is intrinsically real, unlike quantum mechanics vector spaces which are intrinsically complex.

[3] The minus sign is due to the convention that $X^{a'}$ represent a frame moving in the positive x-direction relative to X^a. So, if the particle is a rest in X^a, it is moving in the negative X'-direction in $X^{a'}$.

is Lorentz invariant. So a photon, leaving the origin at $t = 0$ and moving in the direction of the x-axis for example, has $ct = x$, $y = z = 0$ and

$$\underset{\sim}{X}.\underset{\sim}{X} = -c^2t^2 + x^2 = 0.$$

An electron at rest at the origin, $x = y = z = 0$, must move through time, $X^0 = ct$, so

$$\underset{\sim}{X}.\underset{\sim}{X} = -c^2t^2 < 0$$

and the length squared of its 4-dimensional position vector is negative, $\underset{\sim}{X}$ is a time-like vector.

Similarly for the 4-momentum

$$\underset{\sim}{P}.\underset{\sim}{P} = -(P^0)^2 + (P^1)^2 + (P^2)^2 + (P^3)^2 = -m^2c^2$$

has the same value in all inertial reference frames. You can check for yourself that we get the same answer regardless of whether we use the components of $\underset{\sim}{P}$ in the un-primed frame, P^a, or in the primed frame $P^{a'}$. In any frame

$$\underset{\sim}{P}.\underset{\sim}{P} = -\frac{E^2}{c^2} + \mathbf{P}.\mathbf{P} = -m^2c^2,$$

where $\mathbf{P}.\mathbf{P} = (P^1)^2 + (P^2)^2 + (P^3)^2 = P^2$ is just the ordinary dot product in 3-dimensions. Hence we get the general formula

$$E^2 = m^2c^4 + c^2P^2 \tag{3}$$

relating energy to relativistic 3-momentum for a massive particle moving with speed v.

The energy is a minimum in the *rest frame* of the particle, *i.e.* a reference frame in which it is not moving and hence has zero 3-momentum, so

$$E = mc^2.$$