

MP466: Particle Physics

Brian Dolan

Fourier Transforms

You are familiar with Fourier series for a function on an interval $-T < t < T$.

$$f(t) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi t}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{T}\right).$$

where the constants a_n and b_n are determined using orthogonality of the trigonometric functions for positive integers n and n' ,

$$\begin{aligned} \int_{-\pi}^{\pi} \cos(n\theta) \cos(n'\theta) d\theta &= \pi \delta_{nn'} & \int_{-\pi}^{\pi} \sin(n\theta) \sin(n'\theta) d\theta &= \pi \delta_{nn'} \\ \int_{-\pi}^{\pi} \sin(n\theta) \cos(n'\theta) d\theta &= 0, \end{aligned}$$

to be

$$a_n = \frac{1}{T} \int_{-T}^T f(t) \cos\left(\frac{n\pi t}{T}\right) dt, \quad b_n = \frac{1}{T} \int_{-T}^T f(t) \sin\left(\frac{n\pi t}{T}\right) dt$$

for positive n , while

$$a_0 = \frac{1}{2T} \int_{-T}^T f(t) dt.$$

In the Fourier series $\frac{n\pi}{T}$ is a frequency,

$$\omega_n = \frac{n\pi}{T},$$

and for large T the ω_n are close together for successive n , approaching a continuous variable ω as $T \rightarrow \infty$. Define

$$d\omega = \frac{\pi}{T}$$

and the sums go over to Riemann integrals as $T \rightarrow \infty$

$$f(t) = \int_0^{\infty} \left(a(\omega) \cos(\omega t) + b(\omega) \sin(\omega t) \right) d\omega$$

where

$$\begin{aligned} a(\omega_n) &= \frac{T}{\pi} a_n = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos(\omega_n t) dt \\ b(\omega_n) &= \frac{T}{\pi} b_n = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin(\omega_n t) dt. \end{aligned}$$

It is conventional to define the *cosine transform* $\tilde{f}_c(\omega)$ and the *sine transform* $\tilde{f}_s(\omega)$ of $f(t)$ as

$$\begin{aligned}\tilde{f}_c(\omega) &:= \int_0^\infty f(t) \cos(\omega t) dt \\ \tilde{f}_s(\omega) &:= \int_0^\infty f(t) \sin(\omega t) dt.\end{aligned}$$

These integrals certainly exist if $\int_0^\infty |f(t)| dt$ exists and is finite. They are examples of a class of functions called *integral transforms* where, given a function $f(t)$, we construct a new function $\tilde{f}(\omega)$ as an integral

$$\tilde{f}(\omega) := \int_0^\infty f(t) K(\omega, t) dt$$

where $K(\omega, t)$ is called the *kernel* of the integral transform.

If $f(-t) = f(t)$ is an even function then $\tilde{f}_c(\omega) = \frac{\pi}{2} a(\omega)$ and if $f(-t) = -f(t)$ is an odd function then $\tilde{f}_s(\omega) = \frac{\pi}{2} b(\omega)$.

Another type of integral transform that is very useful in physics when periodic phenomena are under consideration is the *Fourier transform*,

$$\tilde{f}(\omega) = \int_{-\infty}^\infty f(t) e^{i\omega t} dt. \quad (1)$$

Specifying $\tilde{f}(\omega)$ is completely equivalent to specifying $f(t)$ because

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^\infty \tilde{f}(\omega) e^{-i\omega t} d\omega, \quad (2)$$

as we shall now show.

Using the definition of $\tilde{f}(\omega)$ in the right hand side of (2) gives

$$\begin{aligned}\frac{1}{2\pi} \int_{-\infty}^\infty \tilde{f}(\omega) e^{-i\omega t} d\omega &= \frac{1}{2\pi} \int_{-\infty}^\infty \left(\int_{-\infty}^\infty f(t') e^{i\omega t'} dt' \right) e^{-i\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty \left(\int_{-\infty}^\infty e^{i\omega(t'-t)} d\omega \right) f(t') dt'.\end{aligned} \quad (3)$$

Now

$$\delta(t - t') = \frac{1}{2\pi} \int_{-\infty}^\infty e^{i\omega(t'-t)} d\omega$$

is a representation of the Dirac δ -function (a heuristic proof is sketched below), so

$$\frac{1}{2\pi} \int_{-\infty}^\infty \tilde{f}(\omega) e^{-i\omega t} d\omega = \int_{-\infty}^\infty \delta(t - t') f(t') dt' = f(t).$$

Hence

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t} dt \quad \Leftrightarrow \quad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega)e^{-i\omega t} d\omega.$$

To see that $\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(t'-t)} d\omega$ can be interpreted as an integral of the Dirac δ -function first note that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(t'-t)} d\omega = 0$$

if $t \neq t'$, because both the real and imaginary parts are just trigonometric functions which oscillate with ω and

$$\int_{-\pi}^{\pi} \cos(\omega(t-t')) d\omega = \int_{-\pi}^{\pi} \sin(\omega(t-t')) d\omega = 0,$$

so the integral over all ω is zero too. Next when $t = t'$, $e^{i\omega(t'-t)} = 1$, so

$$\frac{1}{2\pi} \int_{-\Omega}^{\Omega} e^{i\omega(t'-t)} d\omega = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} d\omega = \frac{\Omega}{\pi} \xrightarrow{\Omega \rightarrow \infty} \infty,$$

so $\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(t'-t)} d\omega$ is zero if $t \neq t'$ and diverges if $t = t'$: these are properties of the Dirac δ -function. We can check the normalisation by setting $t' = 0$ and making sure that

$$\int_{-T}^T \delta(t) dt = 1$$

for any $T > 0$. Evaluating

$$\begin{aligned} \int_{-T}^T \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} d\omega \right) dt &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-T}^T e^{-i\omega t} dt \right) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{i}{\omega} e^{-i\omega t} \right]_{-T}^T d\omega = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(\omega T)}{\omega} d\omega \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin u}{u} du = \frac{2}{\pi} \int_0^{\infty} \frac{\sin u}{u} du \\ &= 1, \end{aligned}$$

where $u = \omega T$ and we have used the definite integral

$$\int_0^{\infty} \frac{\sin u}{u} du = \frac{\pi}{2}.$$

While this is far from a rigorous proof that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} d\omega = \delta(t)$$

it is certainly consistent with this interpretation.

An important Fourier transform that is used in the lecture is the 3-dimensional transform of the Yukawa potential

$$f(\mathbf{r}) = \frac{e^{-\kappa r}}{r}.$$

We are Fourier transforming in three space co-ordinates x_1, x_2, x_3 rather than time, so we shall use wave-vectors k_1, k_2, k_3 rather than angular frequency ω . The transform that we want to calculate is

$$\tilde{f}(\mathbf{k}) = \int e^{i(k_1 x_1 + k_2 x_2 + k_3 x_3)} f(\mathbf{r}) d^3 x = \int e^{i\mathbf{k} \cdot \mathbf{r}} f(\mathbf{r}) d^3 x,$$

where the integral is over all space. It is simplest to use polar co-ordinates

$$\begin{aligned} \tilde{f}(\mathbf{k}) &= \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{e^{i\mathbf{k} \cdot \mathbf{r}} e^{-\kappa r}}{r} r^2 dr \sin \theta d\theta d\phi = 2\pi \int_0^\infty \int_0^\pi \frac{e^{ikr \cos \theta} e^{-\kappa r}}{r} r^2 dr \sin \theta d\theta \\ &= 2\pi \int_0^\infty \frac{e^{-\kappa r}}{r} \left(\int_0^\pi e^{ikr \cos \theta} \sin \theta d\theta \right) r^2 dr = -2\pi \int_0^\infty \frac{e^{-\kappa r}}{r} \left[\frac{e^{ikr \cos \theta}}{ikr} \right]_0^\pi r^2 dr \\ &= 2\pi \int_0^\infty \frac{e^{-\kappa r}}{r} \left(\frac{e^{ikr} - e^{-ikr}}{ikr} \right) r^2 dr = \frac{2\pi}{ik} \int_0^\infty (e^{-\kappa r + ikr} - e^{-\kappa r - ikr}) dr \\ &= \frac{2\pi}{ik} \left(\left[\frac{e^{-\kappa r + ikr}}{-\kappa + ik} \right]_0^\infty - \left[\frac{e^{-\kappa r - ikr}}{-\kappa - ik} \right]_0^\infty \right) = \frac{2\pi}{ik} \left(-\frac{1}{-\kappa + ik} + \frac{1}{-\kappa - ik} \right) \\ &= \frac{2\pi}{ik} \left\{ \frac{-(-\kappa - ik) + (-\kappa + ik)}{\kappa^2 + k^2} \right\} = \frac{4\pi}{\kappa^2 + k^2}. \end{aligned}$$

So we have the following important Fourier transform pairs

$$f(\mathbf{r}) = \frac{e^{-\kappa r}}{r}, \quad \tilde{f}(\mathbf{k}) = \frac{4\pi}{\kappa^2 + k^2}.$$

In the limit $\kappa \rightarrow 0$ we get the Fourier transform of the Coulomb potential,

$$f(\mathbf{r}) = \frac{1}{r}, \quad \tilde{f}(\mathbf{k}) = \frac{4\pi}{k^2}.$$