MP465: Mathematical Physics

Electrodynamics Problem Sheet

1) Starting from the Lorentz force law determine the units, in the MKSA system, of the electric and magnetic fields and also of the electric permittivity of the vacuum, ϵ_0, and the magnetic permeability of the vacuum, μ_0. Show that $\epsilon_0\mu_0$ has the dimensions of $(time)^2/(length)^2$. Show that $\epsilon_0\mathbf{E}.\mathbf{E}$ has the dimensions of energy density and that $\epsilon_0(\mathbf{E}\times\mathbf{B})$ has dimensions of momentum density.

2) Show that the electric field at $\mathbf{r}$ generated by a point charge Q at $\mathbf{r}'$,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{r}-\mathbf{r}}{|\mathbf{r}-\mathbf{r}|^3},$$

satisfy the vector equations

$$\nabla.\mathbf{E}(\mathbf{r}) = \frac{1}{\epsilon_0}\delta(\mathbf{r}-\mathbf{r}'), \qquad \nabla\times\mathbf{E}(\mathbf{r}) = 0,$$

for $\mathbf{r}\neq\mathbf{r}'$.

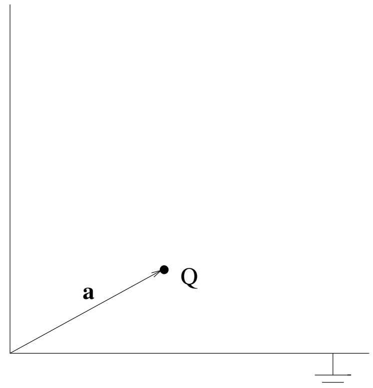
3) A thin uniform shell of electric charge of radius R carries a total charge Q. Find the electrostatic potential at a point $\mathbf{r}$ outside the shell.

4) A solid sphere of radius R contains a spherically symmetric charge density $\rho(r')$ with $0\leq\rho'\leq R$ and total charge $Q=\int_{Sphere}\rho(r')dV'$. Find the electrostatic potential at a point $\mathbf{r}$ outside the sphere in terms of Q.

5) Calculate the electric field at the point $\mathbf{r}$ in questions (3) and (4).

6) An infinite flat conducting sheet is grounded and a point charge Q is placed a distance d away from the sheet. Using the method of images calculate the potential due to this charge at a field point on the same side of the sheet as the charge itself.

7) An infinite flat conducting sheet is grounded and bent through 90° along a straight line. A charge is placed at a point **a** away from the 90° edge as shown in the figure below.



Using the method of images calculate the potential due to this charge at a field point in the same quadrant as the charge itself.

8) Calculate the potential and the electric field generated by a point charge Q placed outside a grounded conducting sphere of radius R, with the charge a distance a from the centre ($a > R$).

Determine the charge density $\sigma(\theta, \phi)$ induced on the surface of the sphere and calculate the total charge induced there.

9) A grounded, conducting sphere of radius R has a charge, Q, placed *inside* it, a distance a from the centre ($a < R$). Using the method of images, calculate the potential everywhere inside the sphere.

10) Check that the potential in question (9) agrees with question (6) in the limit of the radius of the sphere becoming infinite with Q remaining a finite distance from the surface.

11) Using the results from question (9), calculate the Green function for the problem of finding the potential inside a grounded, conducting sphere of radius R due to a charge distribution inside the sphere

12) A grounded, conducting sphere of radius R has a uniform ring of total charge Q and radius $a < R$ placed inside it, centred on the centre of the sphere. Using the Green function found in question (11) deduce the potential at any point inside the sphere on an axis through the centre of the ring and perpendicular to it.

Note: You may find the following form for the charge distribution useful

$$\rho(\mathbf{r}') = \frac{Q}{2\pi a^2}\delta(r' - a)\delta(\cos\theta').$$

13) Calculate the first two non-vanishing terms in the multipole expansions of the following two charge distributions:

$$\begin{array}{llllll} (i) & Q \ \text{at} & z = 3a, & -Q \ \text{at} & z = a, & (x = y = 0) \\ (ii) & Q \ \text{at} & z = a, & -Q \ \text{at} & z = -a, & (x = y = 0). \end{array}$$

Show that the lowest (dipole) term is the same in each case, but that the next to lowest order terms differ.

14) Find the lowest order term in a multipole expansion of the electrostatic potential due to the charge distribution

$$\begin{array}{cccc} Q & \text{at} & (x',y',z') = (a,a,0) & \text{and} \quad (x',y',z') = (-a,-a,0) \\ -Q & \text{at} & (x',y',z') = (a,-a,0) & \text{and} \quad (x',y',z') = (-a,a,0). \end{array}$$

Sketch the electric field lines arising from this potential, in the $z = 0$ plane.

15) Repeat question (14), but with

$$-Q \quad \text{at} \quad z = -a, \qquad 2Q \quad \text{at} \quad z = 0, \qquad -Q \quad \text{at} \quad z = +a, \qquad (x = y = 0).$$

16) Calculate the magnetic induction due to a circular wire of radius a, carrying a current I at a point on the axis perpendicular to the wire, passing through it's centre.

17) Show that the magnetic field

$$\mathbf{B} = B_0\{x(a-z)\hat{\mathbf{x}} - y(a+z)\hat{\mathbf{y}} + \left(a^2 + b - \frac{1}{2}(x^2+y^2) + z^2\right)\hat{\mathbf{z}},$$

with a, b and B_0 constants, satisfies $\nabla.\mathbf{B} = 0$ and $\nabla \times \mathbf{B} = 0$ everywhere. Find the extrema of $|\mathbf{B}|^2$ and identify the minima. Hence show that a magnetic dipole can be held at the origin in stable equilibrium in this field configuration, provided b is negative and and a is not too small.

18) Determine the state of polarisation of an electromagnetic wave which has electric field given by the real part of $\underline{\mathcal{E}} = \underline{\mathcal{E}}_0 e^{-i(\omega t - kz)}$ where $\underline{\mathcal{E}}_0 = \mathcal{E}_0\underline{\hat{x}} + \widetilde{\mathcal{E}}_0\underline{\hat{y}}$ is a constant complex vector with

i) $\mathcal{E}_0/\widetilde{\mathcal{E}}_0$ real

ii) $\mathcal{E}_0/\widetilde{\mathcal{E}}_0 = i|\mathcal{E}_0/\widetilde{\mathcal{E}}_0|$

More generally $\mathcal{E}_0/\widetilde{\mathcal{E}}_0 = e^{i\phi}|\mathcal{E}_0/\widetilde{\mathcal{E}}_0|$ with $0 \leq \phi < 2\pi$. Determine the state of polarisation of such a wave and sketch the shape traced out by the electric field in the plane transverse to the direction of motion. Calculate the angle that the long axis of this shape makes with the x-axis.
Hint: you may find it useful to write the electric field in terms of the basis $\mathbf{e}_+ = \hat{\mathbf{x}} + i\hat{\mathbf{y}}$ and $\mathbf{e}_- = \hat{\mathbf{x}} - i\hat{\mathbf{y}}$ and show that, under a rotation through an angle θ in the $x-y$ plane, $\mathbf{e}_\pm \to e^{\pm i\theta}\mathbf{e}_\pm$.

19) Show that, in the radiation zone $kr >> 1$, the magnetic field $\widetilde{\mathbf{B}} = \nabla \times \widetilde{\mathbf{A}}$ arising from the magnetic vector potential

$$\widetilde{\mathbf{A}}(\mathbf{r}) = -i\omega\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\tilde{\mathbf{p}},$$

with $\omega = ck$, is

$$\widetilde{\mathbf{B}} = \frac{1}{4\pi\epsilon_0}\frac{k^2}{c}\frac{e^{ikr}}{r}(\mathbf{n}\times\tilde{\mathbf{p}}),$$

where $\mathbf{n} = \hat{\mathbf{r}}/r$ is the unit radial vector.

Using Maxwell's equation $\nabla\times\mathbf{B} = \mu_0\epsilon_0\dot{\mathbf{E}}$ show that the electric field can be calculated from

$$\widetilde{\mathbf{E}} = \frac{ic}{k}\nabla\times\widetilde{\mathbf{B}}$$

and is equal to

$$\widetilde{\mathbf{E}} = -c\,\tilde{\mathbf{n}}\times\widetilde{\mathbf{B}}.$$

20) Show that the following field configuration is a solution of Maxwell's equations, where $\mathcal{B}_0$ is a (possibly complex) constant. Calculate the time averaged energy flux and show that it lies purely in the z-direction,

$$\begin{aligned}\mathcal{E}_y &= i\frac{ka}{\pi c}\mathcal{B}_0\sin\left(\frac{\pi x}{a}\right)e^{ikz-i\omega t}\\ \mathcal{B}_x &= -i\frac{ka}{\pi}\mathcal{B}_0\sin\left(\frac{\pi x}{a}\right)e^{ikz-i\omega t}\\ \mathcal{B}_z &= \mathcal{B}_0\cos\left(\frac{\pi x}{a}\right)e^{ikz-i\omega t}\end{aligned}.$$

(all other components zero)

21) Given that $\bigtriangledown^2\left(\frac{1}{r}\right) = -4\pi\delta^{(3)}(\underline{\mathrm{r}})$, prove that $G_k(\underline{\mathrm{r}}) = -\frac{1}{4\pi}\frac{\mathrm{e}^{\pm ikr}}{r}$ is a Green function for the Helmholtz operator, $\bigtriangledown^2 + k^2$.

22) A perfectly absorbing, spherical shell surrounds a radiating dipole at its centre. Calculate the pressure on the shell due to the radiation from the dipole.

23) Suppose a source consists of N rotating dipoles, all of the same magnitude. Compare the total power radiated when they all have the same phase and when they all have random, uncorrelated phases.

24) Using the Lorentz transformation properties of $F_{\mu\nu}$, evaluate the electric and magnetic fields of a uniform line of charge, with charge density λ/unit length, moving in the x-direction with constant velocity v. Add to this the field due to a *static* line of charge superimposed on the first with line density $-\lambda$. Determine the total field due to both lines in the limit of $v/c << 1$. Use your result to derive the Biot-Savart law. (See "The Feynman Lectures On Physics" Vol. II, Chapter 13-6.)

25) Show that the expression

$$F_\mu = e\sum_{\nu=0}^{3}F_{\mu\nu}U^\nu$$

reproduces, in the limit of small velocities, the Lorentz force law for the force on a particle with charge e due to an electromagnetic field (F_μ is the four force on the particle, U_μ is it's four velocity and $F_{\mu\nu}$ the usual electromagnetic field tensor).

26) Use the Lorentz invariants derived in the lectures to show that:

a) If **E** and **B** are perpendicular and of the same magnitude in one inertial reference frame, they are perpendicular and of the same magnitude in *all* inertial reference frames.

b) If **E** vanishes and **B** does not vanish in one inertial reference frame, then there is *no* inertial reference frame in which **B** vanishes.

27) Using the Lorentz transformation rules for $F^{\mu\nu}$ derived in the lectures obtain an expression for the transverse magnetic field components $B_y(\mathbf{x})$ and $B_z(\mathbf{x})$ due to a charged particle moving with uniform velocity $\mathbf{v}$ in the x-direction in an inertial reference frame S.

28) In the lectures, the electric and magnetic fields, $\mathbf{E}'$ and $\mathbf{B}'$ due to an electric charge, e, moving with uniform speed were calculated by Lorentz transforming the electric field

$$\mathbf{E} = \frac{e}{4\pi\epsilon_0}\frac{\mathbf{r}}{r^3}$$

of a charge at rest at the origin, with $\mathbf{B} = 0$. Using the form of $\mathbf{E}'$ and $\mathbf{B}'$ obtained in the lectures show that

$$\frac{1}{c^2}|\mathbf{E}'|^2 - |\mathbf{B}'|^2 = \frac{1}{c^2}|\mathbf{E}|^2 - |\mathbf{B}|^2 = \left(\frac{e}{4\pi c\,\epsilon_0 r^2}\right)^2$$

and

$$\mathbf{E}'.\mathbf{B}' = \mathbf{E}.\mathbf{B} = 0.$$