Mathematical Physics MP202: Mechanics

Problem Sheet 4

25th April 2008

1) The acceleration due to gravity at the equator is slightly less than that at the north pole for two reasons: firstly because of the effect of the centrifugal force at the equator due to the Earth's rotation, tending to reduce the acceleration due to gravity, and secondly because the Earth bulges slightly at the equator (again due to the centrifugal force) making the surface at the equator about 20 km farther away from the centre than the surface at the north pole, thus weakening the gravitational force because of the inverse square law. Given that the radius of the Earth at the north pole is $6,360$ km calculate the difference between the acceleration due to gravity at the north pole and at the equator.

Experimental measurements show that the difference is 0.05 ms^{-2}. How does your answer compare to this? Can you explain any discrepancy?

2) It is often suggested that water will swirl down a plug-hole in a sink or a bath in one direction in the northern hemisphere and in the opposite direction in the southern hemisphere. Do you think this is correct?

3) A particle of mass m moving in two dimensions experiences a frictional force proportional to its velocity, $\mathbf{F} = -c\,\mathbf{v}$, with $c > 0$ a constant. Show that the equations of motion in a rotating reference frame are

$$\begin{aligned} m(\ddot{x}' - 2\omega\dot{y}' - \omega^2 x') &= -c(\dot{x}' - \omega y') \\ m(\ddot{y}' + 2\omega\dot{x}' - \omega^2 y') &= -c(\dot{y}' + \omega x'), \end{aligned}$$

where x' and y' are Cartesian co-ordinates in the rotating frame and ω the angular velocity.

Determine the trajectory of the particle in a rotating reference frame, if it starts out at $x'(0) = x_0$, $y'(0) = 0$ with initial velocity $\dot{x}'(0) = -v_0$, $\dot{y}'(0) = -\omega x_0$.

4) A system of N particles with masses m_i at points $\mathbf{r_i}$, $i = 1, \ldots, N$, and total mass $M = \sum_{i=1}^{N} m_i$ consists of two disjoint parts with masses M_1 and M_2 ($M = M_1 + M_2$) and centres of mass $\mathbf{R}_1$ and $\mathbf{R}_2$ respectively. Show that the centre of mass of the whole system is

$$\mathbf{R} = \frac{M_1\mathbf{R}_1 + M_2\mathbf{R}_2}{M}.$$

Find the centre of mass of the Earth-Moon system, relative to the centre of the Earth, given that mass of the Earth is $M_1 = 6.0 \times 10^{24}$ kg, the mass of the Moon is $M_2 = 7.3 \times 10^{22}$ kg and the distance between the Earth and the Moon is 3.8×10^8 m. How does your answer compare with the radius of the Earth, which is $6,400$ km?

5) Calculate the moment of inertia of a flat circular disc of mass M and radius a about the centre of the disc. The disc is pierced half-way between its centre and the edge and suspended in a vertical plane from a frictionless pivot. Calculate the period of small

oscillations of the disc about its equilibrium position in terms of M, a and the acceleration due to gravity, g.

6) Calculate the period of small oscillations of a pendulum consisting of a point mass m_1 fixed to one end of a uniform rigid rod of mass m_2 and length l free to rotate in a fixed vertical plane about its other end which is a fixed point.

7) Show that the kinetic energy of a particle of mass m moving in a two-dimensional plane, when expressed in two-dimensional polar co-ordinates, is

$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2).$$

Using r and ϕ as generalised co-ordinates determine the generalised momenta associated with them and derive the Euler-Lagrange equations of motion for m. Solve the equations for an arbitrary initial position and velocity and show that the motion is a straight line.

8) A bead of mass m is constrained to move on a frictionless wire with shape $y(x) = a\cos(bx)$ in a vertical plane, with a constant gravitational field and y measured vertically upwards (a and b are positive constants). Find the Lagrangian for the bead and derive the equations of motion.

9) Two masses m_1 and m_2 are hung around a frictionless pulley by a light inelastic cord of length l, each hanging vertically downwards. Ignoring any sideways motion of the mass so they only move vertically and ignoring the mass of the pulley, find the Lagrangian for the system and derive the equations of motion.

10) A bead of mass m slides on a frictionless wire in the shape of a circle of radius a, rotating with constant angular velocity ω about an axis in the same plane as the wire and passing through its centre. The position of the bead is given by

$$x = \sin\psi\cos\omega t, \qquad y = \sin\psi\sin\omega t, \qquad z = a(1-\cos\psi),$$

where $0 \le \psi \le \pi$.

Using ψ as a generalised co-ordinate show that the Lagrangian is

$$L = \frac{m}{2}a^2\dot{\psi}^2 - V_{eff}(\psi)$$

where the effective potential is

$$V_{eff}(\psi) = -mga\cos\psi - \frac{1}{2}ma^2\omega^2\sin^2\psi.$$

Sketch the effective potential for $0 \le \omega^2 < g/a$ and $\omega^2 > g/a$ and identify points of stable and unstable equilibrium in both cases.