Mathematical Physics MP202: Mechanics

Problem Sheet 3

8th April 2008

1) A projectile is fired from a cliff-top 80 meters high, at an angle of 30^o above the horizontal, with speed $60m/s$. How far does it travel before hitting the ground below the cliff? (Ignore air resistance and take $g = 10m/s$.)

2) A mass m is thrown at an angle $\beta > 0$ to the horizontal, with initial speed v_0 and experiences air resistance proportional to its velocity, $\mathbf{F}_{air} = -c\dot{\mathbf{x}}$. Show that the trajectory is described by

$$y(x) = \left(\tan\beta + \frac{mg}{cv_0\cos\beta}\right)x - \frac{m^2g}{c^2}\ln\left(\frac{mv_0\cos\beta}{mv_0\cos\beta - cx}\right).$$

Check that this reduces to a parabola for small c.

3) A particle of mass m is thrown from ground level with initial speed v_0, at an angle β to the horizontal. If the air resistance is proportional to the velocity, $\mathbf{F}_{air} = -c\dot{\mathbf{x}}$, with $c > 0$, show that m hits the ground again at a time t_0 given implicitly as a function of β by

$$t_0(\beta) = \frac{1}{g}\left(v_0\sin\beta + \frac{mg}{c}\right)\left(1 - e^{-ct_0(\beta)/m}\right).$$

4) Under the same conditions as question (3) show that m travels horizontal distance

$$x(\beta) = \left(\frac{mgv_0\cos\beta}{mg + v_0c\sin\beta}\right)t_0(\beta)$$

before striking the ground.

Hence show that the maximum distance, for a fixed v_0, is obtained by choosing β so that

$$\ln\left(1 + \frac{cv_0}{mg\sin\beta}\right) = \frac{cv_0(cv_0\sin\beta + mg)}{mg(cv_0 + mg\sin\beta)}.$$

Does this answer agree with your expectation for small c?

5) Show that the unit basis vectors for spherical polar co-ordinates given in the lectures

$$\begin{aligned}
\boldsymbol{e}_r &= \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k},\\
\boldsymbol{e}_\theta &= \cos\theta\cos\phi\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} - \sin\theta\,\mathbf{k},\\
\boldsymbol{e}_\phi &= -\sin\phi\,\mathbf{i} + \sin\cos\;\mathbf{j},
\end{aligned}$$

are all mutually orthogonal and that

$$\begin{aligned}
\mathbf{e}_r \times \mathbf{e}_\theta &= \mathbf{e}_\phi,\\
\boldsymbol{e}_\theta \times \boldsymbol{e}_\phi &= \boldsymbol{e}_r,\\
\boldsymbol{e}_\phi \times \boldsymbol{e}_r &= \boldsymbol{e}_\theta,
\end{aligned}$$

where the orientation of the Cartesian co-ordinates is chosen so that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$.

6) If the position of a particle $\mathbf{r}(t) = r(t)\boldsymbol{e}_r(t)$ is changing with time, show that

$$\begin{aligned}
\dot{\boldsymbol{e}}_r &= \dot{\theta}\, \boldsymbol{e}_\theta + \dot{\phi} \sin\theta\, \boldsymbol{e}_\phi, \\
\dot{\boldsymbol{e}}_\theta &= -\dot{\theta}\, \boldsymbol{e}_r + \dot{\phi} \cos\theta\, \boldsymbol{e}_\phi, \\
\dot{\boldsymbol{e}}_\phi &= -\dot{\phi}(\sin\theta\, \boldsymbol{e}_r + \cos\theta\, \boldsymbol{e}_\theta).
\end{aligned}$$

Derive the following expressions for the velocity and acceleration:

$$\begin{aligned}
\dot{\mathbf{r}} &= \dot{r}\boldsymbol{e}_r + r\dot{\theta}\, \boldsymbol{e}_\theta + r\sin\theta\dot{\phi}\, \boldsymbol{e}_\phi, \\
\ddot{\mathbf{r}} &= (\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2)\boldsymbol{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin\theta\cos\theta\dot{\phi}^2)\boldsymbol{e}_\theta \\
&\quad + (2\dot{r}\dot{\phi}\sin\theta + 2r\cos\theta\dot{\theta}\dot{\phi} + r\sin\theta\ddot{\phi})\boldsymbol{e}_\phi.
\end{aligned}$$

7) The force between two neutral molecules that causes liquids to condense is called the Van der Waals force and, for large separation, is an attractive central force that falls off as the 7th power of distance from the origin. The potential is therefore of the form

$$V(r) = -\frac{a}{r^6},$$

where $a > 0$ is a constant and r is the separation between two molecules.

If one molecule is very heavy, and considered to be held fixed at the origin, and a light molecule of mass m moves under the influence of this force, show that a circular orbit is possible for

$$r = \left(\frac{6a}{ml^2}\right)^{1/4},$$

where l is the angular momentum per unit mass of the molecule m. Show that m must have total energy

$$E = \frac{l^3}{3}\sqrt{\frac{m^3}{6a}}$$

in order to be in this orbit.

Is the orbit stable?

8) If the potential in question (7) is modified to include a repulsive component at short distances

$$V(r) = \frac{b}{r^{10}} - \frac{a}{r^6},$$

where $b > 0$ is another constant, sketch the effective potential for angular momenta in the ranges: $l^2 < \frac{a^2}{2mb}$, $\frac{a^2}{2mb} < l^2 < \frac{9a^2}{10mb}$ and $\frac{9a^2}{10mb} < l^2$. Show that stable circular orbits are possible, provided, $l^2 < \frac{9a^2}{10mb}$ with radius

$$r = \left(\frac{3a - \sqrt{9a^2 - 10ml^2b}}{ml^2}\right)^{1/4}.$$

9) An ellipse, with its centre at the origin, is described, in Cartesian co-ordinates (x', y'), by the equation

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1$$

with a and b positive constants. When $a > b$ the quantity $\epsilon = \sqrt{1 - b^2/a^2} < 1$ is called the *eccentricity* of the ellipse ($b = a$ is a circle of radius a, with eccentricity zero).

For the Kepler problem it is convenient to shift the origin to $x' = \epsilon a$ and use different co-ordinates (x, y) with $x = x' - \epsilon a$, $y = y'$, in which case the equation is

$$\left(\frac{x}{a} + \epsilon\right)^2 + \frac{y^2}{b^2} = 1.$$

Re-write this in terms of polar co-ordinates, $x = r\cos\phi$ and $y = r\sin\phi$, and eliminate b in favour of a and ϵ to show that it is equivalent to the formula

$$r = \frac{a(1-\epsilon^2)}{1 + \epsilon\cos\phi},$$

which is the form used in the lectures.

Calculate the area, A, of the ellipse and show that it is

$$A = \pi ab = \pi a^2\sqrt{1-\epsilon^2}.$$

10) Solve the central force problem for a repulsive inverse square force,

$$\mathbf{F} = \frac{C}{r^2}\boldsymbol{e}_r,$$

with C a positive constant, and show that the orbits are hyperbolae.