Mathematical Physics MP202: Mechanics

Problem Sheet 2

26th February 2008

1) The position of a body moving round the the Sun can be described using 2-dimensional polar co-ordinates (r, ϕ) with the Sun at the origin. The time dependence of the radial co-ordinated is governed by Newton's second law and is

$$\ddot{r} = -\frac{GM}{r^2} + \frac{l^2}{r^3},$$

where $l = r^2\dot{\phi}$ is the angular momentum, per unit mass, of the body and M is the mass of the Sun (the second term on the right hand side is the centrifugal force due to the planet's angular motion). Find the potential $U(r)$ for this force, as a function of r, and analyse the possible motion of the body, depending on whether its energy is positive, zero or negative where the zero of energy is determined by choosing

$$U(r) \underset{r\to\infty}{\longrightarrow} 0.$$

2) Einstein's general theory of relativity goes beyond Newton's theory of universal gravitation and is applicable when speeds become an appreciable fraction of the speed of light and gravitational fields change as a finstion of time. In Einstein's theory the potential for the orbit equation is modified to

$$U(r) = -\frac{GM}{r} + \frac{l^2}{2r^2} - \frac{GMl^2}{c^2r^3},$$

where c is the speed of light. Repeat question (1) with this potential.

3) A mass hangs vertically from a perfect spring, under the action of gravity. If the mass is pulled down a distance d from equilibrium and released from rest, show that the subsequent motion is simple harmonic with frequency $\omega = \sqrt{(k/m)}$, where k is the spring constant and m the mass.

4) A particle of mass m is attached to the middle of a light, horizontally stretched wire, of length $2l$ under tension T. Show that the vertical force on the mass due to the wire is $2xT/l$ in an upward direction where x is the vertical displacement downwards, provided x is small compared to l.

(Use the fact that $\tan\theta \approx \sin\theta \approx \theta$ for small θ.)

Hence show that, if given a vertical push, such a mass will execute simple harmonic motion with frequency

$$\omega = \sqrt{\frac{2T}{ml}}.$$

5) A mass, m, hangs vertically from a light spring of spring constant k. If the air-friction is proportional to the speed v, cv, show that the one-dimensional equation of motion in the vertical direction is

$$m\frac{d^2y}{dt^2} = -ky - c\frac{dy}{dt}$$

where y is the upwards displacement from the static equilibrium position of the spring and mass and $c > 0$. (Note that the spring will not be relaxed at the static equilibrium position.)

Find the most general solution of this equation and show that, if the mass is given an upward push with speed v_0 at $y = 0$ when $t = 0$, the subsequent motion is

$$y(t) = \frac{v_0}{\omega}\exp\left(-\frac{1}{2}\frac{ct}{m}\right)\sin\omega t$$

where

$$\omega^2 = \frac{k}{m} - \frac{c^2}{4m^2}, \qquad \left(\text{assume} \qquad k > \frac{c^2}{4m}\right).$$

6) Find a particular solution of the damped harmonic oscillator equation,

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = F(t)/m,$$

when $F(t) = F_0(1 - \cos\Omega t)$, where F_0 is a constant.

7) A damped harmonic oscillator, with undamped natural frequency ω_0 and damping γ, is subject to a driving force with angular frequency Ω, $F_0\cos(\Omega t)$:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 = \frac{F_0}{m}\cos(\Omega t).$$

As shown in the lectures, the general solution of this equation is

$$x(t) = A_\gamma e^{-\gamma t/2}\cos(\omega t - \delta_\gamma) + \frac{F_0}{m\sqrt{(\omega_0^2 - \Omega^2)^2 + \gamma^2\Omega^2}}\cos(\Omega t + \phi)$$

where $\tan\phi = -\frac{\gamma\Omega}{\omega_0^2 - \Omega^2}$ and $\omega^2 = \omega_0^2 - \frac{\gamma^2}{4}$, and A_γ and δ_γ are constants to be determined by the initial conditions.

The first term in this solution is exponentially damped and dies away for $t >> 1/\gamma$ and represents the transient part of the behaviour. Determine the unique initial position and velocity of the oscillator that ensures that the transients vanish for all t, not just late t.

8) With the same solution as in question (7) the oscillator is at rest in its equilibrium position when the force is switched on at $t = 0$, $x(0) = \dot{x}(0)$. Show that

$$A_\gamma = \frac{F_0\omega_0}{m\omega\sqrt{(\omega_0^2 - \Omega^2)^2 + \gamma^2\Omega^2}}$$

and

$$\tan\delta_\gamma = \frac{\gamma}{2\omega}\left(\frac{\omega_0^2 + \Omega^2}{\omega_0^2 - \Omega^2}\right).$$