Mathematical Physics MP202: Mechanics

Problem Sheet 1

14th February 2008

(1) A particle of mass m moves in one dimension and is subject to the following time-dependent forces:

i) $F(t) = at^2 + bt + c$, with a, b and c positive constants;

ii) $F(t) = a\cos(\omega t)$, with a and ω constants;

iii) $F(t) = ae^{-bt}$, with a and b positive constants .

In each case calculate the particle's position and velocity as a function of time, assuming that it starts from the origin at $t = 0$, $x(0) = 0$, with speed $\dot{x}(0) = v_0$.

(2) An iPod, weighing $0.1kg$, is dropped from the top of the spire ($120m$ high). On the way down it is subject to a retarding force, due to air resistance, of the form $F = -cv$ where $c = 0.1kg/s$ is a constant. Calculate the terminal velocity of the iPod and the time it takes to reach the ground. How long would it take in the absence of air friction?

(3) If air resistance is proportional to the square of the velocity for a falling body of mass m, $F_{resistance} = cv^2$ with $c > 0$ for downward motion, show that the terminal velocity is $\sqrt{mg/c}$ and that, if m is dropped from rest at time $t = 0$, then its velocity at time t will be

$$v(t) = -\sqrt{\frac{gm}{c}}\tanh\left(\sqrt{\frac{gc}{m}}t\right)$$

.

Note: $\int \frac{du}{1-u^2} = \tanh^{-1} u$ where $\tanh(u) = \frac{1-e^{-2u}}{1+e^{-2u}}$ is the hyperbolic tangent function.

(4) A particle of mass m subject to a constant gravitational force falls through a resistive medium in which the resistance is proportional to the square of the speed, $F_{resistance} = cv^2$ with $c > 0$ for downward motion. Show that the distance d that the particle falls through in accelerating from v_0 to v_1 is

$$d = \frac{m}{2c}\ln\left(\frac{mg - cv_0^2}{mg - cv_1^2}\right),$$

where g is the acceleration due to gravity.

(5) A body of mass m is projected verticlally upwards from ground level, $y = 0$, with initial velocity v_0. If the air resistance is proportional to velocity, cv, show that the height as a function of velocity is

$$y = \frac{m}{c}(v_0 - v) + \frac{m^2 g}{c^2}\ln\left[\frac{mg + cv}{mg + cv_0}\right]$$

where g is the acceleration due to gravity.

Show that the maximum height is

$$y_{max} = \frac{mv_0}{c} - \frac{m^2 g}{c^2} \ln\left(1 + \frac{cv_0}{mg}\right).$$

(6) There are some fluids which display the unusual property of becoming *less* viscous when flowing fast (e.g. tomato ketchup!). Such fluids are called thixotropic. A particle moving through such a fluid would feel the resistive force becoming *less* as its speed increased.

Suppose a particle of mass m moves freely (i.e. ignoring gravity) through such a fluid in one dimension with frictional force inversely proportional to its speed squared ($F = -c/v^2$, $c > 0$). Show that the position as a function of velocity is

$$x = \frac{m}{4c}(v_0^4 - v^4)$$

where m has speed v_0 at $x = 0$. How far does m move before coming to rest?

(7) A man of mass m jumps, with speed v, from the front to the rear of a stationary boat of mass M and length L. Ignoring friction show that the boats stops when the man lands back on the boat and that the boat moves a distance $d = \frac{mL}{(m+M)}$, independent of the man's speed v.

If the boat is subject to friction as it moves through the water, proportional to its velocity with co-efficient $c > 0$, show that when the man lands the boat has moved a distance $d = \frac{mv}{c}\left(1 - \mathrm{e}^{-cT/M}\right)$ and is pushed backwards at a speed $V = \frac{mv}{(M+m)}\left(1 - \mathrm{e}^{-cT/M}\right)$, where T is the time the man spends in the air. Show that when the boat finally comes to rest it has returned to its initial position for *any* value of c, even $c = 0$ (assuming that c does not change when the man lands back in the boat). Can you reconcile this with your answer to the first part?