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THE NATIONAL UNIVERSITY OF IRELAND MAYNOOTH

MATHEMATICAL PHYSICS

Year 2

SEMESTER 2

AUTUMN REPEAT EXAMINATION

2006-2007

Mathematical Methods 2
MP362

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Time allowed: $1\frac{1}{2}$ hours

Answer two questions

All questions carry equal marks

1) Define what is meant by the **Wronskian** of two differentiable functions, $y_1(x)$ and $y_2(x)$, on an interval I of the real line. Prove that, if the Wronskian does not vanish on I, then $y_1(x)$ and $y_2(x)$ are linearly independent.

Determine the Wronskian of the two functions

$$\begin{aligned} y_1(x) &= \frac{\sin x}{x^2} - \frac{\cos x}{x} \\ y_2(x) &= -\frac{\cos x}{x^2} - \frac{\sin x}{x} \end{aligned}.$$

Are these two functions linearly independent on the interval $I = (0, \infty)$?

Show that these functions are eigenfunctions of the differential equation

$$x^2 \frac{d^2 y(x)}{dx^2} + 2x \frac{dy(x)}{dx} + x^2 y(x) = 0$$

and determine their eigenvalues.

2) Briefly explain the meaning of the term **normal** as applied to a second-order, linear ordinary differential operator acting on a function $y(x)$ on an interval I. Show that any such operator can be always be cast into self-adjoint form,

$$\mathcal{L} y(x) = \frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right) + q(x) y(x).$$

Let $\phi_n(x)$ be a complete set of orthogonal eigenfunctions for $\mathcal{L}$ and $h(x)$ an integrable function on I. Determine the co-efficients of $\phi_n(x)$ in an orthogonal function expansion of $h(x)$, in terms of an integral over I.

Find a particular solution of the inhomogeneous equation

$$y'' - 2xy' + 8y = (x+4)(x-1)$$

on the interval $(-\infty, \infty)$.
Note: The first three Hermite polynomials are

$$H_0(x) = 1, \qquad H_1(x) = 2x, \qquad H_2(x) = 4x^2 - 2,$$

with eigenvalue equation

$$H_n'' - 2xH_n' = -2nH_n.$$

3) Using separation of variables find the most general solution of the one-dimensional heat equation

$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t}$$

for a differentiable function $u(t,x)$, with boundary conditions $u(t,0) = 0$, $u(t,L) = T$, where $T > 0$ is a constant, and $u(0,x) = 0$ for $0 \leq x < L$.

Note: You may assume that

$$\int_{-\pi}^{\pi} \sin(n\theta)\sin(m\theta)d\theta = \pi\delta_{m,n}$$

for n and m positive integers.

4) Derive the Cauchy-Riemann conditions for a complex function

$$f(z,\bar{z}) = u(x,y) + iv(x,y)$$

of the complex variable $z = x + iy$, with $\bar{z} = x - iy$ the complex conjugate variable,

$$\frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y}, \qquad \frac{\partial u(x,y)}{\partial y} = -\frac{\partial v(x,y)}{\partial x},$$

that ensure that the complex derivative of $f(z)$ with respect to z is well defined.

Find the regions of the complex plane in which the following functions satisfy the Cauchy-Riemann conditions,

$$f(z,\bar{z}) = (\bar{z})^2,$$
$$f(z,\bar{z}) = \sqrt{z}.$$