MP362: Mathematical Methods II

Problem Sheet 4

Brian Dolan 24th April 2007

(1) Evaluate the real and imaginary parts of the following complex functions, where a is a real constant:

(*i*) z^3

(*ii*) $\dfrac{1}{(z-a)(z+a)}$

(*iii*) $\sinh z$

(*iv*) e^{-az}

(*v*) $\ln z$

(*vi*) z^a

where a is a real constant.

(2) Show that the complex functions

$$f(z) = e^z \qquad \text{and} \qquad f(z) = \cos(z)$$

satisfy the Cauchy-Riemann conditions for $|z| < \infty$, and are thus analytic entire functions.

(3) Determine the points of non-analyticity of the functions in question (1).

(4) Develop Taylor expansions of the following functions about the given points, z_0:

(*i*) $\dfrac{1}{(z-i)}, \qquad z_0 = 0$

(*ii*) $\dfrac{1}{(z-i)}, \qquad z_0 = 1$

(*iii*) $\ln(1+z), \qquad z_0 = 0$

Determine the radii of convergence in examples (*i*) and (*ii*).

(5) Determine the Laurent expansions of the following functions about the points z_0, in the regions indicated:

(*i*) $\dfrac{1}{(z-i)}, \qquad z_0 = i \qquad$ over the whole complex plane except the point $z = i$

(*ii*) $\dfrac{1}{(z-1)(z+1)}, \qquad z_0 = 1 \qquad 0 < |z-1| < 2$

(*iii*) $\dfrac{1}{z^2(z-1)}, \qquad z_0 = 0 \qquad 0 < |z| < 1$

(*iv*) $\dfrac{\sin z}{z^2} \qquad z_0 = 0 \qquad 0 < |z| < \infty$

(6) Find all the poles of the functions in question (5), determine their order, and calculate their residues.

(7) Assuming that $f(z)$ is analytic within and on the contour C, show that

$$\oint_C \frac{f'(z)dz}{(z-z_0)} = \oint_C \frac{f(z)dz}{(z-z_0)^2}.$$

(8) Show that

$$\oint_C \frac{dz}{(z-a)(z+a)} = 0,$$

where C is a contour which encloses both the points $z = \pm a$, where a is real. Hence deduce that

$$\oint_C \frac{dz}{z^2} = 0$$

for any contour C that encloses the origin.

(9) Determine the nature of the singularities of each of the following functions and evaluate the residues ($a > 0$):

(i) $$\frac{1}{z^2+a^2}$$

(ii) $$\frac{1}{(z^2+a^2)^2}$$

(iii) $$\frac{\sin(1/z)}{z^2+a^2}$$

(iv) $$\frac{e^{iz}}{z^2-a^2}$$

(10) Using the calculus of residues, establish the following definite integrals:

(i) $$\int_0^{2\pi} \frac{d\theta}{1+a\cos\theta} = \frac{2\pi}{\sqrt{1-a^2}} \qquad \text{where} \quad -1 < a < 1$$

(ii) $$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\,dx = \frac{\pi}{a}e^{-a} \qquad a \quad \text{real and positive}$$

(iii) $$\int_0^{\infty} \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2} \qquad \textit{Hint}: \quad \sin^2 x = (1/2)(1-\cos 2x)$$