MP362: Mathematical Methods II

Problem Sheet 3

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(1) Find the unique solution of the three dimensional equation, using spherical polar co-ordinates, for a function $u(r,\theta)$, which is independent of the azimuthal angle ϕ, inside a sphere of radius a, with the boundary condition that

$$u(a,\theta) = 1 + \cos(2\theta).$$

(2) In electrostatics the electric field $\mathbf{E}$ can be written as the gradient of the electric potential $u(\mathbf{r})$, $\mathbf{E}(\mathbf{r}) = -\nabla u(\mathbf{r})$, and satisfies the differential form of Gauss' law

$$\nabla.\mathbf{E}(\mathbf{r}) = 0 \qquad \Rightarrow \qquad \nabla^2 u(\mathbf{r}) = 0$$

at any point $\mathbf{r}$ where the electric charge density vanishes.

Solve this equation to find the electrostatic potential outside a conducting sphere of radius a when the sphere is placed in an external electric field that is constant before the sphere is introduced. One boundary condition in this instance is that the potential should be constant on the surface of a conductor, which can be taken to be zero, so $u(a,\theta,\phi) = 0$.

Note: a constant electric field can always be chosen to point in the z-direction,

$$\mathbf{E} = E_0\hat{\mathbf{z}} = -\nabla(E_0 z) = -E_0\nabla(r\cos\theta),$$

where $E - 0$ is a constant, and the sphere should not disturb the field at infinity. We should also expect, from the symmetry of the problem, that $u(r,\theta,\phi)$ will be independent of ϕ. So another condition is that $u(r,\theta) \to E_0 r\cos\theta$, for $r \to \infty$.

(3) Demonstrate explicitly that the associated Legendre functions $P_l^m(x)$

$$\begin{aligned}
P_1^1(x) &= \sqrt{1-x^2},\\
P_2^1(x) &= 3x\sqrt{1-x^2},\\
P_2^2(x) &= 3(1-x^2),\\
P_3^1(x) &= \frac{3}{2}(5x^2-1)\sqrt{1-x^2},\\
P_3^2(x) &= 15(1-x^2),\\
P_3^2(x) &= 15(1-x^2)^{3/2},
\end{aligned}$$

satisfy the associated Legendre equation

$$\frac{d}{dx}\left((1-x^2)\frac{dP_l^m(x)}{x}\right) - \frac{m^2}{1-x^2}P_l^m(x) + l(l+1)P_l^m(x) = 0.$$

Confirm the normalisation condition

$$\int_{-1}^{1} P_l^{m'}(x)P_l^m(x)dx = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{m,m'}.$$

for P_2^1 and P_2^2.