MP362: Mathematical Methods II

Problem Sheet 1

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(1) Solve the equation

$$y'' = \lambda y$$

with the following boundary conditions

$$\begin{array}{lll} (i) & y(0) = 0, & y(\pi) = 0 \\ (ii) & y(0) = 0, & y'(0) = 1. \end{array}$$

In each case determine any restrictions on the eigenvalue λ.

(2) Show that the eigenfunctions bellow satisfy the eigenvalue equations for the corresponding differential operators, $\mathcal{L}$, and determine the eigenvalues in each case,

i) Legendre's equation: $\mathcal{L} = (1 - x^2)D^2 - 2xD$, with Legendre polynomials $P_n(x)$
ii) Laguerre's equation: $\mathcal{L} = xD^2 + (1 - x)D$, with Laguerre polynomials $L_n(x)$
iii) Hermite's equation: $\mathcal{L} = D^2 - 2xD$, with Hermite polynomials $H_n(x)$

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}(5x^3 - 3x)$$
$$L_0(x) = 1, \quad L_1(x) = 1 - x, \quad L_2(x) = \frac{1}{2}(x^2 - 4x + 2), \quad L_3(x) = \frac{1}{6}(-x^3 + 9x^2 - 18x + 6)$$
$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2, \quad H_3(x) = 8x^3 - 12x.$$

(3) By computing Wronskians, show that the following sets of functions are linearly independent

$$\begin{array}{ll} (i) & e^x, \quad e^{-x} \\ (ii) & \ln x, \quad x \ln x \qquad (x > 0) \\ (iii) & 1, \quad x, \quad x^2, \ldots, \quad x^n. \end{array}$$

(4) Given that the hyperbolic trigonometric functions $\cosh(x)$ and $\sinh(x)$ are related by differentiation, in that

$$\frac{d}{dx}\cosh(x) = \sinh(x) \qquad \frac{d}{dx}\sinh(x) = \cosh(x),$$

calculate the Wronskian of the three functions $\cosh(x)$, $\sinh(x)$ and $\exp(x)$. Do you think these functions are linearly dependent on $(-\infty, \infty)$?

(5) Express the following differential equations in self-adjoint form and determine the weight functions in each case:

i) Harmonic Oscillator Equation

$$y'' = \lambda y, \qquad x \in (-\infty, \infty) \quad \text{(or some subset} \quad I \subseteq (-\infty, \infty))$$

ii) Legendre's Equation

$$(1 - x^2)y'' - 2xy' = \lambda y, \qquad x \in [-1, 1] \quad (x = \pm 1 \quad \text{are singular points})$$

iii) Laguerre's equation

$$xy'' + (1 - x)y' = \lambda y, \qquad x \in [0, \infty) \quad (x = 0 \quad \text{is a singular point})$$

iv) Hermite's equation

$$y'' - 2xy' = \lambda y \qquad x \in (-\infty, \infty) \quad \text{(or some subset} \quad I \subseteq (-\infty, \infty))$$

v) Bessel's equation

$$x^2y'' + xy' + x^2y = \lambda y \qquad x \in [0, \infty) \quad \text{(or some subset} \quad I \subseteq [0, \infty)).$$

(6) Chebyshev's equation is given by

$$(1 - x^2)y'' - xy' - \lambda y = 0.$$

Determine over which range of x the equation is normal, and rewrite it in self-adjoint form over this range. Determine the weight function.

(7) Prove that the following sets of functions from question (2) are orthogonal, with the given weight functions and intervals of integration,

$$\begin{array}{llll}
(i) & P_0(x), \quad P_1(x), \quad P_2(x); & r(x) = 1, & I = [-1, 1] \\
(ii) & L_0(x), \quad L_1(x), \quad L_2(x); & r(x) = e^{-x}, & I = [0, \infty) \\
(iii) & H_0(x), \quad H_1(x), \quad H_2(x); & r(x) = e^{-x^2}, & I = (-\infty, \infty).
\end{array}$$

(8) Express the following polynomial functions as linear combinations of: (a) Legendre polynomials and (b) Hermite polynomials,

$$\begin{array}{ll}
(i) & (x - 2)^2 \\
(ii) & x^3 - 4x^2 + x - 2 \\
(ii) & x^2 + bx + c.
\end{array}$$

(9) Derive the Fourier expression for the function $h(x) = x$ in the interval $x \in I = [-\pi, \pi]$,

$$h(x) = 2\sum_{n=1}^{\infty} \frac{(-1)}{n}^{n+1} \sin(nx).$$

What happens if this series expansion is extended outside of the interval $[-\pi, \pi]$?

(10) Given that

$$\frac{2}{(5-4x)^{1/2}} = \sum_{n=0}^{\infty} \frac{1}{2^n} P_n(x)$$

for $|x| \leq 1$ find a particular solution of the equation

$$(1-x^2)y'' - 2xy' = \frac{2}{(5-4x)^{1/2}} - 1.$$

(11) Given

$$G(x,t) = (1-2xt+t^2)^{-1/2}, \qquad x \in [-1,1], \qquad t \in [0,1)$$

prove that

$$\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial G}{\partial x}\right] + \frac{\partial}{\partial t}\left[t^2\frac{\partial G}{\partial t}\right] = 0.$$

Expanding G in powers of t as $G(x,t) = \sum_{n=0}^{\infty} P_n(x)t^n$, show that $P_n(x)$ satisfy Legendre's equation order n.

Use this result to derive the expansion given in equation (10),

$$\frac{2}{(5-4x)^{1/2}} = \sum_{n=0}^{\infty} \frac{1}{2^n} P_n(x)$$

(12) The Legendre polynomials can be derived from

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$$

(this is known as Rodrigues' formula). Prove by induction that these functions satisfy Legendre's equation

$$(1-x^2)\frac{d^2P_n}{dx^2} - 2x\frac{dP_n}{dx} + n(n+1)P_n = 0.$$

(13) Using Rodrigues' formula for the Legendre polynomials given in question (12) show that

$$\int_{-1}^{1} P_n(x)P_{n'}(x)dx = \frac{2}{2n+1}\,\delta_{n,n'}.$$