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THE NATIONAL UNIVERSITY OF IRELAND MAYNOOTH

MATHEMATICAL PHYSICS

Year 2

SEMESTER 2
2006-2007

Mathematical Methods 2
MP362

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Time allowed: $1\frac{1}{2}$ hours
Answer two questions
All questions carry equal marks

1) Define what is meant by the **Wronskian** of three differentiable functions, $y_1(x)$, $y_2(x)$ and $y_3(x)$, on an interval I of the real line. Prove that, if the Wronskian does not vanish on I, then $y_1(x)$, $y_2(x)$ and $y_3(x)$ are linearly independent.

Determine the Wronskian for the three functions

$$y_1(x) = \cos^2 x, \qquad y_2(x) = \sin^2 x \qquad \text{and} \qquad y_3(x) = \sin x \cos x.$$

Are these three functions linearly independent on the interval consisting of the whole real line $(-\infty, \infty)$?

2) i) Explain briefly what is meant by the term **self-adjoint** as applied to a linear second order ordinary differential equation. Prove that any two eigenfunctions for such an equation with periodic boundary conditions, corresponding to distinct eigenvalues, are orthogonal.

ii) Find a particular solution of the equation

$$(1-x^2)\frac{d^2y(x)}{dx^2} - 2x\frac{dy(x)}{dx} - 4y(x) = \frac{1}{\sqrt{a-x}}$$

where $-1 \leq x \leq 1$ and $a > 1$ is a constant.

Note: you may assume that

$$\int_{-1}^{1} \frac{1}{\sqrt{a-x}} P_n(x)dx = \frac{2^{n+2}}{(2n+1)} \frac{1}{(\sqrt{a-1}+\sqrt{a+1})^{2n+1}}$$

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \frac{2}{2n+1}\delta_{n,m}$$

where $P_n(x)$ are Legendre polynomials, with eigenvalues $-n(n+1)$.

3) Show that Laplace's equation in spherical polar co-ordinates

$$\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{1}{r^2}\cot\theta\frac{\partial}{\partial\theta}\right)u(r,\theta)=0$$

for a differentiable function $u(r,\theta)$, which is independent of the azimuthal angle ϕ, can be split into two ordinary differential equations using separation of variables.

Solve this equation inside a sphere of radius a with the boundary conditions

$$u(a,\theta)=\cos\theta+\sin^2\theta.$$

Note: the first three Legendre polynomials are

$$P_1(\cos\theta)=1,\qquad P_1(\cos\theta)=\cos\theta,\qquad P_2(\cos\theta)=\frac{1}{2}(3\cos^2\theta-1).$$

4) Explain briefly what is the difference between a **Laurent expansion** and a **Taylor expansion** of a complex function about a point z_0 and define what is meant by a **simple pole**.

Determine the poles and residues of the function

$$f(z)=\frac{z}{(1-a^2z^2)(z^2-a^2)},$$

where $0<a<1$ is real constant. Hence, or otherwise, show that

$$\int_0^{2\pi}\frac{d\theta}{\left(1-2a^2\cos(2\theta)+a^4\right)}=\frac{2\pi}{(1-a^4)}.$$

Note: you may assume the residue theorem for the integral of a complex function $f(z)$ around a closed contour, C, in the complex z-plane,

$$\oint_C f(z)dz=2\pi i\sum_{\text{residues}}a_{-1}(z_i),$$

where the sum is over all residues $a_{-1}(z_i)$, of poles z_i of $f(z)$, enclosed by the contour C.