Schrödinger invariant solutions of M-theory

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Motivation

- AdS/CFT : various guises

1) AdS/CFT maps a *weakly* coupled string theory (supergravity solutions) to strongly coupled field theory, and vice versa; most **confident when supersymmetry present** - powerful non-renormalisation theorems; many non-trivial checks over 10 years.

2) Original incarnation $AdS_5 \times S^5 \Leftrightarrow \mathcal{N} = 4$ SYM, since extended to less supersymmetry, running couplings $\Rightarrow$ study theories with qualitative similarity to QCD via weak-strong duality; many successes, η/s, meson spectra, overlap with lattice data.

3) However, only one QCD, finding gravity dual for QCD $\sim$ manned Mars mission? Large motivation for hunt for holographic dual shifting to CM; odds much better - many pre-existing materials and may be possible to engineer a dual experimentally.

Motivation

- AdS/CMT

1) Computational tool to model strongly coupled systems, in particular quantum critical points; only alternative is Lattice (less suited to dynamics)

2) Chances of finding an experimental set-up much greater; many effective Hamiltonians and an increasing number may be engineered via optical lattices.

3) Tantalisingly, if an experimental set-up was realised, face prospect of a laboratory experiment describing quantum gravity $\Rightarrow$ richer understanding of black holes?

4) In general for CM need $z \neq 1$ e.g. $z = 2$, symmetry group of free Schrödinger equation; dilute gas of lithium-6 or potassium-40 with fermionic interaction strength tuned by external B, approximate $z = 2$. Symmetry group of NRABJM. Clear motivation - first example of NR "AdS/CFT".

Outline

- Review NR ABJM
- Geometric realisation of NR symmetry
- Killing spinor equation, G-structures
- Solution
- Conclusion

NR ABJM

- ABJM

1) ABJM : $\mathcal{N}=6$ supersymmetric Chern-Simons-matter theory, $U(N)_k \times U(N)_{-k}$ gauge group with gauge fields A_μ and $\tilde{A}_\mu$ with Chern-Simons levels $(k,-k)$. The matter fields consist of bi-fundamental complex scalars Z^α and fermions Ψ_α ($\alpha = 1, ..., 4$), transform under global $SU(4)_R \times U(1)_B$ as **4** and $\bar{\mathbf{4}}$, respectively.The $U(1)_B$ charge $\leftrightarrow$ number operator - counts bosons and fermions. Also exists $\mathbb{Z}_2$-symmetry (parity) $Z_\alpha, \Psi^\alpha, A_\mu, \tilde{A}_\mu \leftrightarrow \bar{Z}^\alpha, \bar{\Psi}_\alpha, \tilde{A}_\mu, A_\mu$.

2) Theory dual to M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$. Can regard S^7 as circle-fibre over $\mathbb{CP}^3$; $\mathbb{Z}_k$ acts on fibre; breaks $SO(8)$ of S^7 to $SU(4)_R \times U(1)_B$. $U(1)_B \simeq$ M-theory circle, can reduce to IIA on $AdS_4 \times \mathbb{CP}^3$.

NR ABJM

- Mass-deformation
  3)ABJM Lagrangian has several parts

$$\mathcal{L} = \mathcal{L}_{\rm CS} + \mathcal{L}_{\rm kin} + \mathcal{L}_{\rm Yukawa} + \mathcal{L}_{\rm potential} \,,$$

where (for example)

$$\mathcal{L}_{\rm kin} = -\mathrm{Tr}(D_\mu \bar{Z}^\alpha D^\mu Z_\alpha + i\bar{\Psi}_\alpha \gamma^\mu D_\mu \Psi^\alpha) \,,$$

with $D_\mu Z_\alpha = \partial_\mu Z_\alpha - iA_\mu Z_\alpha + iZ_\alpha \tilde{A}_\mu$. Lagrangian invariant under $\mathcal{N} = 6$ (Poincaré) supersymmetry. Theory admits (equal) mass deformation

$$\mathcal{L}_m = -Tr(M^\alpha{}_\gamma M^\gamma{}_\beta \bar{Z}^\beta Z_\alpha + i\bar{\Psi}_\alpha M^\alpha{}_\beta \Psi^\beta) + \cdots \tag{1}$$

which breaks the $SU(4)_R$ down to $SU(2) \times SU(2) \times U(1)$ through the choice

$$M = \frac{mc}{\hbar} \mathrm{diag}(1, 1, -1, -1). \tag{2}$$

Preserves $\mathcal{N} = 6$ once $\delta_m \Psi^\alpha$ contribution added. *Hosomichi et al.*, *Gomis et al.*

NR ABJM

- NR Limit

  Given the mass-deformed theory, many possible non-rel systems; preserved symmetries and supersymmetry depend on limit i.e. (anti)-particles, or both.

  1) Example: Begin with scalar Lagrangian

$$\mathcal{L}_{\text{scalar}} = \frac{1}{c^2} D_t \bar{Z}^{\alpha} D_t Z_{\alpha} - D_i \bar{Z}^{\alpha} D_i Z_{\alpha} - \frac{m^2 c^2}{\hbar^2} \bar{Z}^{\alpha} Z_{\alpha} \tag{3}$$

  Taking just particle modes

$$Z_{\alpha} = \frac{\hbar}{\sqrt{2m}} z_{\alpha} e^{-imc^2 t/\hbar} \tag{4}$$

  get in $c \to \infty$ limit (correction terms suppressed $O(1/c^2)$)

$$\mathcal{L}^{NR}_{\text{scalar}} = \bar{z}^{\alpha} \left(i\hbar D_t + \frac{\hbar^2}{2m} D_i^2 \right) z_{\alpha}$$

NR ABJM

- Full NR Lagrangian
  Following process through, one finds...

$$\mathcal{L}_{\text{scalar}} = \text{Tr}\Big[i\hbar \bar{z}^{\alpha} D_t z_{\alpha} - \frac{\hbar^2}{2m} D_i \bar{z}^{\alpha} D_i z_{\alpha} - \frac{\pi \hbar^2}{mk} \left(z_a \bar{z}^{\alpha} z_{\beta} \bar{z}^{\gamma} \Omega^{\beta}{}_{\gamma} - \bar{z}^{\alpha} z_{\alpha} \bar{z}^{\beta} z_{\gamma} \Omega^{\gamma}{}_{b} \right) \Big],$$

where $\Omega^{\alpha}{}_{\beta} = \text{diag}(1, 1, -1, -1)$.

$$\begin{aligned}
\mathcal{L}_{\text{fermion}} &= \text{Tr}\Big[i\hbar \bar{\psi}_{\alpha} D_t \psi^{\alpha} - \frac{\hbar^2}{2m} D_i \bar{\psi}_{\alpha} D_i \psi^{\alpha} + \frac{\hbar^2}{2m} \Omega^{\alpha}{}_{\beta} \left(\bar{\psi}_{\alpha} F_{12} \psi^{\beta} - \tilde{F}_{12} \bar{\psi}_{\alpha} \psi^{\beta} \right) \Big], \\
\mathcal{L}_{\text{Yukawa}} &= \frac{\pi \hbar^2}{mk} \text{Tr}\Big[\bar{z}^{\alpha} z_{\alpha} \left(\bar{\psi}_a \psi^a - \bar{\psi}_i \psi^i \right) + z_{\alpha} \bar{z}^{\alpha} \left(\psi^a \bar{\psi}_a - \psi^i \bar{\psi}_i \right) \\
&\quad -2 \left(z_a \bar{z}^b \psi^a \bar{\psi}_b + \bar{z}^a z_b \bar{\psi}_a \psi^b \right) + 2 \left(z_i \bar{z}^j \psi^i \bar{\psi}_j + \bar{z}^i z_j \bar{\psi}_i \psi^j \right) \\
&\quad -2 \epsilon_{ab} \epsilon_{ij} \left(\bar{z}^a \psi^b \bar{z}^i \psi^j + \bar{z}^a \psi^i \bar{z}^j \psi^b \right) - 2 \epsilon^{ab} \epsilon^{ij} \left(z_a \bar{\psi}_b z_i \bar{\psi}_j + z_a \bar{\psi}_i z_j \bar{\psi}_b \right) \Big], \\
\mathcal{L}_{\text{NR}} &= \frac{k\hbar}{4\pi} \epsilon^{\mu\nu\rho} \text{Tr}\Big(A_{\mu} \partial_{\nu} A_{\rho} - \frac{2i}{3} A_{\mu} A_{\nu} A_{\rho} - \tilde{A}_{\mu} \partial_{\nu} \tilde{A}_{\rho} + \frac{2i}{3} \tilde{A}_{\mu} \tilde{A}_{\nu} \tilde{A}_{\rho} \Big),
\end{aligned}$$

2) Lagrangian is invariant under full Schrödinger algebra, e.g. D

$$(t, x\,;\; z, \psi) \;\rightarrow\; (\lambda^{-2} t, \lambda^{-1} x\,;\; \lambda z, \lambda \psi)\,.$$

NR ABJM

- Preserved Symmetries

1) R-symmetry $SU(4)_4 \to SU(2)_1 \times SU(2)_2 \times U(1)_R$. Retain original $U(1)_B$.

2) Poincaré symmetry replaced by Galilean $\{H, P_i, J, G_i\}$.

3) Though mass-deformation breaks rel. conformal symmetry, NR conformal symmetry is restored $\{D, C\}$.

4) Generalisation as super-Schrödinger algebra, 2 types of supercharges (dynamical $\{Q, \bar{Q}\} \sim H$, kinematical $\{q, \bar{q}\} \sim M$). Also have another set of conformal supercharges S $[C, Q] \sim S$. Q, q can be identified by expanding the susy transformations:

$$\delta z = \sqrt{\frac{2mc}{\hbar}}\delta_K z + \sqrt{\frac{\hbar}{2mc}}\delta_D z.$$

5) Identifies 12 original susy parameters split into 10 kinematical and 2 dynamical. 4 singlets under $SU(2) \times SU(2)$ and 8 "spectators" that transform in $(\mathbf{2}, \mathbf{2})$.

NR ABJM

- Super-Schrödinger symmetry (Bosonic part)

The Schrödinger algebra $Sch(d)$ contains an $SO(2,1)$ subalgebra among the time-translation (H), dilatation (D) and special conformal (C) generators.

$$[D,H] = +2H\,, \qquad [D,C] = -2C\,, \qquad [H,C] = -D\,,$$

as well as the $SO(d)$ subalgebra,

$$[M^{ij}, M^{kl}] = +\delta^{jk} M^{il} + \delta^{il} M^{jk} - \delta^{ik} M^{jl} - \delta^{jl} M^{ik}\,.$$

The remaining generators are space-translations (P^i) and Galilean boosts (G^i). They are vectors under the $SO(d)$,

$$[M^{ij}, P^k] = +\delta^{jk} P^i - \delta^{ik} P^j\,, \qquad [M^{ij}, G^k] = +\delta^{jk} G^i - \delta^{ik} G^j\,,$$

and satisfy the following commutation relations:

$$\begin{aligned} & & [D, P^i] = +P^i\,, \qquad & [D, G^i] = -G^i\,, & \\ [H, P^i] = 0\,, \qquad & & [C, P^i] = +G^i\,, \qquad & [H, G^i] = -P^i\,, \qquad & [C, G^i] = 0\,. \end{aligned} \tag{5}$$

Finally, we have the central extension with the "rest-mass" or the particle number,

$$[P^i, G^j] = -\delta^{ij} M\,.$$

NR ABJM

- Super-Schrödinger symmetry (Global frame)

Can also introduce Virasoro-like notation (*Blau et al.*),

$$L_0 \equiv \tfrac{1}{2} D\,,\;\; L_{-1} \equiv H\,,\;\; L_{+1} \equiv C\,, \quad P^i_{-1/2} \equiv P^i\,,\;\; P^i_{+1/2} \equiv G^i\,, \quad M_0 \equiv M\,.$$

Then, the commutation relations can be compactly written as

$$[L_m, L_n] = (m-n) L_{m+n}\,, \quad [L_m, P^i_r] = \left(\tfrac{1}{2} m - r\right) P^i_{m+r}\,, \quad [P^i_r, P^j_s] = (r-s)\delta^{ij} M_{r+s}\,.$$

The operator-state map naturally introduces the following recombination of generators:

$$\widehat{L}_0 \equiv \frac{1}{2}(-iH - iC)\,, \quad \widehat{L}_{\pm 1} \equiv \frac{1}{2}(-iH + iC \pm D)\,,$$
$$\widehat{P}^i_{\pm 1/2} = \frac{1}{\sqrt{2}}(-iP^i \mp G^i)\,, \quad \widehat{M}_0 = -iM_0\,.$$

The new generators also satisfy Virasoro-like commutation relations,

$$[\widehat{L}_m, \widehat{L}_n] = (m-n) L_{m+n}\,, \quad [\widehat{L}_m, \widehat{P}^i_r] = \left(\tfrac{1}{2} m - r\right) \widehat{P}^i_{m+r}\,, \quad [\widehat{P}^i_r, \widehat{P}^j_s] = (r-s)\delta^{ij} \widehat{M}_{r+s}\,,$$

as well as the conjugation relations

$$(\widehat{L}_m)^\dagger = L_{-m}\,, \quad (\widehat{P}^i_r)^\dagger = P^i_{-r}\,, \quad (\widehat{M}_0)^\dagger = \widehat{M}_0\,.$$

NR ABJM

- $\mathcal{N} = 2$ super-Sch. algebra

1) $\mathcal{N} = 2$ refers to the supersymmetry of the rel. parent theory - kinematical $(q, \bar{q})$, dynamical $(Q, \bar{Q})$ and conformal $(S, \bar{S})$ supercharges (Poincaré frame).

2) Virasoro-like notation the supercharges are denoted by q, $Q_{-1/2} \equiv Q$, $Q_{+1/2} \equiv S$ and their conjugates. They transform under the $SO(2,1) \times U(1)_J \times U(1)_R$ subalgebra as

$$[L_m, Q_r] = \left(\tfrac{1}{2}m - r\right) Q_r \,, \quad [L_m, q] = 0 \,,$$

$$[J, Q_r] = +\tfrac{1}{2} Q_r \,, \quad [R, Q_r] = +Q_r \,, \quad [J, q] = +\tfrac{1}{2} q \,, \quad [R, q] = -q \,.$$

$$[\bar{P}_r, Q_s] = (r - s)\bar{q} \,, \qquad [\bar{P}_r, q] = 0 \,,$$

and anti-commutators among supercharges give

$$\{\bar{Q}_r, Q_s\} = L_{r+s} + \tfrac{1}{2}(r - s)\left(J - \tfrac{3}{2}R\right) , \qquad \{q, Q_r\} = P_r \,, \qquad \{\bar{q}, q\} = 2M \,.$$

NR ABJM

- $\mathcal{N} = 6$ super-Sch. algebra

  1) The additional eight *spectator* supercharges satisfy the following:

$$
\begin{aligned}
&[L_m, q_{a\dot{a}}] = 0\,, \quad [P_r, q_{a\dot{a}}] = 0 = [\bar{P}_r, q_{a\dot{a}}]\,, \\
&\{Q_r, q_{a\dot{a}}\} = 0 = \{\bar{Q}_r, q_{a\dot{a}}\}\,, \quad \{q, q_{a\dot{a}}\} = 0 = \{\bar{q}, q_{a\dot{a}}\}\,, \\
&[J, q_{a\dot{a}}] = +\tfrac{1}{2} q_{a\dot{a}}\,, \quad [R, q_{a\dot{a}}] = 0\,, \\
&[R^a{}_b, q_{c\dot{c}}] = -\delta^\alpha_\gamma q_{b\dot{c}} + \tfrac{1}{2}\delta^a_b q_{c\dot{c}}\,, \quad [R^{\dot{a}}{}_{\dot{b}}, q_{c\dot{c}}] = -\delta^a_c q_{\dot{c}\dot{b}} + \tfrac{1}{2}\delta^{\dot{a}}_{\dot{b}} q_{c\dot{c}}\,, \\
&\left\{\bar{q}^{a\dot{a}}, q_{b\dot{b}}\right\} = \frac{1}{2}\delta^a_b \delta^{\dot{a}}_{\dot{b}} M - \delta^a_b R^{\dot{a}}{}_{\dot{b}} + \delta^{\dot{a}}_{\dot{b}} R^a{}_b\,,
\end{aligned}
$$

  where $R^a{}_b$, $R^{\dot{a}}{}_{\dot{b}}$ are the $SU(2)$ generators defined by

$$
[R^a{}_b, R^c{}_d] = \delta^c_b R^a{}_d - \delta^a_d R^c{}_b\,, \quad (R^a{}_b)^\dagger = R^b{}_a\,.
$$

  2) The $\mathcal{N} = 2$ subalgebra still holds, except generator R replaced by $\tilde{R}(4/3)R - (2/3)\Sigma$. From commutation relations, the shift is needed to make $q_{a\dot{a}}$ neutral under $J - \frac{3}{2}\tilde{R}$, which should hold because $q_{a\dot{a}}$ commutes with Q_r.

NR Geometric duals

- Realising Schrödinger geometrically

1) Embed $Sch(d) \subset O(d+2,2)$ rel. conformal group in $d+2$.
Son;Balasubramanian,McGreevy

2)Analogy: massless KG in $(d+1)+1)$-dim Minkowski spacetime

$$\Box\Phi \equiv -\partial_t^2\Phi + \partial_i^2\Phi = 0, \quad i = 1, \ldots, d+1$$

3) Define LC coords $x^{\pm} = \frac{1}{\sqrt{2}}(t \pm x^{d+1})$. Get

$$2iM\partial_+\Phi + \partial_i^2\Phi = 0$$

Identified $\partial_- = -iM$, $Sch(d)$ with x^+ playing the role of time. Compactification of x^- direction $\rightarrow$ M takes discrete values.

NR Geometric duals

- Realising Schrödinger geometrically

  4) Deformation of AdS metric in Poincaré

$$ds^2 = -2r^z(dx^+)^2 + r^2(2dx^+(dx^- + C) + dx_i dx^i),$$

  5) z dynamical exponent. scaling symmetry
  $(x^+, x^-, x^i, r) \to (\lambda^z x^+, \lambda^{2-z} x^-, \lambda x^i, \lambda^{-1} r)$

  6) Toy model: gravity, massive vector, negative c.c.

$$S = \int d^{d+2}x dr \sqrt{-g}\left(\frac{1}{2}R - \Lambda - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{m^2}{2}A_\mu A^\mu\right)$$

NR Geometric duals

- *Recap* : $Sch(d)$ supergravity solutions

1) Discrete light-cone quantisation (DLCQ). Identifying x^- direction, if p_- sufficiently large, gravity system can be trusted.

2) Generating gravity solutions. 2 approaches i) TsT ii) Consistent truncation. Both seen in work of *Maldacena,Martelli,Tachikawa*.

3) TsT: 2 commuting isometries x^-, φ, T-dualise φ, shift along $x^- \to x^- + \sigma\tilde{\varphi}$, T-dualise back. Simple. $AdS_5 \times SE_5$ breaks susy.

4) Consistent truncations. Truncate higher-dim theory to lower-dim toy model of Son. Gives embedding in string theory (more than just holography). Examples $AdS_4 \times SE_7$ *Gauntlett et al.*; $AdS_5 \times KE_6$ *EÓC,Varela, Yavartanoo*.

5) Other work - *Hartnoll, Yoshida*, *Donos, Gauntlett* SE_5 and CY cones over SE_5.

NR Geometric duals

- Deformation of $AdS_5 \times M_6$ (Ooguri-Park)

  1) Begin with the most general warped $AdS_5 \times M_6$ solutions dual to $\mathcal{N} = 1$; M_6 is topologically $\mathbb{CP}^1$-bundle over base $M_4 \in \{KE_4, \mathcal{C}_1 \times \mathcal{C}_2\}$; Replace AdS_5 with NR metric

$$ds_5^2 = -f(y)\frac{(dx^+)^2}{r^4} + \frac{-2dx^+(dx^- + A) + dx_i dx^i + dr^2}{r^2}.$$

  2) Solve $dG_{(4)} = d * G_{(4)} - \frac{1}{2}G_{(4)}^2 = 0 \Rightarrow J_4 \wedge dA = 0, (dA = - *_4 dA)$. Einstein $\Rightarrow f(y) = \beta y$.

  3) Supersymmetry: Original 4 Poincaré and 4 SC. Get two conditions: $\beta\Gamma^+\epsilon = F^{(2)}\epsilon = 0$. Effect of $\Gamma^+\epsilon = 0$ seen through

$$\gamma^+ r^{-1/2}\psi_0^+ = \gamma^+[r^{1/2} + ir^{-1/2}(x^i\gamma^i - x^+\gamma^- - x^-\gamma^+)]\psi_0^- = 0$$

  $\beta = 0$ 6 susy, otherwise 2.

NR Limit

- BW/LLM and NR limit:

  1) Gravity dual of the ABJM is $AdS_4 \times S^7/\mathbb{Z}_k$; need to carry over the mass deformation and NR limit to the gravity side.

  2) Gravity dual of mass deformed theory is well known - *Bena, Warner* (later LLM, see also *Pope et. al*); BW stack of M2, turn on transverse 4-form flux; in process breaks R-symmetry from $SO(8)$ to $SO(4) \times SO(4)$

  3) Example of Myers dielectric effect where M2 polarised into M5 wrapping S^3.

  4) Approach: Reminiscent of DLCQ (also BMN), but LC momentum must be taken transverse to M2.

NR Limit

- BW/LLM and NR limit:

  5) In principle, one may be able to proceed as follows: modify the BW/LLM solution by adding the particle number M; in IIA, it amounts to turning on the flux counting the D0-brane charge; perform standard coordinate change of the DLCQ procedure:

$$\tilde{\phi} = \phi - \alpha t\,, \quad \tilde{t} = t$$

$$\Rightarrow \qquad \widetilde{H} \equiv i\partial_{\tilde{t}} = i\partial_t - \alpha(-i\partial_\phi) \equiv H - \alpha M\,, \quad \widetilde{M} \equiv -i\partial_{\tilde{\phi}} = -i\partial_\phi \equiv M\,.$$

  6) With suitable constant α and an appropriate scaling limit, the LC Hamiltonian is identified with the Hamiltonian of the NR theory. Unfortunately, hindered by a technical difficulty; not clear how to turn on D0-charge and get back-reacted solution; the $U(1)_B$ circle is fibered non-trivially along the $\mathbb{CP}^3$ base. Need other approach.

Ansatz

- Ansatz

  1) Recall R-symmetry breaking,

$$SO(8) \supset U(1)_B \times SU(4) \supset U(1)_B \times SU(2)_1 \times SU(2)_2 \times U(1)_R \,.$$

2) To see how these R-symmetries are realised geometrically, consider S^7 as a warped product of S^3's,

$$ds^2_{S^7} = d\alpha^2 + \cos^2\alpha \, d\Omega_1^2 + \sin^2\alpha \, d\Omega_2^2 \,.$$

3)Use Euler-angle coordinates (θ, ϕ, ψ) for each S^3:

$$d\Omega_i^2 = \frac{1}{4}\left[d\theta_i^2 + \sin^2\theta_i d\phi_i^2 + (d\psi_i - \cos\theta_i d\phi_i)^2\right] \qquad (i = 1, 2, \text{ no sum}).$$

4) We choose the orientations of the 3-spheres such that the $U(1)_R$ acts diagonally on $\psi_{1,2}$ and the $U(1)_B$ acts with an opposite relative sign.

Ansatz

- Metric

  1) Begin with $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ and imagine mass deformation and then NR limit, preserving R-symmetry and fibration of $U(1)_B$ and $U(1)_R$ angles over the two S^2's; $w = \frac{1}{2}(\psi_1 + \psi_2)$, $v = \frac{1}{2}(\psi_1 - \psi_2)$, $Dw = dw - \frac{1}{2}(\cos\theta_1 d\phi_1 + \cos\theta_2 \phi_2)$, $Dv = dv - \frac{1}{2}(\cos\theta_1 d\phi_1 - \cos\theta_2\phi_2)$.

  2) Led to general ansatz

$$ds^2 = e^{2c_1}\left(-c_2\frac{dt^2}{r^4} + \frac{2dt(Dv + c_3 Dw) + dr^2 + d\vec{x}^2}{r^2} + \frac{4}{9}e^{2h_2}(Dw)^2\right) + e^{-4c_1}\left(e^{-2h_2}dy^2 + \frac{4}{3}e^{2h_1}(e^{+2h_3}d\omega_1^2 + e^{-2h_3}d\omega_2^2)\right).$$

  3) $(c_{1,2,3},\ h_{0,1,2,3})$ depend only on y, which is the only coordinate not constrained by symmetries; numerical factors $4/9$ and $4/3$ for later convenience.

Ansatz

- Orthonormal frame

The metric ansatz admits a natural orthonormal frame,

$$e^{+} = \frac{e^{2c_1}}{r^2} dt \,, \quad e^{-} = -\frac{c_2}{2r^2} dt + Dv + c_3 Dw \,,$$
$$e^1 = \frac{e^{c_1}}{r} dx^1 \,, \quad e^2 = \frac{e^{c_1}}{r} dx^2 \,, \quad e^7 = \tfrac{2}{3} e^{c_1 + h_2} Dw \,, \quad e^8 = \frac{e^{c_1}}{r} dr \,, \quad e^9 = e^{-2c_1 - h_2} dy \,,$$
$$(e^3, e^4 \,;\, e^5, e^6) = \frac{1}{\sqrt{3}} e^{-2c_1 + h_1} \left(e^{+h_3} (\sigma_1, \sigma_2) \,;\, e^{-h_3} (\tau_1, \tau_2) \right) \,.$$

Here, σ_A, τ_A are invariant one forms of S^3's.

Ansatz

- Flux

$$
\begin{aligned}
F \quad = \quad & e^{-3c_1} e^{+8} \left[e^{-2c_1} k_1 e^{12} + e^{4c_1 - 2h_1} (e^{-2h_3} k_{4,1} e^{34} + e^{+2h_3} k_{4,2} e^{56}) \right] \\
& + e^{h_2} e^{+9} \left[e^{-2c_1} k_2 e^{12} + e^{4c_1 - 2h_1} (e^{-2h_3} k_{5,1} e^{34} + e^{+2h_3} k_{5,2} e^{56}) \right] \\
& + e^{c_1} e^{97} \left[e^{-3c_1} k_3 e^{+8} + e^{4c_1 - 2h_1} (e^{-2h_3} k_{6,1} e^{34} + e^{+2h_3} k_{6,2} e^{56}) \right] \\
& + e^{8c_1 - 4h_1} k_7 e^{3456} \, .
\end{aligned}
$$

Earlier compensating factors in metric lead to simple for of Bianchi identity ($dF = 0$)

$$
k_1' + 4k_2 = 0\,, \quad k_{4,1}' + 2k_{5,1} - k_3 = 0\,, \quad k_{4,2}' + 2k_{5,2} - k_3 = 0\,, \quad k_7' - (k_{6,1} + k_{6,2}) = 0\,. \tag{6}
$$

There is a discrete $\mathbb{Z}_2$ symmetry exchanging the two 2-spheres, acts as a parity $y \rightarrow -y$

$$
\begin{aligned}
\text{Even} \quad &: \quad c_1, c_2, h_1, h_2, k_1, (k_{4,1} + k_{4,2}), (k_{5,1} - k_{5,2}), (k_{6,1} + k_{6,2}) \, . \\
\text{Odd} \quad &: \quad c_3, h_3, k_2, k_3, (k_{4,1} - k_{4,2}), (k_{5,1} + k_{5,2}), (k_{6,1} - k_{6,2}), k_7 \, .
\end{aligned}
$$

KSE

- Methods

1) Approach hinges upon two techniques: spinorial Lie derivative, G-structure.

2) Lie derivative of a spinor ϵ w.r.t. a Killing vector K

$$\mathfrak{L}_K \epsilon = K^m \nabla_m \epsilon + \frac{1}{4} (\nabla_a K_b) \Gamma^{ab} \epsilon.$$

spinorial Lie derivative gives a geometric realisation of algebra,

$$[K, Q_1] = Q_2 \quad \Longleftrightarrow \mathfrak{L}_K \epsilon_{Q_1} = \epsilon_{Q_2} .$$

3) From metric, write out all $\mathfrak{L}_K \epsilon$ associated with Killing directions; super Schrödinger algebra determines coordinate dependence of dynamical supercharges $Q(y)$; with Q determined, q and S follow from algebra.

4) From $\{\epsilon_i\}$ can construct differential forms

$$K_{ij} = (\bar{\epsilon}_i \Gamma_a \epsilon_j) e^a, \quad \Omega_{ij} = \frac{1}{2} (\bar{\epsilon}_i \Gamma_{ab} \epsilon_j) e^{ab} \quad \Sigma_{ij} = \frac{1}{5!} (\bar{\epsilon}_i \Gamma_{abcde} \epsilon_j) e^{abcde}$$

KSE

- Killing spinors
  1) Use spinorial Lie derivative. Find from algebra that

$$\mathfrak{L}_H \epsilon_Q = \mathfrak{L}_{P_i} \epsilon_Q = \mathfrak{L}_{V_A} \epsilon_Q = \mathfrak{L}_{V'_A} \epsilon_Q = 0\,, \;\; \mathfrak{L}_D \epsilon_Q = \epsilon_Q \quad \Rightarrow \quad \epsilon_Q = \frac{e^{c_1}}{r} \eta(y)\,,$$

Q singlet under $SU(2) \times SU(2)$ also implies Q independent of "three-sphere" (v, w, θ_i, ϕ_i)

2)Next, use $[G, \bar{Q}] = q$ to get

$$\epsilon_q = \Gamma^+ \left(\frac{\Gamma^1 + i\Gamma^2}{2} \right) \eta^c\,,$$

where η^c denotes charge conjugation. Note $\Gamma^+ \epsilon_q = 0$ automatic.

3) Use $[C, Q] = S$ to get

$$\epsilon_S = \left[\frac{t}{r} e^{c_1} - \frac{1}{2} \Gamma^+ (x_i \Gamma^i + r \Gamma^8) \right] \eta\,.$$

Note all independent of $(v,\ w,\ \theta_i,\ \phi_i)$ coordinates.

KSE

- Methods

  1) Want to ensure ansatz admits 6 supercharges of $\mathcal{N} = 2$ super-Sch : kin $(q, \bar{q})$ [*null*]; dyn $(Q, \bar{Q})$ [*time-like*]; see *Gauntlett et al.* "The geometry of D=11 (null) Killing spinors".

  2) Use null paper results for single (real) null spinor $\epsilon = \frac{1}{2}(q + \bar{q})$; satisfies

$$\Gamma^{3456}\epsilon = -\epsilon, \Gamma^{+}\epsilon = 0$$

  defines $SU(4) \in Spin(7)$ structure.
  3) As we started from ansatz, we require small frame rotation to make explicit the canonical G-structure frame.

KSE

- G-structure
  1) Geometry of Null KS. Orthonormal frame

$$ds^2 = 2e^+e^- + e^ie^i + e^9e^9\,,$$

$K = e^+$
2) Killing spinor to satisfies

$$\Gamma_{1234}\epsilon = \Gamma_{3456}\epsilon = \Gamma_{5678}\epsilon = \Gamma_{1357}\epsilon = -\epsilon\,, \quad \Gamma^+\epsilon = 0\,.$$

with $\Gamma^9\epsilon = \epsilon$ by construction.

3) Spinor defines a Spin(7) structure with $\Omega = e^+ \wedge e^9\,, \quad \Sigma = e^+ \wedge \Phi\,,$

4) Our ansatz have e^+, also $\Gamma^{3456}\epsilon = -\epsilon$ and $\Gamma^+\epsilon = 0$ for kin supercharges. One finds that two frames are related by a y-dependent rotation in (89)-plane.

KSE

- Killing spinor equation
  1) Start from KSE:

$$\delta\psi_m = \nabla_m \epsilon + \frac{1}{12}(\Gamma_m \mathbf{F} - 3\mathbf{F}_m)\epsilon = 0.$$

2) From earlier work

$$\Gamma^+ \epsilon_q = 0\,, \quad \Gamma^{3456}\epsilon_q = -\epsilon_q\,, \quad \partial_m \epsilon_q = 0 \quad (m \neq y)\,.$$

3) Only have 1-form,2-forms,5-forms; by sandwiching deduce Ω has only two non-zero components Ω_{+9}, Ω_{+8}. This identifies frame rotation to go to canonical frame with only $\Omega_{+9'}$ non-zero.

4) Using important result that ϵ' is *constant* in canonical frame $\Rightarrow$ one differential condition becomes algebraic in new frame.

KSE

- Dynamical supercharges
  1) As $\{\bar{Q}, Q\} = H$, have time-like KS. General analysis - "Geometry of time-like KS". Metric takes form

$$ds^2 = -\Delta^2(dt+\omega)^2 + \Delta^{-1} g_{mn} dx^m dx^n.$$

  2) Base manifold has $SU(5)$ structure given by pair of spinors $\epsilon_d \equiv \frac{1}{\sqrt{2}}(\epsilon_{\bar{Q}} + \epsilon_Q)$. Can decompose ϵ by e-values of Γ^{+-}:

$$\epsilon_d = \frac{e^{c_1}}{r}\left(\Gamma^+\eta_1 + \eta_2\right), \quad \epsilon_k = \frac{1}{2}\left(\epsilon_q + \epsilon_{\bar{q}}\right) = \frac{1}{\sqrt{2}}\Gamma^{+1}\eta_2.$$

  3) Proceed as before to determine Ω, before using G-structure equations

$$\begin{aligned} {\Omega_a}^c {\Omega_c}^b &= -K_a K^b + {\delta_a}^b K^2 \text{ where } K = \Delta^2(dt+\omega), \\ d\Omega &= i_K F. \end{aligned}$$

  4) In general KSE do not restrict every component of metric, flux. However, flux independent of x^-, so KSE determine all components.

Solution

- Equations

Block A : The equations for (c_1, h_1, h_2, h_3) decouple from all other variables.

$$4h_1' - h_2' = -c_1'(2h_1' + h_2')^2 e^{6c_1+2h_2}\,, \qquad 9c_1' = (9c_1' - 4h_1' + h_2')e^{2h_2}\,,$$
$$2h_1' + h_2' = 6(h_1' + h_3')e^{-6c_1+2h_1-2h_2+2h_3}\,, \quad h_3'\cosh(2h_3) = -h_1'\sinh(2h_3)\,.$$

The following auxiliary equations will also be useful,

$$\cos\zeta = e^{h_2}\,, \qquad \sin\zeta = -\tfrac{1}{3}(2h_1' + h_2')e^{3c_1+2h_2} = \frac{1}{3c_1'}(-\zeta'\cos\zeta + 2e^{-3c_1})\,.$$

Block B : With the solutions of Block A as an input, we can solve the equations for (c_3, k_1, k_2, k_3).

$$k_2 = -k_3\,, \quad k_1 = -\frac{6c_3}{\sin\zeta}e^{3c_1}\,, \quad 3c_3' = 2\left(k_1 e^{-6c_1}\frac{h_1' - h_2'}{2h_1' + h_2'} - k_3 e^{-3c_1}\sin\zeta\right)$$
$$3c_3' + k_1 e^{-6c_1} = 6\sin\zeta(c_3\cosh(2h_3) - \sinh(2h_3))e^{3c_1-2h_1}\,,$$

Solution

- Equations

Block C : The last metric component c_2 and all the remaining flux components are determined algebraically by the solutions of Block A and Block B.

$$
\begin{aligned}
c_2 &= \left(\tfrac{1}{4} k_1 e^{-3c_1}\right)^2 , \\
k_{4,1} &= -\tfrac{3}{2}(c_3+1)e^{3c_1}\sin\zeta - \tfrac{1}{4}k_1(2e^{-6c_1+2h_1+2h_3} - e^{2h_2}) , \\
k_{4,2} &= -\tfrac{3}{2}(c_3-1)e^{3c_1}\sin\zeta - \tfrac{1}{4}k_1(2e^{-6c_1+2h_1-2h_3} - e^{2h_2}) , \\
k_{5,1} &= -\tfrac{3}{2}(c_3-1)e^{+4h_2} , \\
k_{5,2} &= -\tfrac{3}{2}(c_3+1)e^{-4h_2} , \\
k_{6,1} &= -\frac{h_1' + 2h_2' + 3h_3'}{3(h_1' + h_3')} e^{2h_2} , \\
k_{6,2} &= -\frac{h_1' + 2h_2' - 3h_3'}{3(h_1' - h_3')} e^{2h_2} , \\
k_7 &= 6c_1' e^{-6c_1+4h_1} .
\end{aligned}
$$

Solution

- Solution

The final form of the solution may be most neatly captured in terms of two quadratic polynomials,

$$g_1 = 1 - y^2\,, \quad g_2 = 1 + \tfrac{1}{2} c y + y^2\,.$$

The metric components are

$$\begin{aligned} e^{6c_1} &= g_1^2 g_2^{-1}\,, & c_2 &= b^2 g_1^{-2} g_2^{-1}\,, & c_3 &= \tfrac{4}{3} b y g_1^{-2}\,, \\ e^{2h_1} &= g_1\,, & e^{2h_2} &= 1 - 4y^2 e^{-6c_1}\,, & e^{2h_3} &= 1\,, \end{aligned}$$

and the flux components are

$$\begin{aligned} k_1 &= -4 b g_2^{-1}\,, \quad k_2 = -b g_2' g_2^{-2}\,, & k_3 &= b g_2' g_2^{-2}\,, \\ k_{4,1} &= -3y + b(2 g_1^{-1} - g_2^{-1}) & k_{4,2} &= +3y + b(2 g_1^{-1} - g_2^{-1})\,, \\ k_{5,1} &= +\tfrac{3}{2} - 2 y b g_1^{-2} & k_{5,2} &= -\tfrac{3}{2} - 2 y b g_1^{-2}\,, \\ k_{6,1} &= k_{6,2} = 1 - 4 g_2 g_1^{-2}\,, & k_7 &= -4 g_2' g_1^{-1} + 2 g_1' + 3 g_2'\,. \end{aligned}$$

Discussion

- Discussion

1) Solution is a one-parameter generalisation of OP solution ($c_3 = 0$).

2) We have succeeded in finding a geometric realisation of $\mathcal{N} = 2$ super-Sch algebra.

3) However, we cannot realise additional 8 "spectator" charges. From algebra

$$\{\bar{q}^{a\dot{a}}, q_{b\dot{b}}\} = \frac{1}{2}\delta^a_b \delta^{\dot{a}}_{\dot{b}} M - \delta^a_b {R^{\dot{a}}}_{\dot{b}} + \delta^{\dot{a}}_{\dot{b}} {R^a}_b \,,$$

but only $\bar{\epsilon}^{a\dot{a}}\Gamma^{-}\epsilon_{b\dot{b}} \neq 0$, so $SU(2) \times SU(2)$ generators cannot be produced.

4) Recent study of susy vacua of mass deformed theory *Kim*^{2}: vacuum dynamically breaks susy unless $N \leq k$. Signals holographic dual should be highly stringy as the 't Hooft coupling is small $\frac{N}{k} \leq 1$. Also (*Rey,Nakayama*)