Efficient descriptions of many-body systems using tensor network states

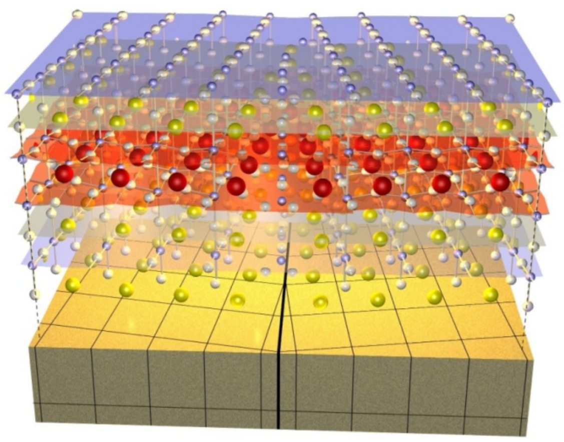
MPQ

Max-Planck-Institut für Quantenoptik

Maynooth University, October 14th, 2014

MANY-BODY QUANTUM SYSTEM

PHYSICAL SYSTEM

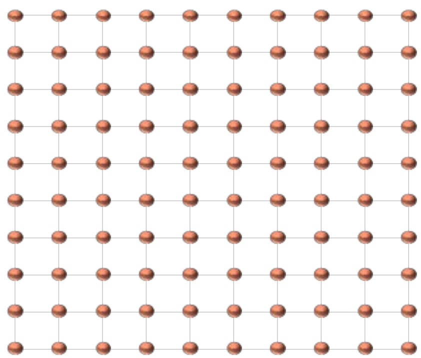


- Dynamics
- Thermal equilibrium (T)

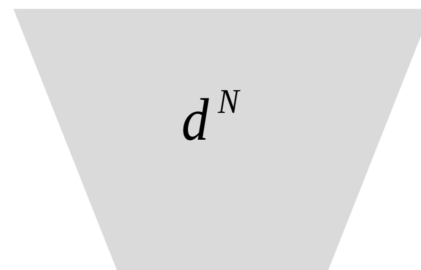
Computation time/memory scales exponentially with the number of constituents

MANY-BODY QUANTUM SYSTEM

MODEL



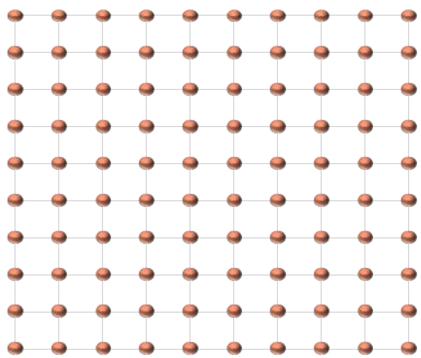
HILBERT SPACE



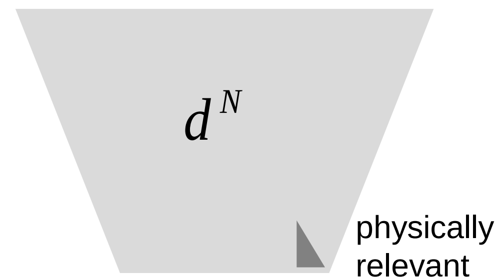
$$c_1\,|00..0\rangle + c_2\,|00..1\rangle + .. + c_{2^N}\,|11..1\rangle$$

MANY-BODY QUANTUM SYSTEM

MODEL



HILBERT SPACE



$$c_1 \,|00..0\rangle + c_2 \,|00..1\rangle + .. + c_{2^N} \,|11..1\rangle$$

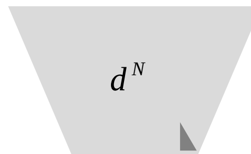
HOWEVER:

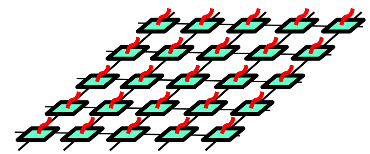
- Interactions in nature are very special:
  few-body Hamiltonians, local
- Physical states occupy a small corner of Hilbert space
- Tensor networks: efficient description of that corner

PROJECTED ENTANGLED-PAIR STATES

- PEPS provide efficient descriptions of states
  with local interactions and in thermal equilibrium

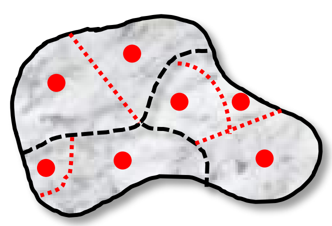
$$D = poly(N)$$





A. Molnar, N. Schuch, F. Verstraete, IC, arXiv:1406.2973

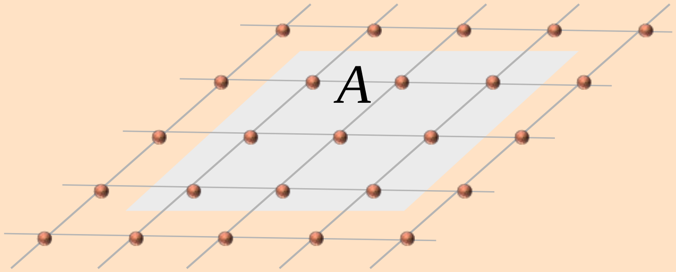
- PEPS provide simple descriptions of complex many-body states



OUTLINE

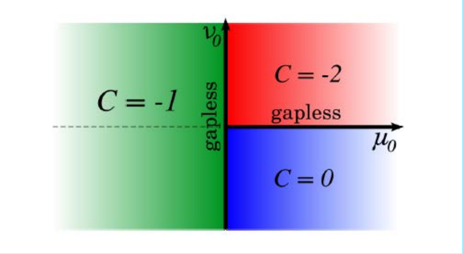
- Bulk-boundary correspondence in lattice systems at T=0:

  T. Walh, H.H. Tu, S. Yang (MPQ),
  N. Schuch (Aachen), D. Perez-Garcia (Madrid), D. Poilblanc (Tolouse)+ F. Verstraete (Vienna)



- Chiral topological models:

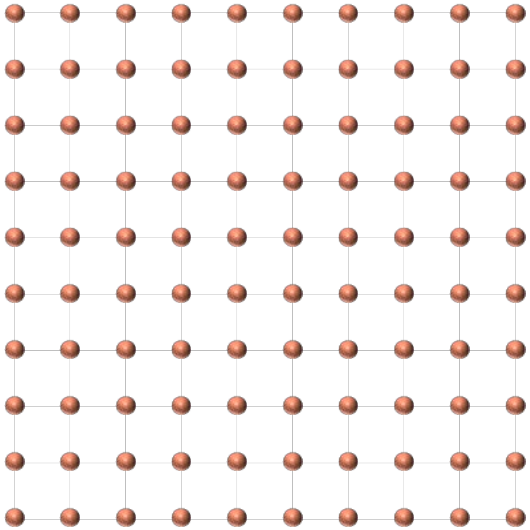
  T. Wahl, S. Yang, H.H. Tu (MPQ), N. Schuch (Aachen)



PROJECTED ENTANGLED PAIR STATES (PEPS)

SPIN LATTICES

- Spins on a lattice in 2D at zero temperature:

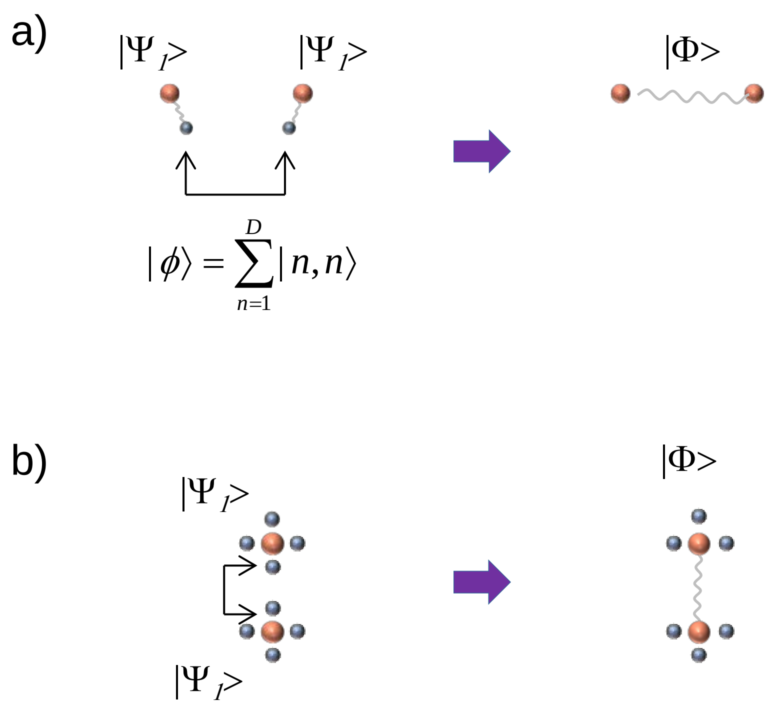


- Many-body state: $|\Psi\rangle$
- Parent Hamiltonian (local)

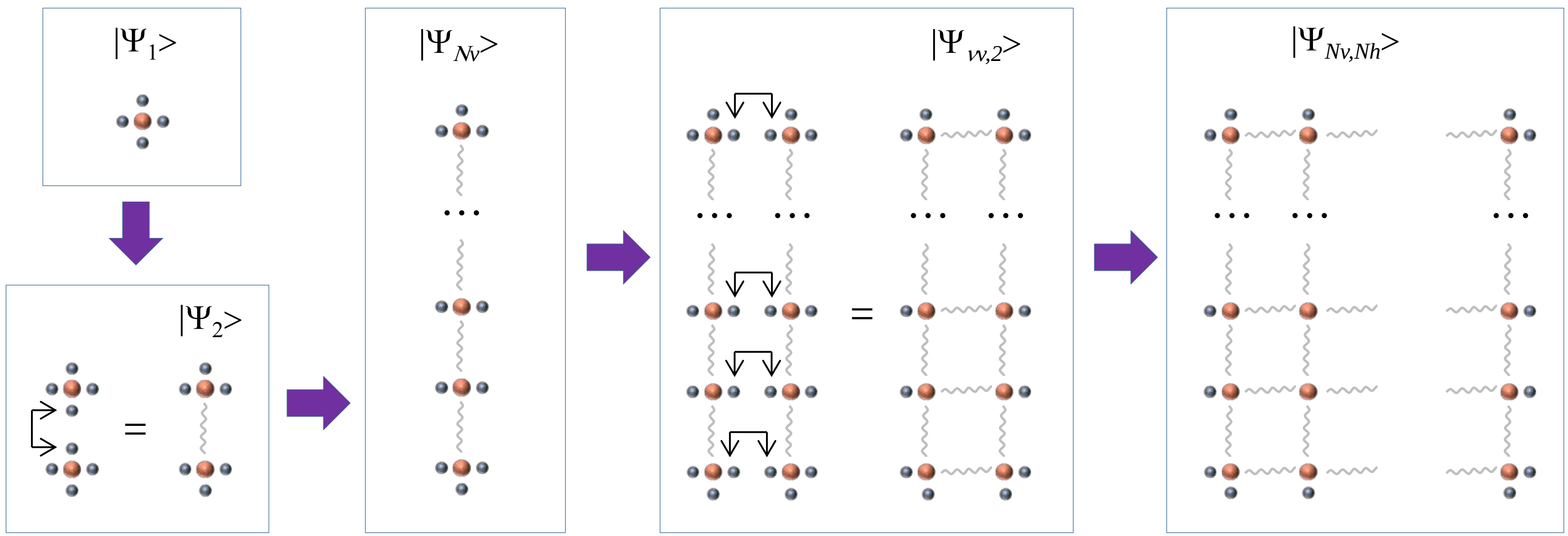
$$H|\Psi\rangle = E_0|\Psi\rangle$$

PROJECTED ENTANGLED-PAIR STATES

- Entanglement swapping

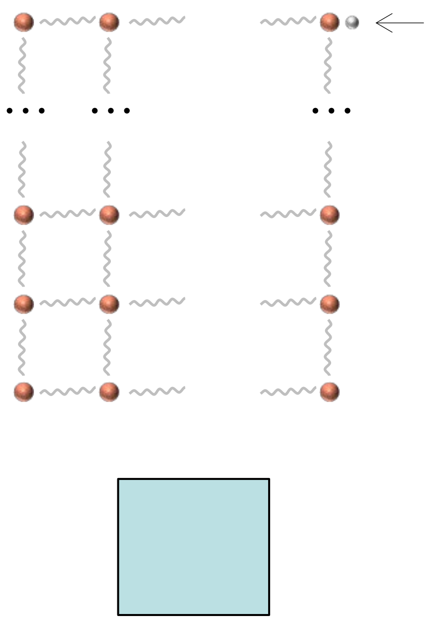


PROJECTED ENTANGLED-PAIR STATES

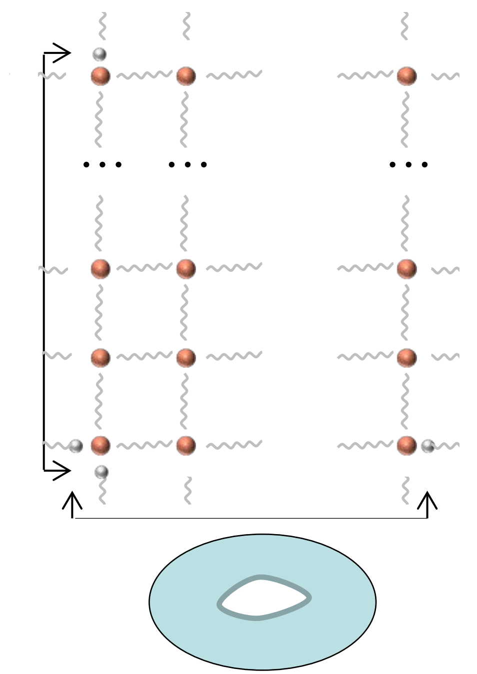


PROJECTED ENTANGLED-PAIR STATES

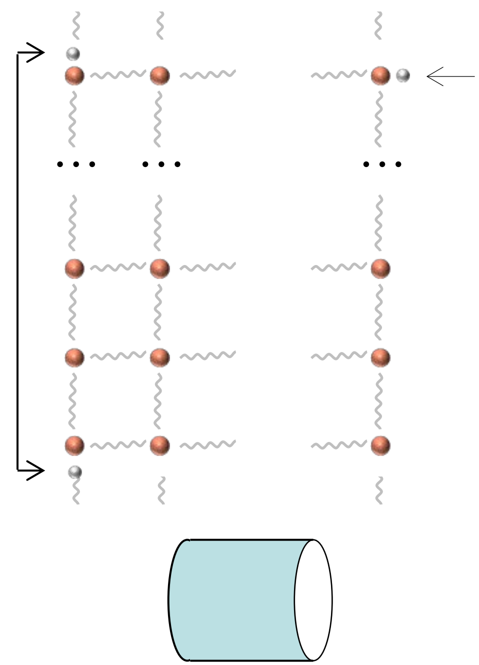
PLANE



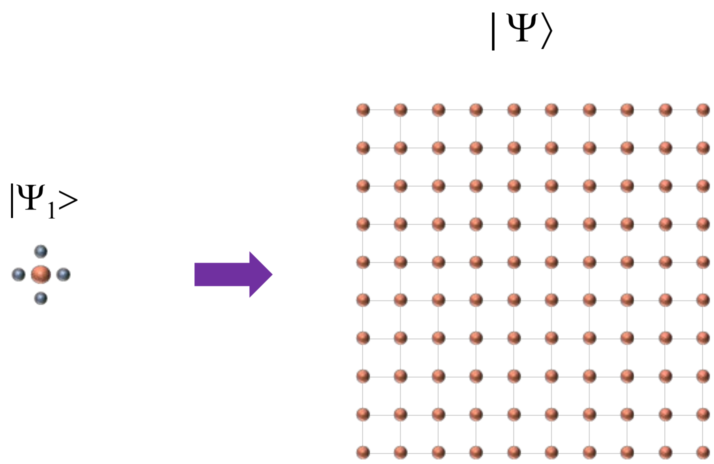
TORUS



CYLINDER



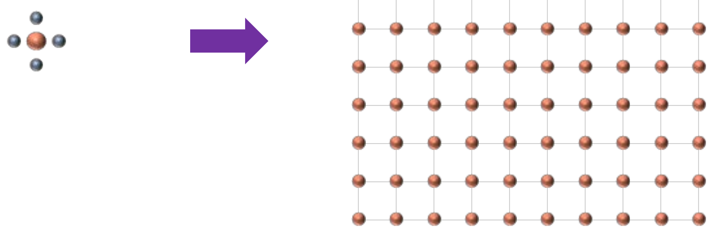
PROJECTED-ENTANGLED PAIR STATES



„PARENT“ HAMILTONIANS

$|\Psi\rangle$

$|\Psi_1>$

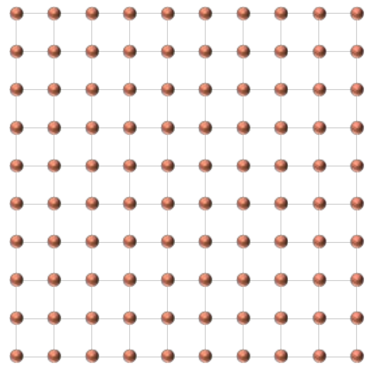


$$H\,|\Psi\rangle = E_0\,|\Psi\rangle$$

- Local: $H = \sum_n h_n$
- Frustration-free: $h_n\,|\Psi\rangle = 0$
- Degeneracy: g

PROJECTED-ENTANGLED PAIR STATES

$|\Psi\rangle$

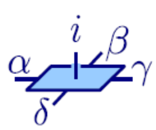


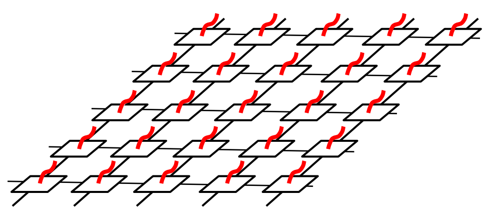
$|\Psi_1>$

$$|\Psi_1\rangle = \sum A^i_{\alpha\beta\gamma\delta} \, |i;\alpha,\beta,\chi,\delta\rangle$$

Tensor network

$A^i_{\alpha\beta\gamma\delta}$



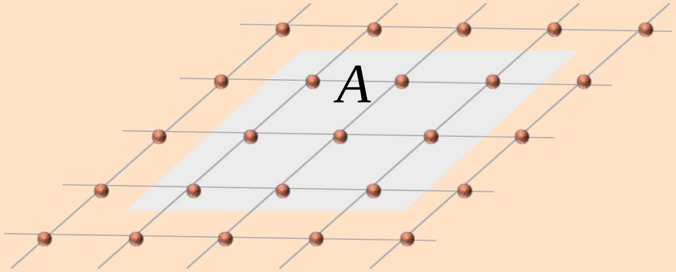


Easy to handle

PEPS give a natural playground to investigate many-body systems

- Bulk-boundary correspondence in lattice systems at T=0:

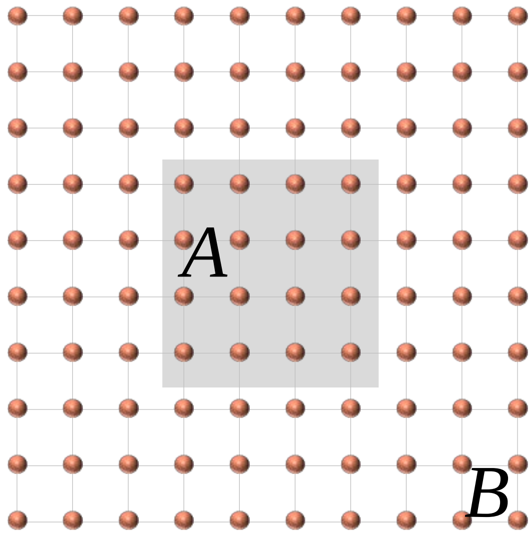
T. Walh, H.H. Tu, S. Yang (MPQ),
N. Schuch (Aachen), D. Perez-Garcia (Madrid), D. Poilblanc (Tolouse)+ F. Verstraete (Vienna)



Related work: Dubail, Read, Rezayi
Qi, Katsura, and Ludwig
Chen, Gu, Wen

SPIN LATTICES

- Spins on a lattice in 2D at zero temperature:



- Many-body state: $|\Psi\rangle$
- Parent Hamiltonian (local)

$$H|\Psi\rangle = E_0|\Psi\rangle$$

- Reduced state in region A:

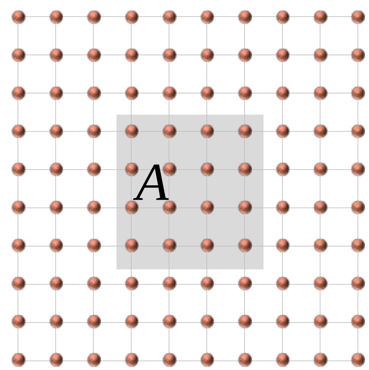
$$\rho_A = \mathrm{tr}_B\left[|\Psi\rangle\langle\Psi|\right]$$

SPIN LATTICES

- Area law: (Srednicky 93):

$$S(\rho_A) \prec N_{\partial A}$$

degrees of freedom $\prec$ # particles at boundary



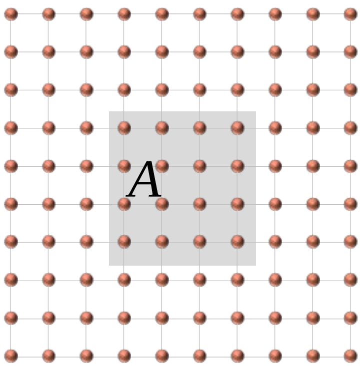
$$\rho_A = \mathrm{tr}\left[|\Psi\rangle\langle\Psi| \right]$$

- Entanglement spectrum: (Li and Haldane, 2008; Peschel, Kitaev and Preskill):

$$\rho_A = e^{-H_A} \qquad\qquad \sigma(H_A)$$

The low energy sector has the same structure as that for a lower dimensional theory (edge states)

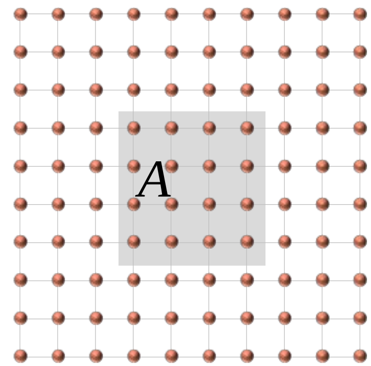
SPIN LATTICES



$$\rho_A = \mathrm{tr}\left[|\Psi\rangle\langle\Psi|\right]$$

- THE (REDUCED) STATE OF THE BULK CAN BE DESCRIBED BY A LOWER DIMENSIONAL THEORY
- THAT THEORY IS SOMEHOW RELATED TO THE BOUNDARY OF THE REGION

SPIN LATTICES

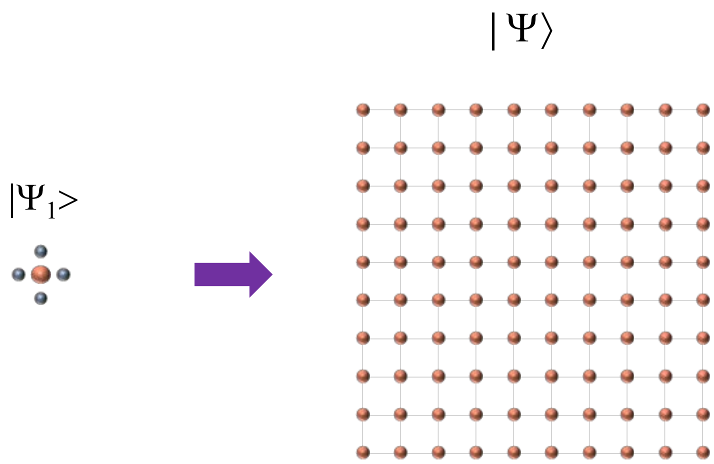


$$\rho_A = \mathrm{tr}\left[|\Psi\rangle\langle\Psi| \right]$$

QUESTIONS:

- What is that theory? Where does it act?
- Is the Hamiltonian H_A local?
- What are the symmetries of H_A, and how are they related to those of Ψ?
- How does the topological character of Ψ manisfest itself?
- What happens at quantum phase transitions?
- Is there any relation to a dynamical Hamiltonian?
- What is the relation to chiral edges for topological insulators?

PROJECTED-ENTANGLED PAIR STATES



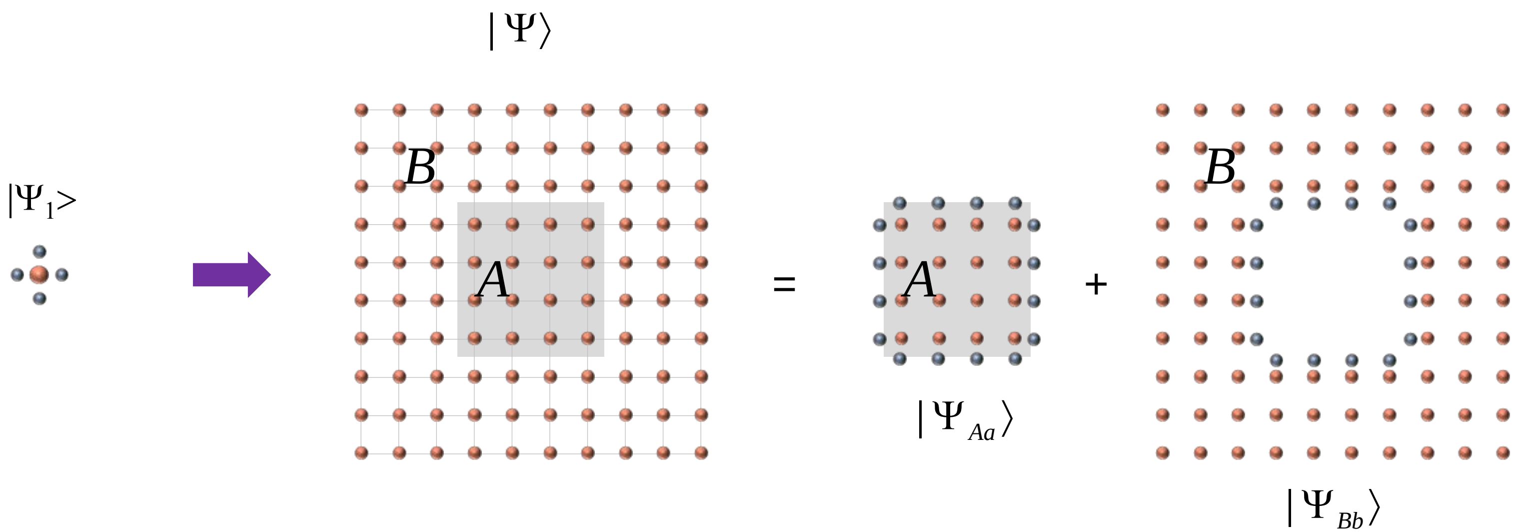
- Approximate well ground states (of gapped phases)
- Fulfill the area law
- There exist numerical techniques

PEPS give a natural playground to investigate this subject

PROJECTED ENTANGLED-PAIR STATES

BULK-BOUNDARY CORRESPONDENCE

IC, Poilblanc, Schuch, and Verstraete, PRB 83, 245134 (2011)



$$\sigma_a = tr_A\left(|\Psi_{Aa}\rangle\langle\Psi_{Aa}|\right)$$

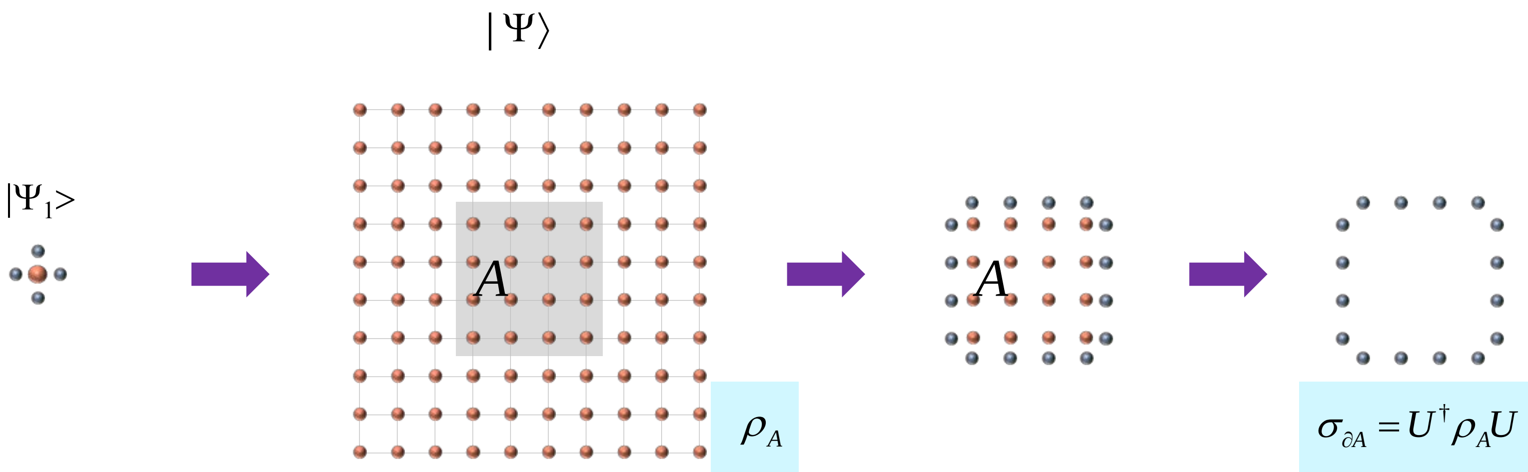
$$\sigma_b = tr_B\left(|\Psi_{Bb}\rangle\langle\Psi_{Bb}|\right)$$

$$\sigma_{\partial A} = \sqrt{\sigma_a^T}\,\sigma_b\sqrt{\sigma_a^T}$$

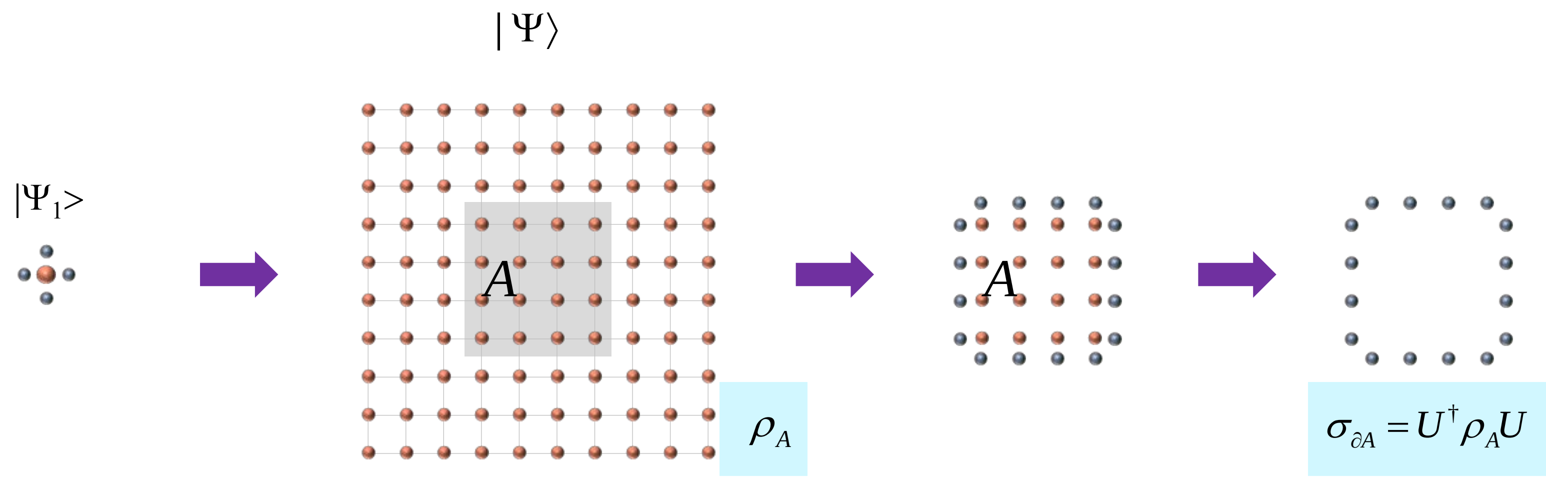
$$\rho_A = tr_B\left(|\Psi\rangle\langle\Psi|\right) = U\sigma_{\partial A}U^\dagger$$

PROJECTED ENTANGLED-PAIR STATES

BULK-BOUNDARY CORRESPONDENCE



PROJECTED ENTANGLED-PAIR STATES
BULK-BOUNDARY CORRESPONDENCE



- The theory corresponds to the auxiliary particles living in the boundary
- Isommetry between the spins in the bulk and the auxiliary ones in the boundary

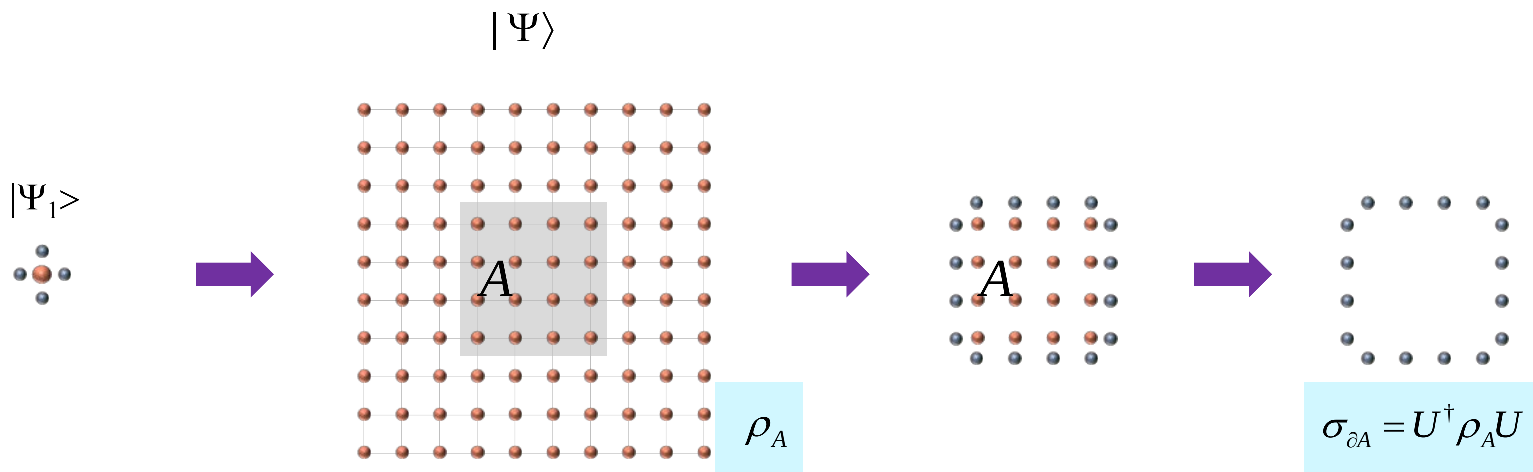
$$\sigma_{\partial A} = U^{\dagger}\rho_A U$$

↑ isommetry

- It „compresses“ the degrees of freedom
- Implies area law
- Allows to determine expectation values in the boundary

$$x_{\partial A} = U^{\dagger} X_A U \quad \Rightarrow \quad \mathrm{tr}(\sigma_{\partial A} x_{\partial A}) = \mathrm{tr}(\rho_A X_A)$$

PROJECTED ENTANGLED-PAIR STATES
BULK-BOUNDARY CORRESPONDENCE



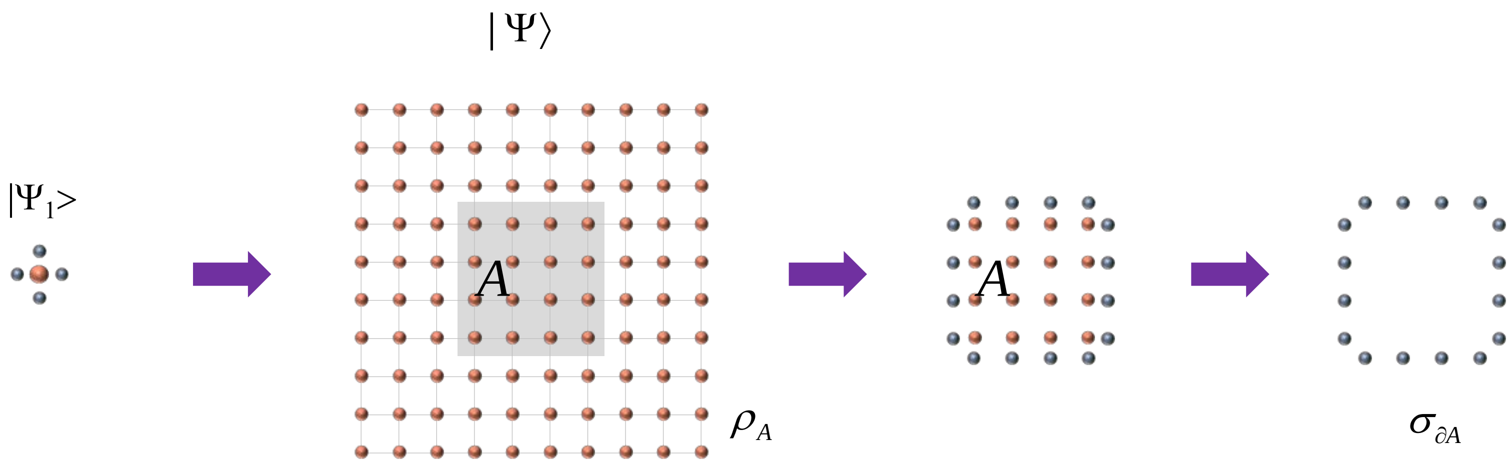
- The theory corresponds to the auxiliary particles living in the boundary
- Isommetry between the spins in the bulk and the auxiliary ones in the boundary

BOUNDARY HAMILTONIAN

$$\sigma_{\partial A} = e^{-H_{\partial A}}$$

- Has the same entanglement spectrum $\sigma(H_{\partial A}) = \sigma(H_A)$
- It can be easily determined (exactly or approximately)

BULK-BOUNDARY CORRESPONDENCE



$$\sigma_{\partial A} = e^{-H_{\partial A}}$$

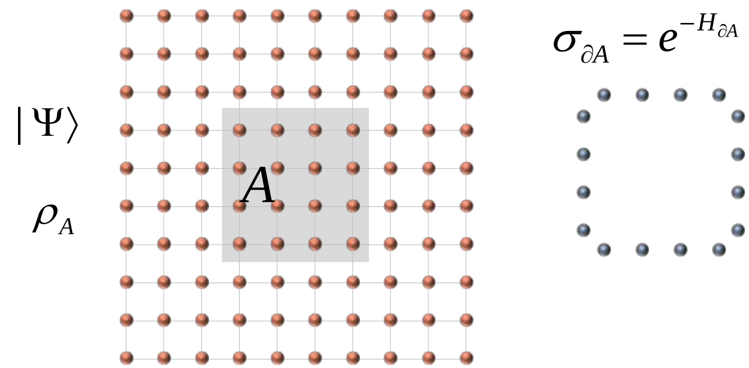
What can we say starting from the boundary Hamiltonian?
(beyond the entanglement spectrum)

Other approaches: Qi, Katsura, and Ludwig, 2012, Dubail, Read, and Rezayi, 2012

PROJECTED ENTANGLED-PAIR STATES

BULK-BOUNDARY CORRESPONDENCE

- Results:



- Symmetries: The boundary Hamiltonian inherits the symmetries

$$u_g |\Psi\rangle = e^{i\theta_g} |\Psi\rangle \quad \Rightarrow \quad U_g H_{\partial A} U_g^\dagger = H_{\partial A}$$

- Locality:

  - For gapped systems, it is local
  - For critical systems, it becomes non-local

- Quantum phase transitions:

  -They are reflected in the boundary Hamiltonian

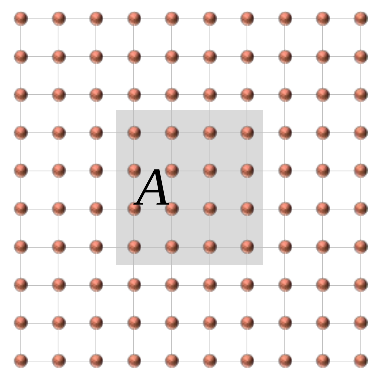
BULK-BOUNDARY CORRESPONDENCE
TOPOLOGICAL PHASES

Schuch, Poilblanc, IC, Perez-Garcia, PRL **111**, 090501 (2013)

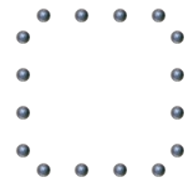
Gapped topological phases in 2D

$|\Psi\rangle$

ρ_A



$\sigma_{\partial A} = e^{-H_{\partial A}}$



PROPERTIES

- Boundary state
- Boundary Hamiltonian

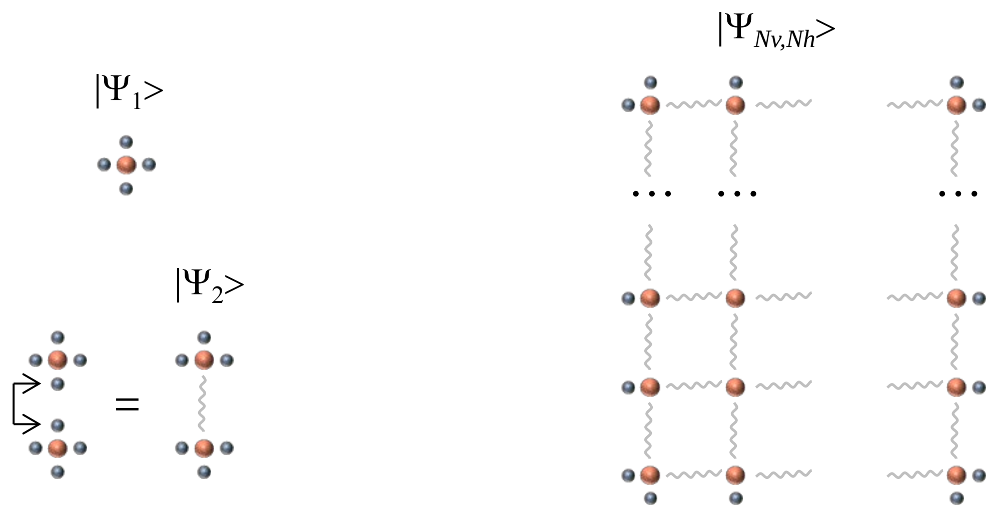
EXAMPLES

- Toric code (Kitaev)
- RVB states
- Phase transitions

BULK-BOUNDARY CORRESPONDENCE

TOPOLOGICAL PHASES

Topological properties are reflected in symmetries of the virtual particles:



$$v_g\,|\Psi_1\rangle = |\Psi_1\rangle \quad \Rightarrow \quad v'_g\,|\Psi_2\rangle = |\Psi_2\rangle \quad \Rightarrow \quad \ldots \quad \Rightarrow \quad u_g\,|\Psi_{Nv,Nh}\rangle = |\Psi_{Nv,Nh}\rangle$$

$$g \in G \qquad\qquad g \in G$$

- Results:
  - The boundary theory develops an extra symmetry

$$\sigma_{\partial A} = U_g \sigma_{\partial A} = \sigma_{\partial A} U_g^\dagger$$

- In general, the boundary operator is block diagonal $\sigma_{\partial A} = \sigma_{\partial A}^1 \oplus \sigma_{\partial A}^2 \oplus \ldots$
- The projector, P_i, on each subspace is higly non-local

  - The boundary Hamiltonian splits

$$H_{\partial A} = H_{\partial A}^{\text{topo}} + H_{\partial A}^{\text{non-universal}}$$

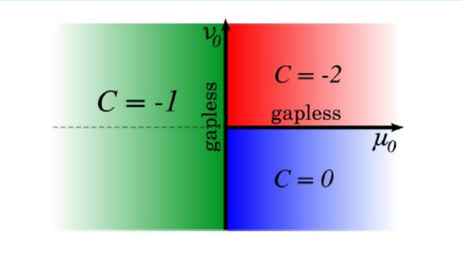
- $H_{\partial A}^{\text{topo}}$ is universal (only depends on the boundary conditions): $H_{\partial A}^{\text{topo}} = \oplus c_i P_i$
- $H_{\partial A}^{\text{non-universal}}$ is local and depends on the details of the state (but not on the boundary conditions)

  - Phase transition

- $H_{\partial A}^{\text{non-universal}}$ becomes non-local
- It can eventually compensate the universal part $H_{\partial A}^{\text{topo}}$

- Chiral topological models:

  T. Wahl, S. Yang, H.H. Tu (MPQ), N. Schuch (Aachen)

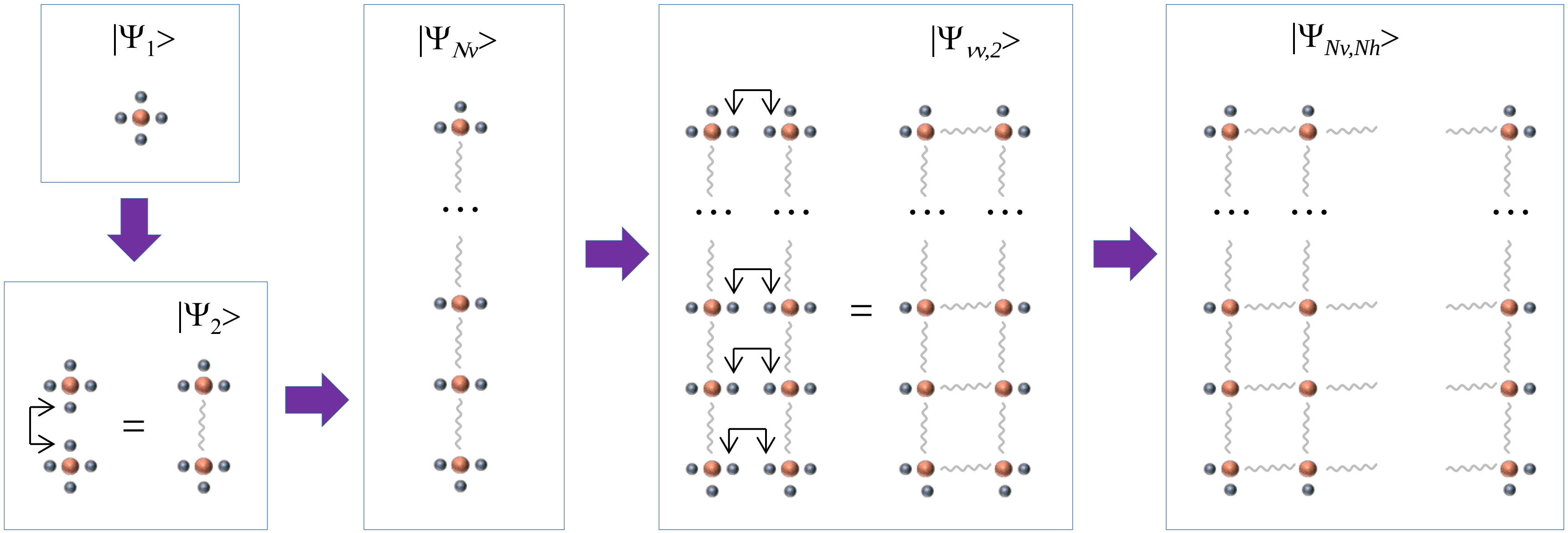


See also:

Dubail and Read

FERMIONS

Kraus,Schuch,Verstraete,IC, Phys. Rev. A 81, 052338 (2010)



Fermionic mode

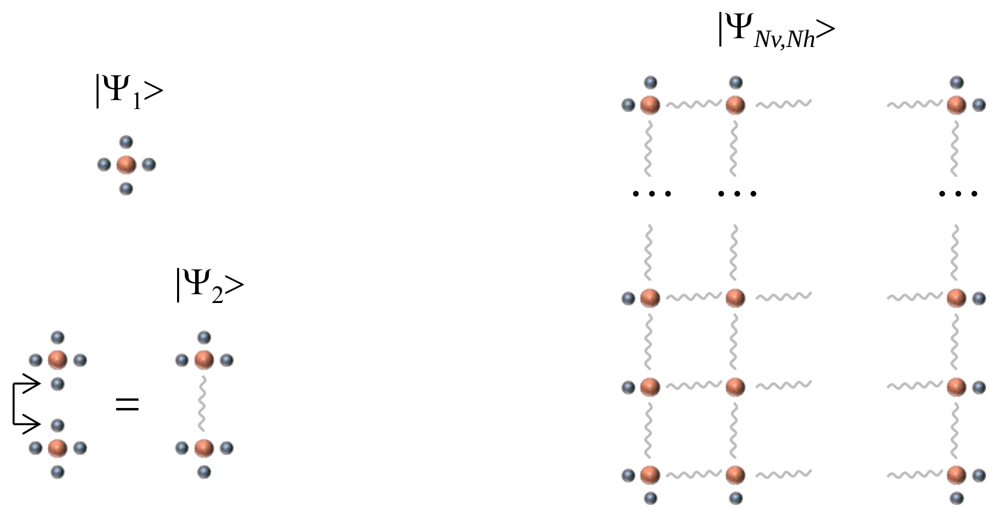
Majorana

$$|\phi\rangle = (1 + icd)\,|\Omega\rangle$$

$$\langle \phi \,|\, \Psi\rangle = |\Psi'\rangle$$

„GAUGE“ SYMMETRIES
STRINGS

Topological properties are reflected in symmetries of the virtual particles:

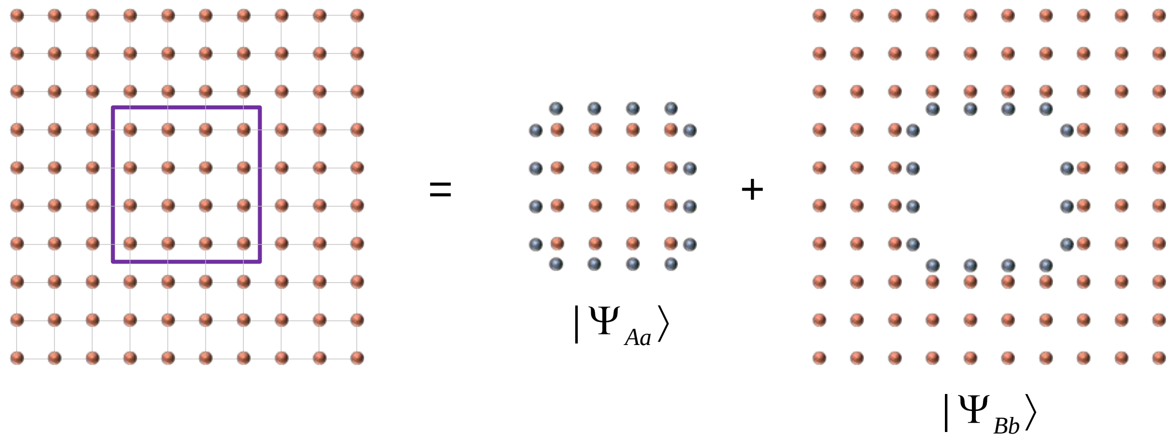


$S_1 | \Psi_1 \rangle = 0$ ➡ $S_2 | \Psi_2 \rangle = 0$ ➡ … ➡ $S_N | \Psi_{Nv,Nh} \rangle = 0$

„GAUGE“ SYMMETRIES

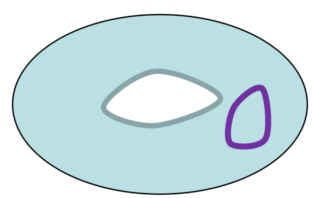
STRINGS

$|\Psi_{\square}\rangle$



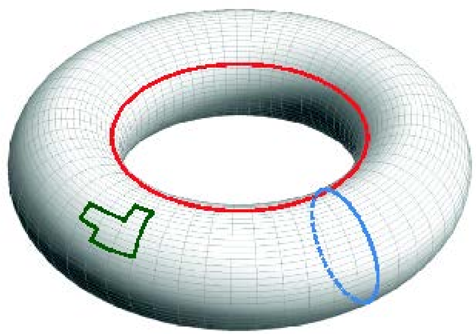
$$|\Psi_{\square}\rangle = \langle \phi_{ab} | S | \Psi_{Aa} \rangle | \Psi_{Bb} \rangle$$

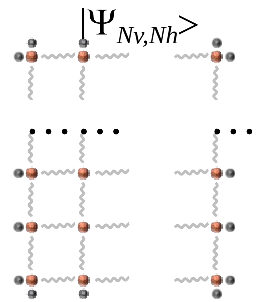
$$S = \sum x_{n,\alpha} c_{n,\alpha}$$



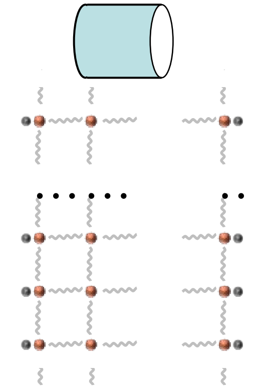
„GAUGE“ SYMMETRIES
STRINGS

$$S\,|\Psi_{N_h,N_v}\rangle = \sum x_{n\alpha} c_{n,\alpha}\,|\Psi_{N_h,N_v}\rangle = 0$$





$\sum x_{n\alpha} c_{n,\alpha}\,|\Psi_{N_h,N_v}\rangle = 0$



$\left[\sum x_{n,\alpha} c^{L}_{n,\alpha} + \sum y_{n,\alpha} c^{R}_{n,\alpha}\right]|\Phi_{N_v}\rangle = 0$

- States with strings along contractible regions vanish
- Strings wrapping up the cylinder can be deformed and moved

Strings in topological models: degeneracy, anyons, brading,

N. Schuch, IC, D. Perez-Garcia, Annals of Physics 325, 2153 (2010)

CHIRAL FERMIONIC QUASI-FREE PEPS

CASE STUDY 1
GAUSSIAN STATE

- Fiducial state:

$$|\Psi_1\rangle = \left(1 + a^\dagger b^\dagger\right)|\Omega\rangle$$

$$b = (c_L - ic_R)e^{i\pi/4} + c_U - ic_D$$

Is a topological superconductor (class D, $p+ip$)

Schnyder et al, Kitaev, Altland and Zirnbauer

- „Gauge“ symmetry:

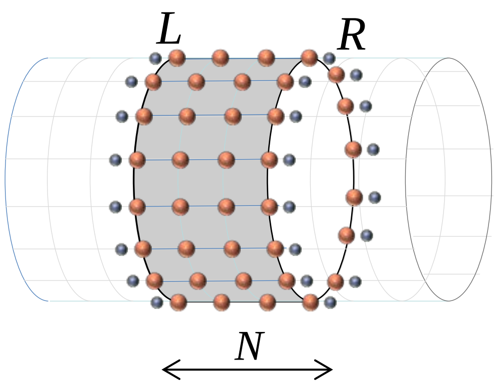
$$d\,|\Psi_1\rangle = 0$$

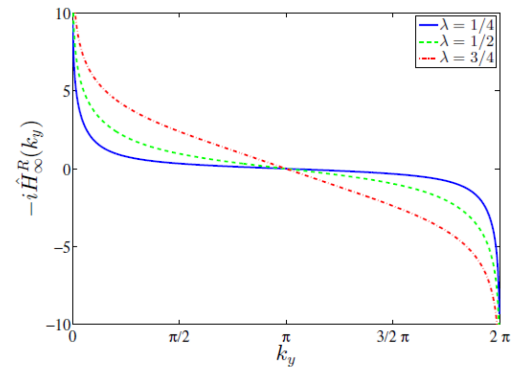
$$d = (c_L - ic_R)e^{i\pi/4} + c_U - ic_D$$

CASE STUDY 1
GAUSSIAN STATE

BOUNDARY THEORY

$$H^{\mathrm{b}}_{\infty} = \bigoplus_{k_y \neq 0,\pi} \left(\hat{H}^{L}_{\infty}(k_y) \oplus \hat{H}^{R}_{\infty}(k_y) \right) \oplus \hat{H}^{LR}_{\infty}(0) \oplus \hat{H}^{LR}_{\infty}(\pi)$$





- Chiral modes
- Right boundary is entangled to the left boundary
- Zeroth Renyi entropy: topological correction:

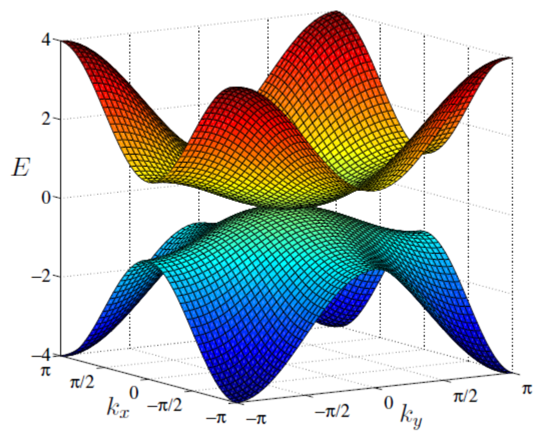
$$S_0(N_v) = aN_v - \log(2)$$

- This is a consequence of the symmetry

$$\left[\sum x_{n,\alpha} c^{L}_{n,\alpha} + \sum y_{n,\alpha} c^{R}_{n,\alpha}\right] |\Phi_{N_v}\rangle = 0 \quad \Rightarrow \quad d\sigma_{LR} = \sigma_{LR} d = 0$$

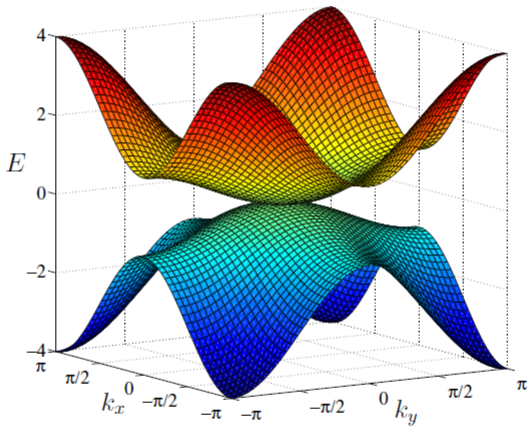
CASE STUDY 1
GAUSSIAN STATE

PARENT HAMILTONIAN: LOCAL



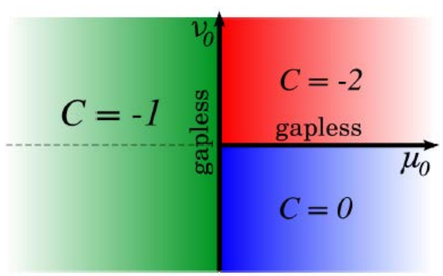
- Degenerate on the torus
- Continuous spectrum (therm. Limit)
- Power-law correlation functions

PARENT HAMILTONIAN: LOCAL



- Degenerate on the torus
- Continuous spectrum (therm. Limit)
- Power-law correlation functions

- It is at a phase transition



CASE STUDY 1
GAUSSIAN STATE

PARENT HAMILTONIAN: GAPPED

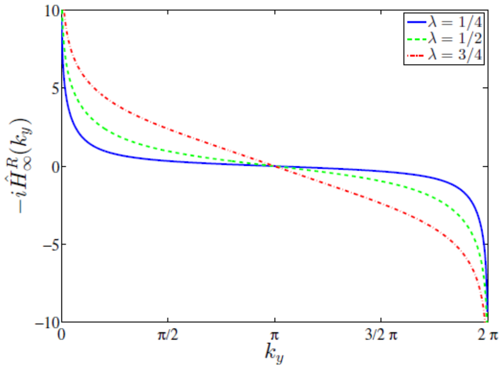
- Flat band Hamiltonian
- Long-range hoppings

$$h_{n,m} \approx 1/|n-m|^3$$

- Robust against local perturbations
- Chern number

$$C = \frac{1}{4\pi} \int_{\mathrm{BZ}} d^2k\, \hat{\mathbf{d}}(\mathbf{k}) \cdot \left(\frac{\partial \hat{\mathbf{d}}(\mathbf{k})}{\partial k_x} \times \frac{\partial \hat{\mathbf{d}}(\mathbf{k})}{\partial k_y} \right)$$

It agrees with the boundary theory



SUMMARY

- Family of FGPEPS
- Smallest bond dimension: one majorana „mode“ per bond
- Chiral:
  - Chiral edge modes
  - Gapped Parent Hamiltonian ($1/r^3$ hopping)
  - Robust against perturbations
  - Chern superconductor
  - c=1/2 and symmetry class D (like $p+ip$ superconductor)
- Topological at a phase transition:
  - Gapless local Hamiltonian
  - Degeneracy depends on topology
  - String operators describe the states
  - Left and right boundaries are entangled

CHIRAL FERMIONIC INTERACTING PEPS

CASE STUDY 2
INTERACTING CHIRAL STATES

IDEA

H.H. Tu, Phys. Rev. B 87, 041103(R) (2013)

$$|\Psi\rangle = \prod_n \mathbf{P}_n^{Gutz} |\Phi,\Phi\rangle$$

← $p+ip$ superconductor

← Gutzwiller projector

➡ Top QFT $SO(2)_1$

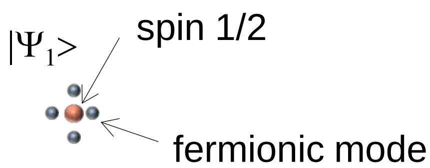
Four primary fields with $h=0,1/8,1/8,1/2$

- Replace $p+ip$ by the chiral FGPEPS:
  - Two copies: two Majorana =1 Fermion mode per bound
  - The Gutzwiller projector does not change the bond dimension

CASE STUDY 2
INTERACTING CHIRAL STATES

SYMMETRIES

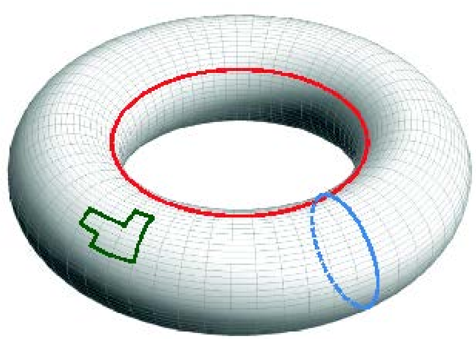
- The state develops a new „gauge“ symmetry:



$$(-1)^{\sum_\alpha x_\alpha^\dagger x_\alpha} |\Psi_1\rangle = |\Psi_1\rangle$$

- The symmetry is inherited for larger regions:

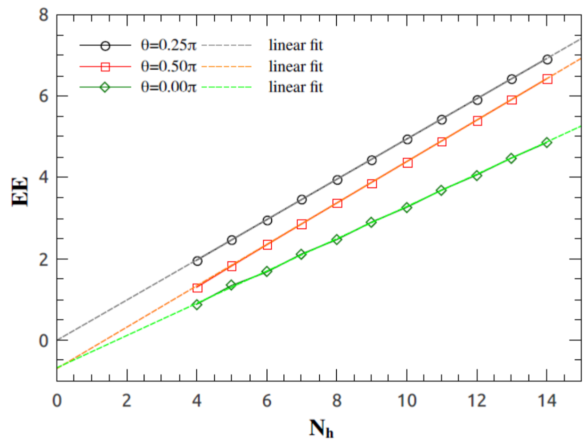
It corresponds to a flux, akin to the toric code.



CASE STUDY 2
INTERACTING CHIRAL STATES

BOUNDARY THEORY

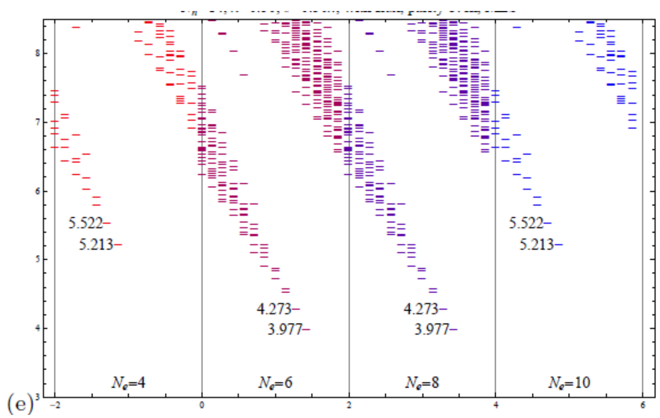
- Entanglement Entropy:



CASE STUDY 2
INTERACTING CHIRAL STATES

BOUNDARY THEORY

- Entanglement spectrum:

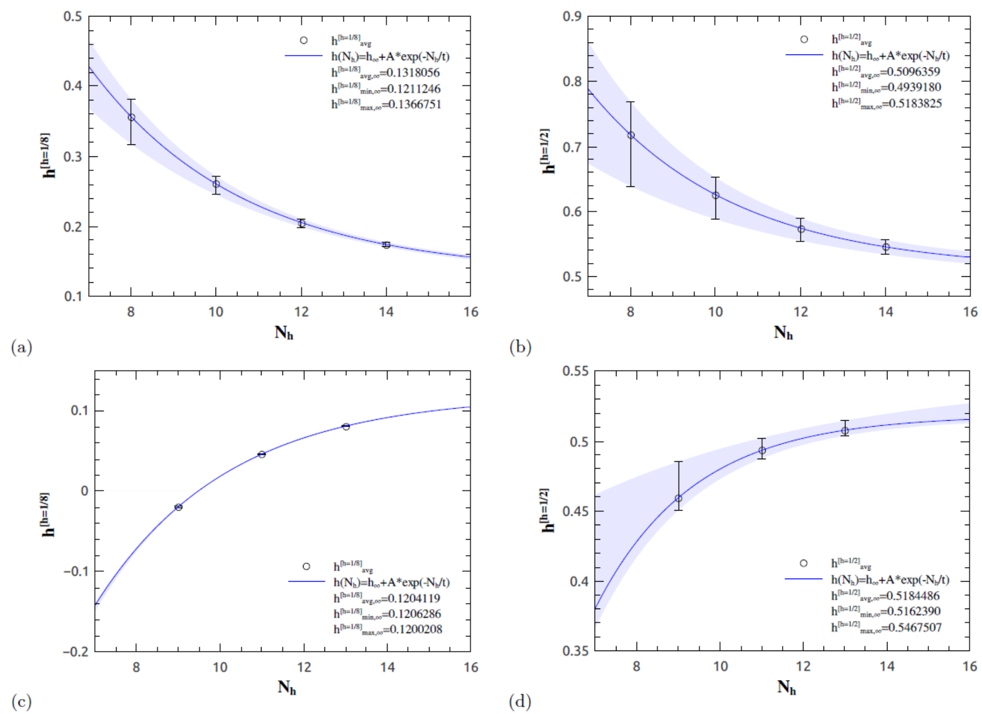


- There are four sectors (MES)
- Degeneracy according to $SO(2)_1$

CASE STUDY 2
INTERACTING CHIRAL STATES

BOUNDARY THEORY

- Entanglement spectrum:

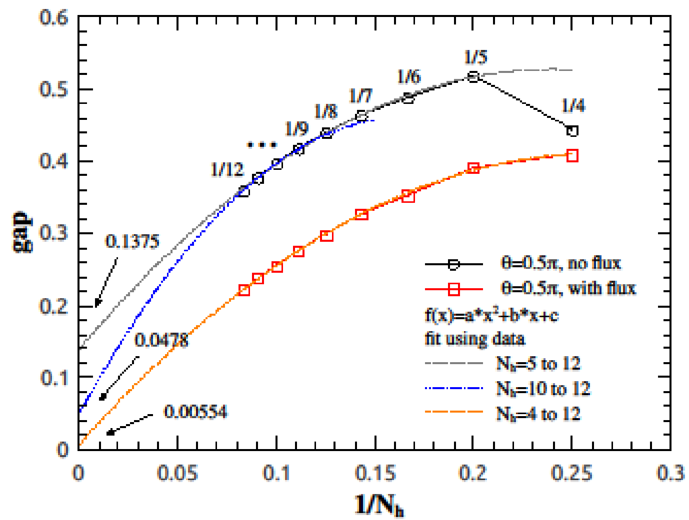


Consistent with $h=0,1/8,1/8,1/2$

CASE STUDY 2
INTERACTING CHIRAL STATES

TRANSFER MATRIX

- Gap in the transfer matrix gives correlation length:

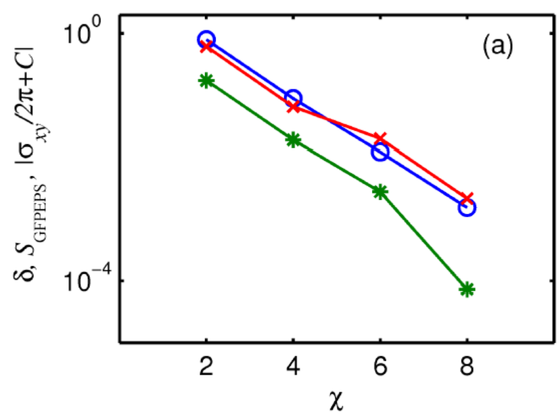


Consistent with infinite correlation length

SUMMARY

- PEPS can describe chiral phases
- Non-interacting and interacting systems
- Can they describe ground states of gapped local Hamiltonians?

Finite temperature:



MPQ