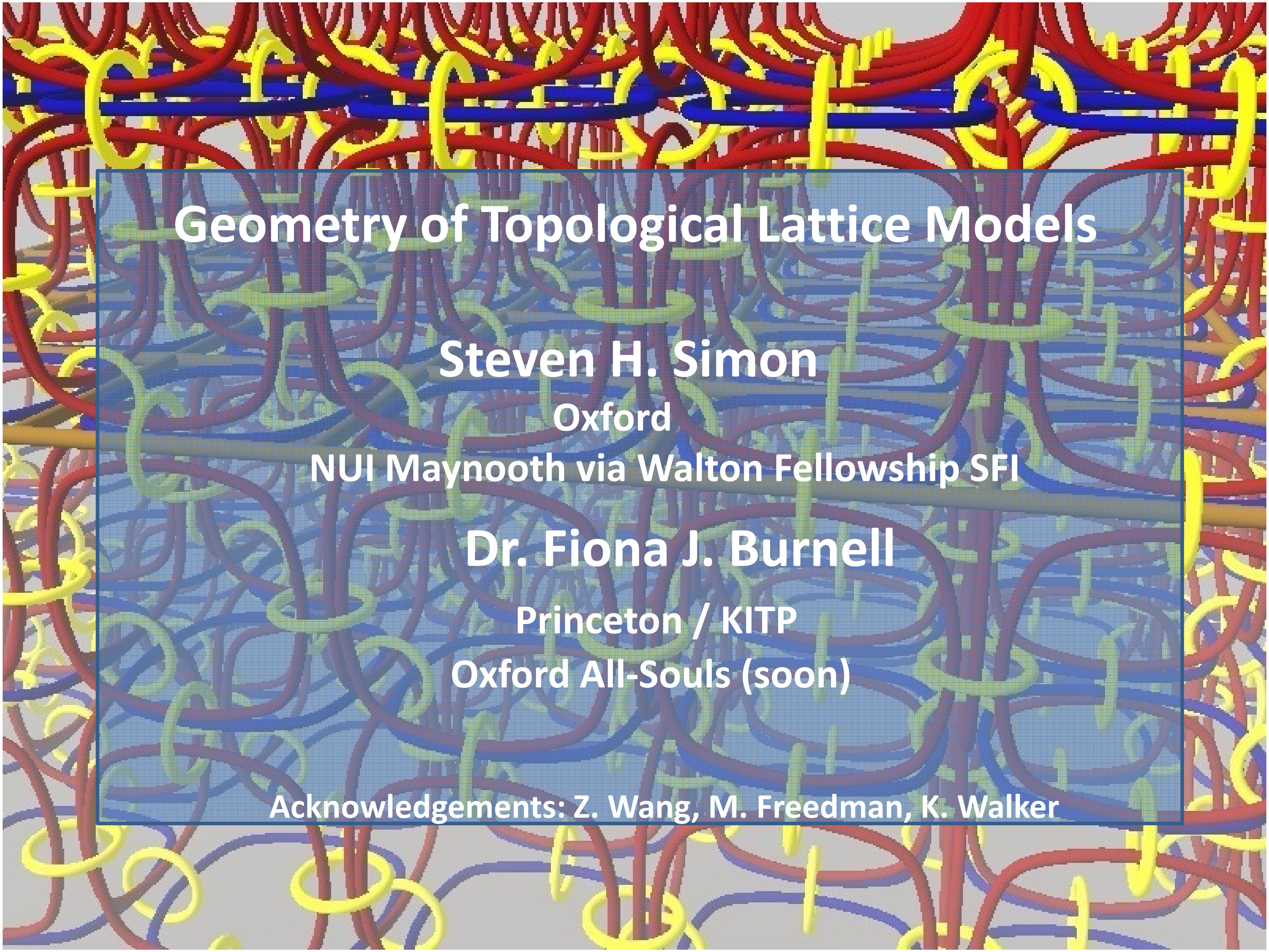

Geometry of Topological Lattice Models

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Oxford All-Souls (soon)

Acknowledgements: Z. Wang, M. Freedman, K. Walker

OUTLINE

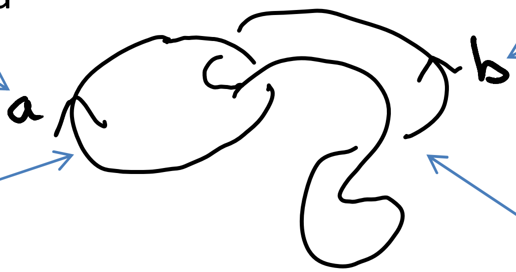
Introductory Material

- Primer on 2+1 D Topological Quantum Field Theories

Rough Definition of Topological Quantum Field Theory:

TQFT =

Mapping from worldlines of particles in a 3d spacetime manifold M to an output that depends only on topology of input.



particle type a

particle type b

path P_1

path P_2

$$\longrightarrow \quad Z[M, Link]$$

Wilson Loop Operator:
particle type (rep) = b
along curve P_2

Ex: Chern-Simons Theory (Pick a gauge group, and a "level")

$$Z_{CS}[M, Link] \text{ ``}=\text{''} \int \mathcal{D}[A]\; W_a(P_1, A) W_b(P_2, A)\; e^{iS_{CS}[A,M]}$$

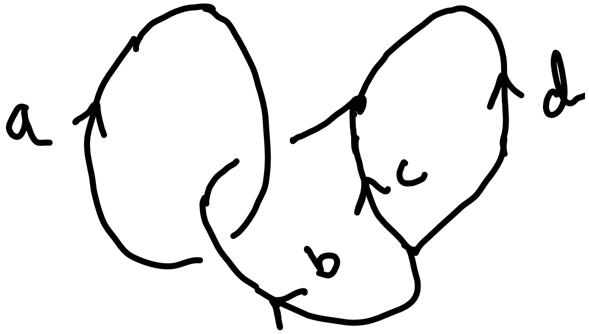
Topological invariant (generalized Jones polynomial of "colored" link in manifold M)

Can also have vacuum partition function an invariant of M

(Witten-Jones)

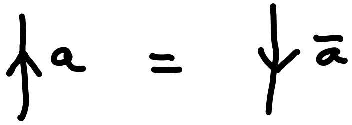
Some more properties of TQFTs

Particles can come together to form other particles.
Can calculate a "value" for any branched link ("graphs")



Other things:

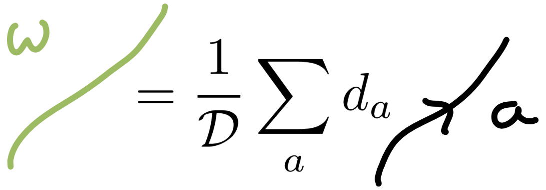
antiparticles



"identity particle"
= "vacuum particle"
= no particle

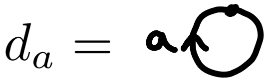
here a and $\bar{a}$
fuse to the vacuum

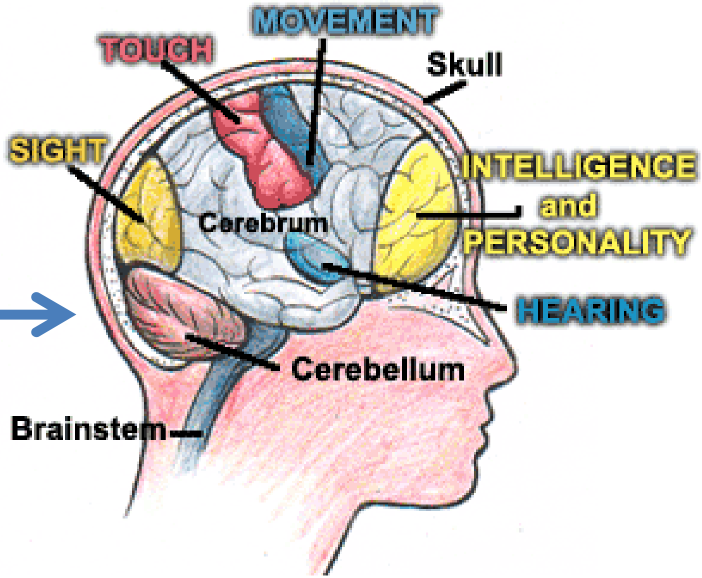
The ω-strand



$$= \frac{1}{\mathcal{D}} \sum_a d_a$$

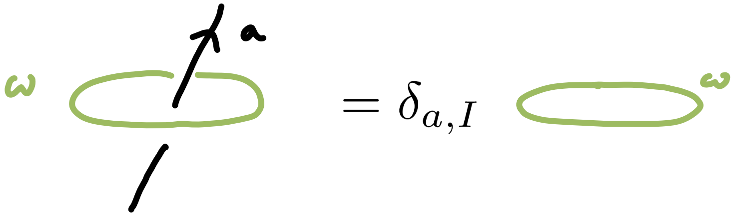
$d_a =$





ω-loop *projects* onto the identity particle (vacuum) through the loop

Killing property:



$= \delta_{a,I}$

$\delta_{a\bar{b}}$

OUTLINE

Introductory Material

- Primer on 2+1 D Topological Quantum Field Theories
  - Nontrivial TQFTs probably Exist!   (Fractional Quantum Hall + …)
  - Could enable “topological” quantum computers
  - Just Interesting

Why should I

Can we find simpler realizations of TQFTs ?

Topological Lattice Models?

- Can we construct a TQFT in “experiment” ?
- Can we regularize Chern-Simons theory?

OUTLINE

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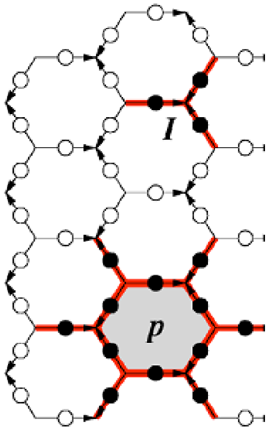
- Primer on 2+1 D Topological Quantum Field Theories
- Topological Lattice Models

Levin-Wen approach – based on Toric code

Levin-Wen Approach to Chern-Simons Theory

$$H = -\sum_{vertices=i} V_i - \sum_{plaquettes=j} P_j$$

All V, P
commuting
projectors



- Bonds labeled with particle types (quantum #s) from the Chern-Simons theory
- Vertex term – Gives energy penalty unless the quantum numbers coming into the vertex "fuse to identity".
- Plaquette term – flips plaquette quantum numbers so ground state is a weighted sum of all configurations admissible to vertex term
  - Each plaquette term is product of 6 F matrices each with 6 (or 10) indices coupling 12 bonds.

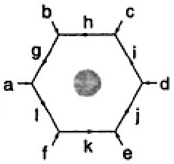
- (Hidden) Vertex-Plaquette Duality

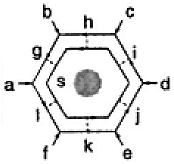
- Quasiparticle Excitations: "Violations" of a vertex term, a plaquette term, or both.

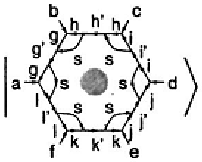
Results in the **Double** of the original Chern-Simons theory (two copies with opposite chiralities)

Some Tensor Calculus From Levin-Wen

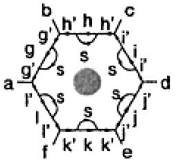
$$B^{s,g'h'i'j'k'l'}_{p,ghijkl}(abcdef)$$
$$= F^{al^*g}_{s^*g'l'^*} F^{bg^*h}_{s^*h'g'^*} F^{ch^*i}_{s^*i'h'^*} F^{di^*j}_{s^*j'i'^*} F^{ej^*k}_{s^*k'j'^*} F^{fk^*l}_{s^*l'k'^*}$$

$B^s_{\boldsymbol{p}}\Big|$


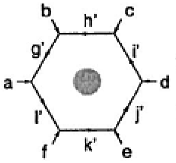
$\Big\rangle = \Big|$


$\Big\rangle = \sum_{g'h'i'j'k'l'} F^{gg^*0}_{s^*sg'^*} F^{hh^*0}_{s^*sh'^*} F^{ii^*0}_{s^*si'^*} F^{jj^*0}_{s^*sj'^*} F^{kk^*0}_{s^*sk'^*} F^{ll^*0}_{s^*sl'^*}\Big|$


$\Big\rangle$

$= \sum_{g'h'i'j'k'l'} F^{gg^*0}_{s^*sg'^*} F^{hh^*0}_{s^*sh'^*} F^{ii^*0}_{s^*si'^*} F^{jj^*0}_{s^*sj'^*} F^{kk^*0}_{s^*sk'^*} F^{ll^*0}_{s^*sl'^*} F^{bg^*h}_{s^*h'g'^*} F^{ch^*i}_{s^*i'h'^*} F^{di^*j}_{s^*j'i'^*} F^{ej^*k}_{s^*k'j'^*} F^{fk^*l}_{s^*l'k'^*} F^{al^*g}_{s^*g'l'^*}\Big|$


$\Big\rangle$

$= \sum_{g'h'i'j'k'l'} F^{bg^*h}_{s^*h'g'^*} F^{ch^*i}_{s^*i'h'^*} F^{di^*j}_{s^*j'i'^*} F^{ej^*k}_{s^*k'j'^*} F^{fk^*l}_{s^*l'k'^*} F^{al^*g}_{s^*g'l'^*}\Big|$


$\Big\rangle$

$$W^{i'_1 i'_2 \ldots i'_N}_{i_1 i_2 \ldots i_N}(e_1 e_2 \ldots e_N) = \sum_{\{s_k\}} \left(\prod_{k=1}^{N} F^{e_k i_k^* i_{k-1}}_{s_k^* i'_{k-1} i'^*_k} \right) \mathrm{Tr}\left(\prod_{k=1}^{N} \Omega^{s_k}_k \right)$$

$$\sum_{s=0}^{N} \overline{\Omega}^m_{rsj} F^{sl^*i}_{kjm^*} \Omega^l_{sti} \frac{v_j v_s}{v_m} = \sum_{n=0}^{N} F^{ji^*k}_{t^*nl^*} \Omega^n_{rtk} F^{jl^*n}_{krm^*}$$

$$\overline{\Omega}^j_{sti} = \sum_{k=0}^{N} \Omega^k_{sti^*} F^{it^*k}_{i^*sj^*}.$$

$1{:}n_{1,0} = 1, \quad n_{1,1} = 0, \quad \Omega^0_{1,000} = 1, \quad \Omega^1_{1,001} = 1,$

$2{:}n_{2,0} = 0, \quad n_{2,1} = 1, \quad \Omega^1_{2,110} = 1,$

$\Omega^0_{2,111} = -\gamma_+^{-1} e^{\pi i/5}, \quad \Omega^1_{2,111} = \gamma_+^{-1/2} e^{3\pi i/5},$

$3{:}n_{3,0} = 0, \quad n_{3,1} = 1, \quad \Omega^1_{3,110} = 1,$

$\Omega^0_{3,111} = -\gamma_+^{-1} e^{-\pi i/5}, \quad \Omega^1_{3,111} = \gamma_+^{-1/2} e^{-3\pi i/5},$

$4{:}n_{4,0} = 1, \quad n_{4,1} = 1, \quad \Omega^0_{4,000} = 1, \quad \Omega^1_{4,110} = 1,$

$\Omega^1_{4,001} = -\gamma_+^{-2}, \quad \Omega^0_{4,111} = \gamma_+^{-1}, \quad \Omega^1_{4,111} = \gamma_+^{-5/2},$

$$\Omega^1_{4,101} = (\Omega^1_{4,011})^* = \gamma_+^{-11/4}(2 - e^{3\pi i/5} + \gamma_+ e^{-3\pi i/5}). \quad (51)$$

This is one reason why we need another way of understanding this construction

OUTLINE

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- Primer on 2+1 D Topological Quantum Field Theories
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  Levin-Wen approach – based on Toric code

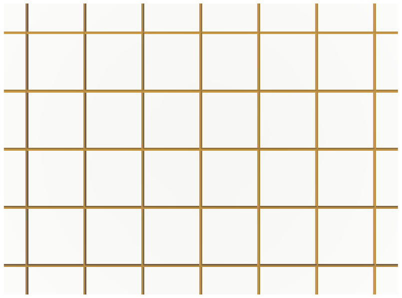
Geometric Approach to Topological Lattice Models

Build Up a Lattice Model from the Continuum Piece by Piece

Chern-Simons theory is a theory of loops in 2+1 d

How to build a lattice model from loops?

Building a Topological Lattice Model

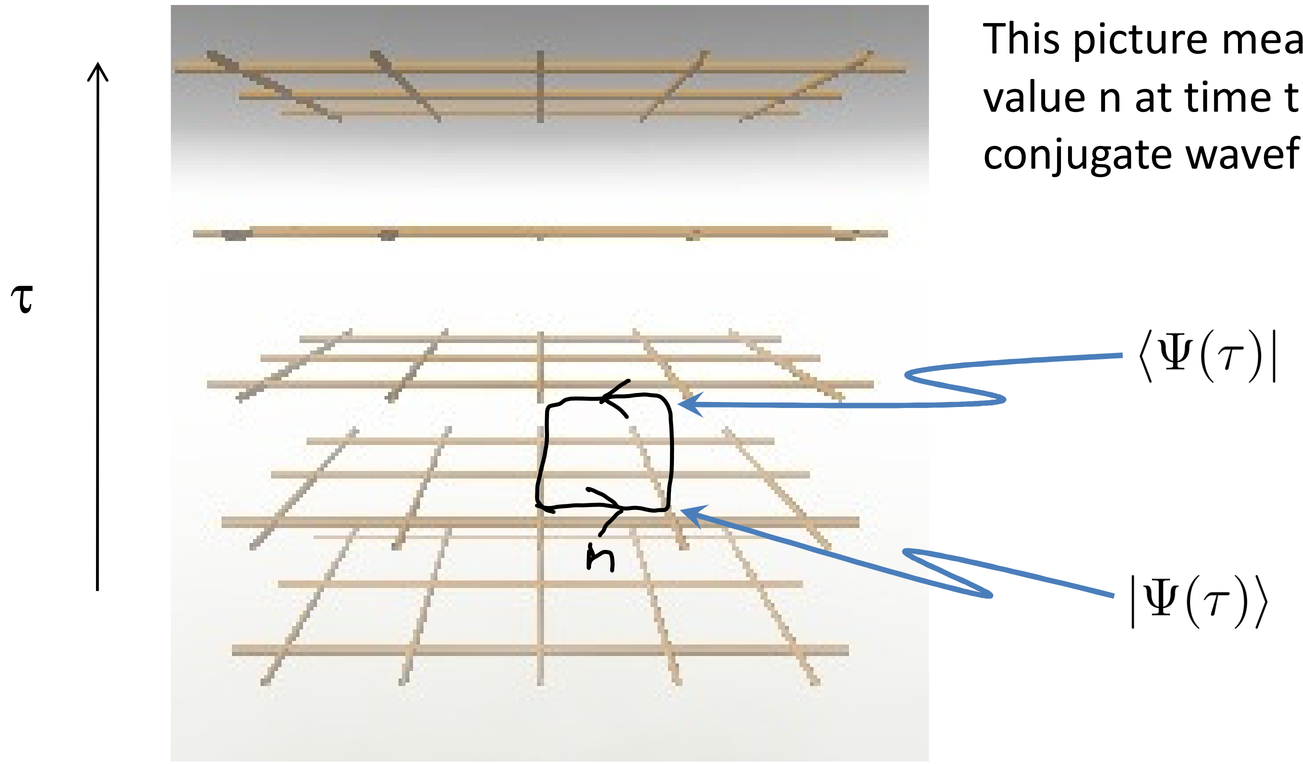


- Pick a 2d lattice
- Pick a (chiral) Chern-Simons theory
  - The quantum numbers we put on the bonds of the lattice are the quantum numbers of Chern-Simons theory
  - BUT ALSO Chern-Simons theory allows us to evaluate knots

Building a Topological Lattice Model

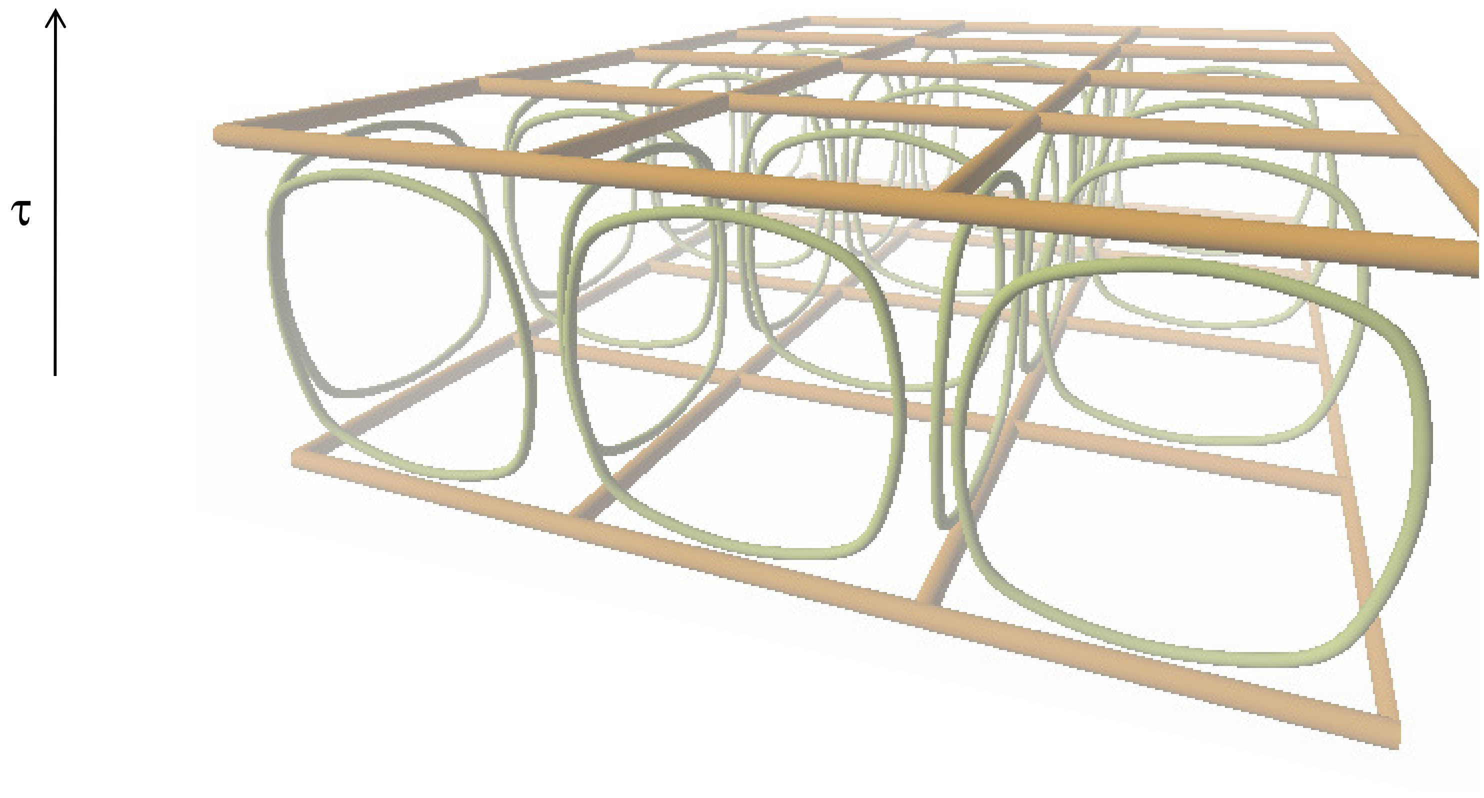
Duplicate lattice to represent many time steps

This picture means that this bond has value n at time time τ, and the conjugate wavefunction has value n̄



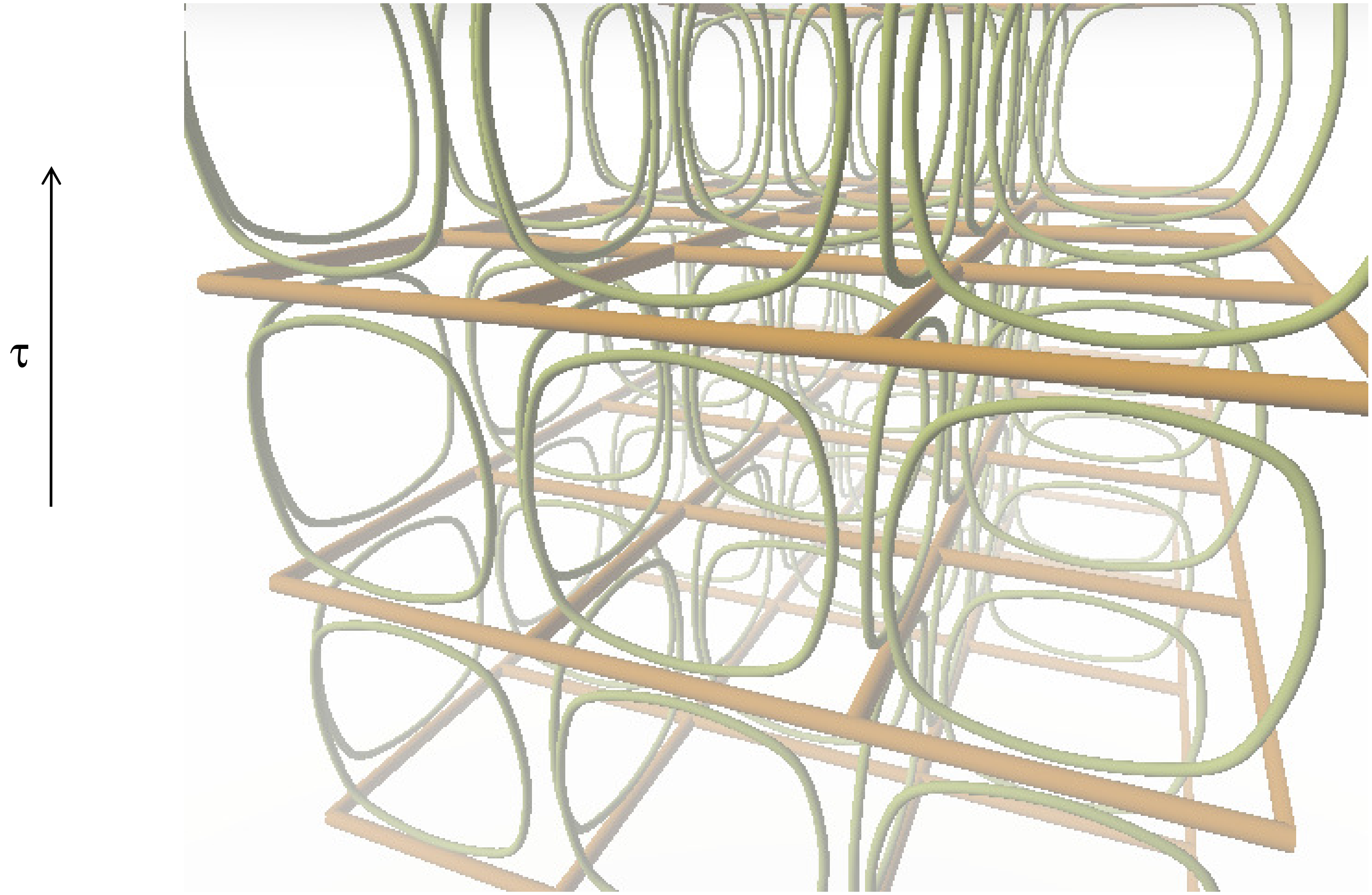
n is a particle type in our chiral Chern-Simons theory
(Chern-Simons is theory of loops and knots)

Building a Topological Lattice Model



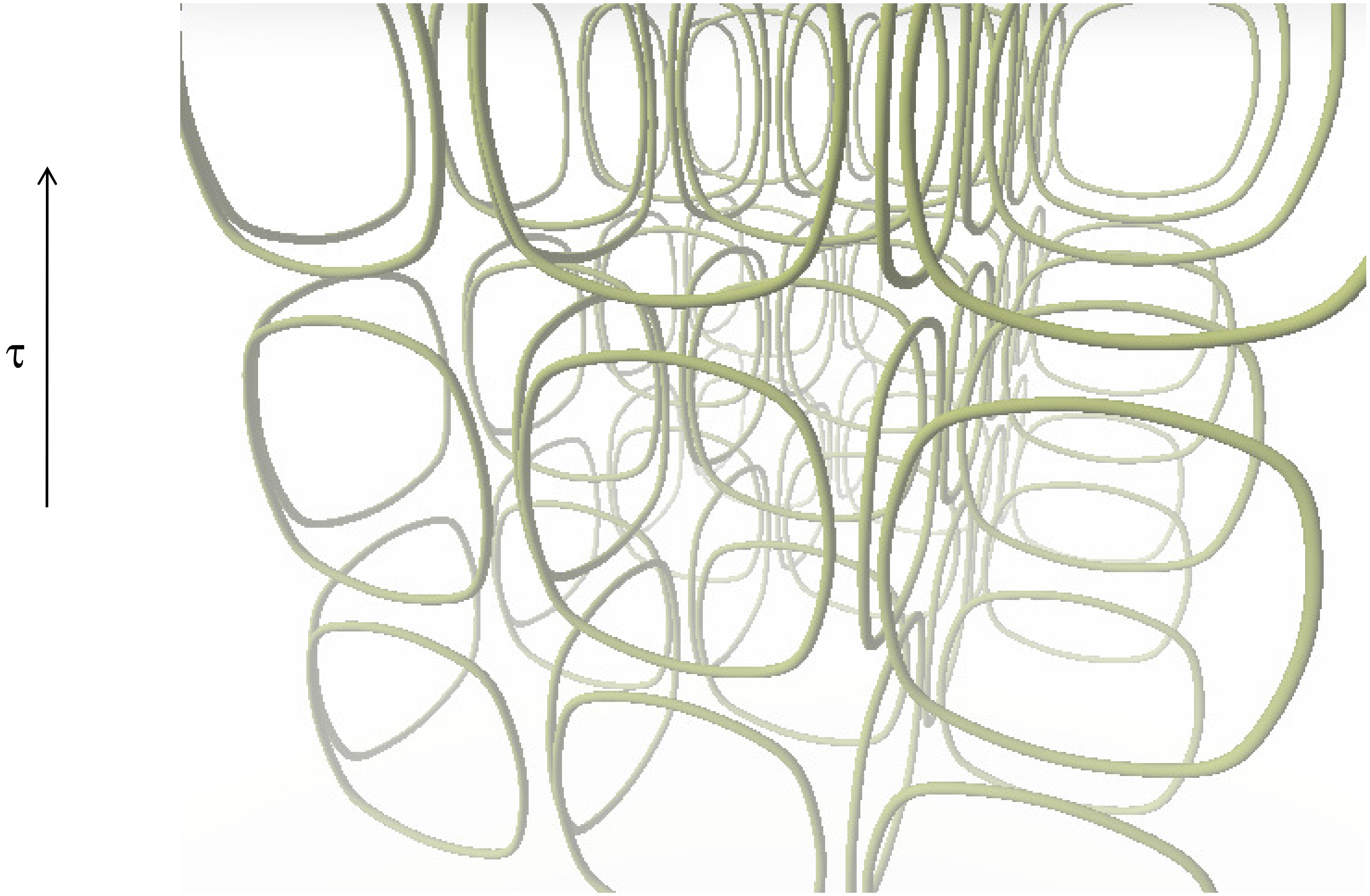
This is how we represent the state of the system at one time step (each green loop has a label)

Building a Topological Lattice Model



This is how we represent the state of the system at all times (each green loop has a label)

Building a Topological Lattice Model



This is how we represent the state of the system at all times (each green loop has a label)

We will want to sum over all possible quantum numbers at all times:

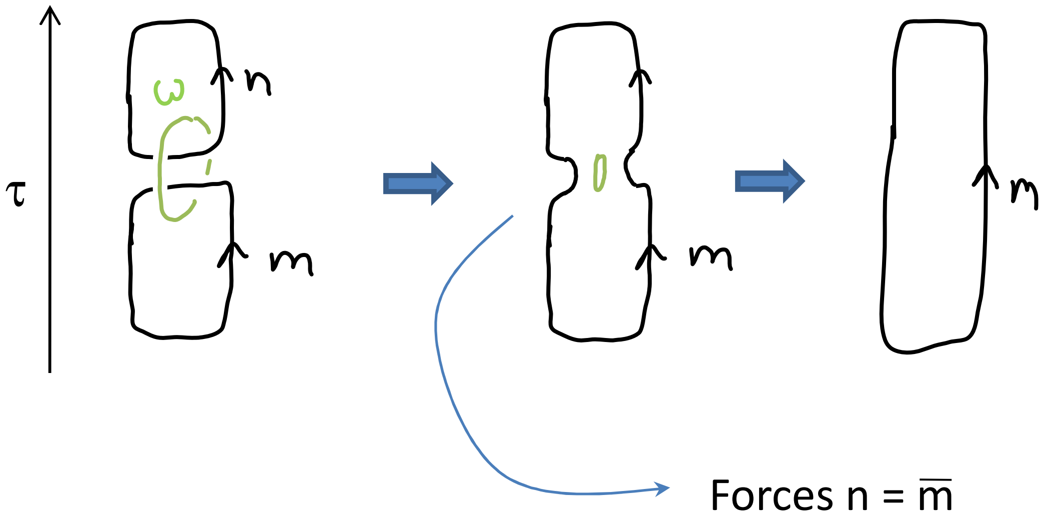
ω is a sum over quantum numbers

Building a Topological Lattice Model

Construct a Hamiltonian : H = 0 means free propagation

For H=0. Quantum numbers should be conserved in time.

Between time slices, we want to “insert a complete set”. Use an ω to do this



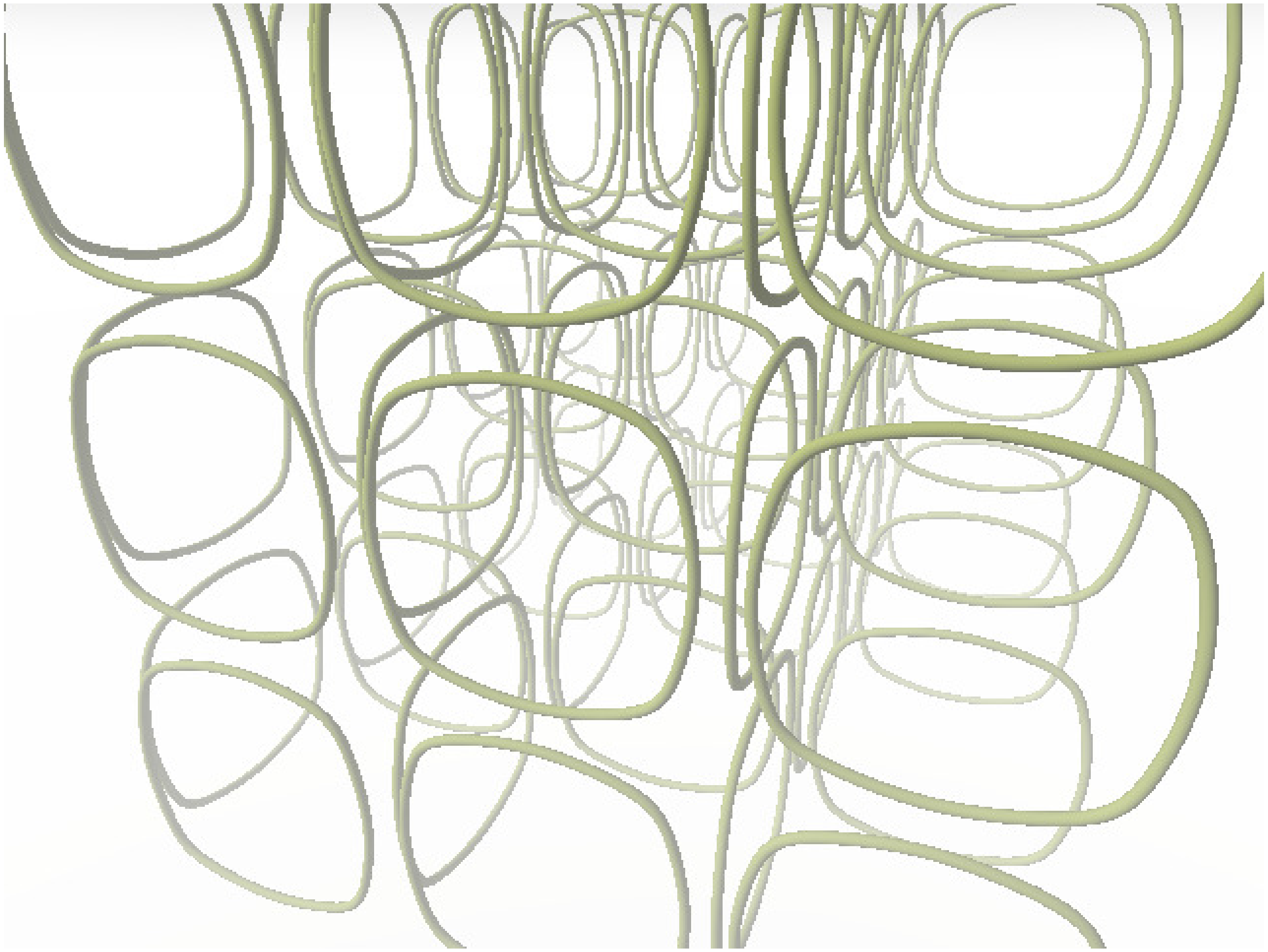
Forces n = $\overline{m}$

$$\sum_n |n\rangle\langle n|$$

This “transfers quantum numbers faithfully up in time”

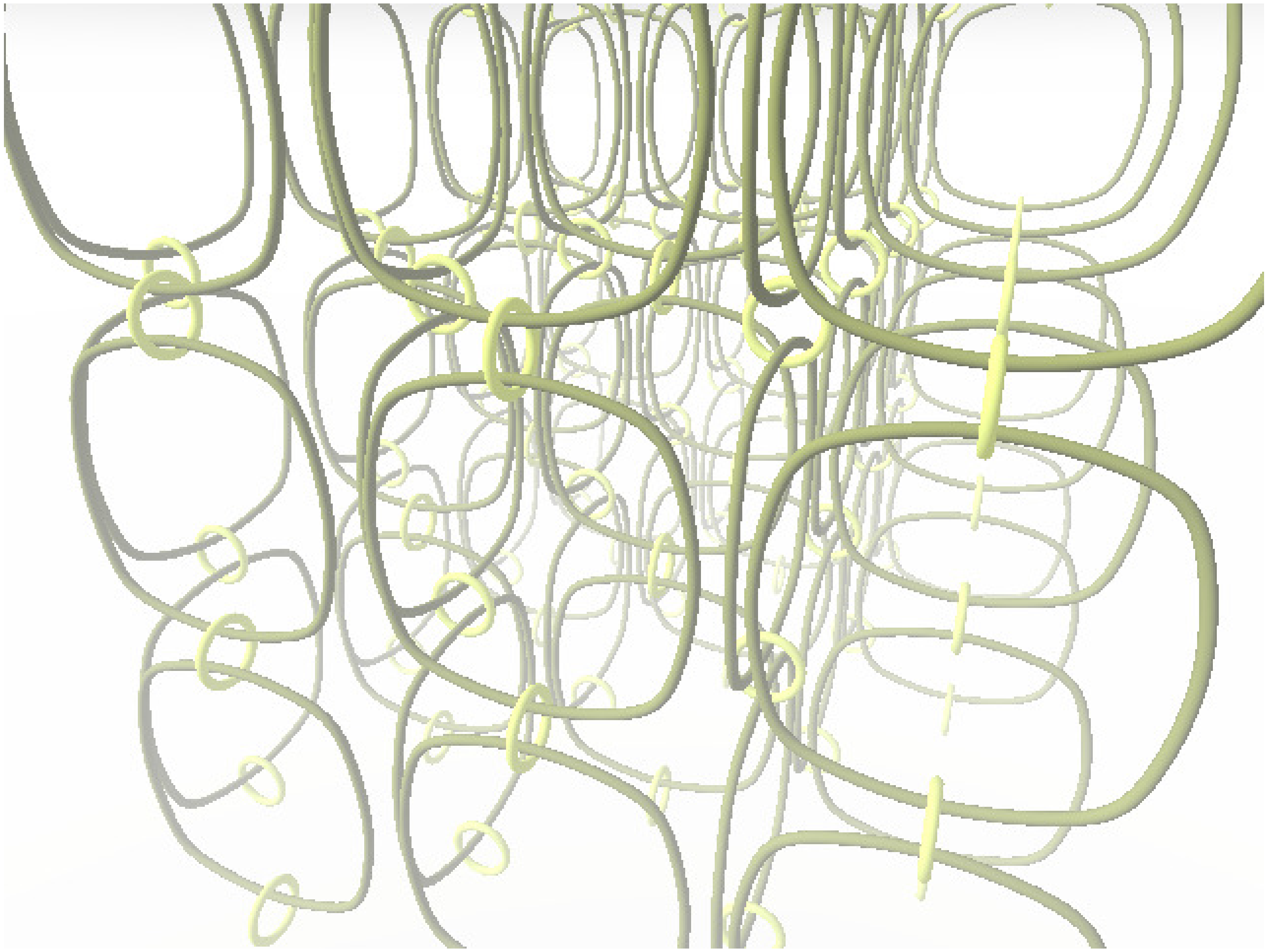
Building a Topological Lattice Model

Construct a Hamiltonian : H = 0 means free propagation



Building a Topological Lattice Model

Construct a Hamiltonian : H = 0 means free propagation

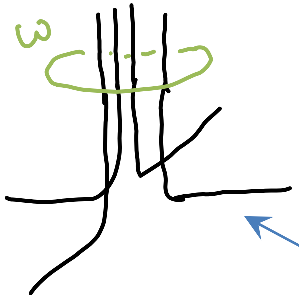


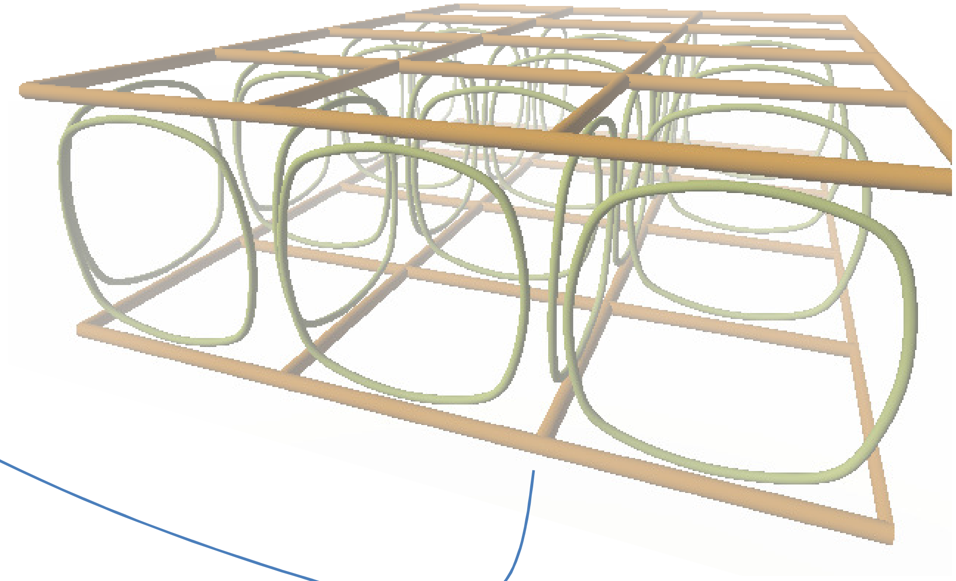
This “transfers quantum numbers faithfully up in time”

The terms of the Hamiltonian (1) The vertex term

$$H = -\sum_{vertices=i} V_i - \sum_{plaquettes=j} P_j$$

Vertex condition: bonds must fuse to identity





Killing property : Only nonzero if all strings coming into vertex fuse to identity

The terms of the Hamiltonian (1) The vertex term

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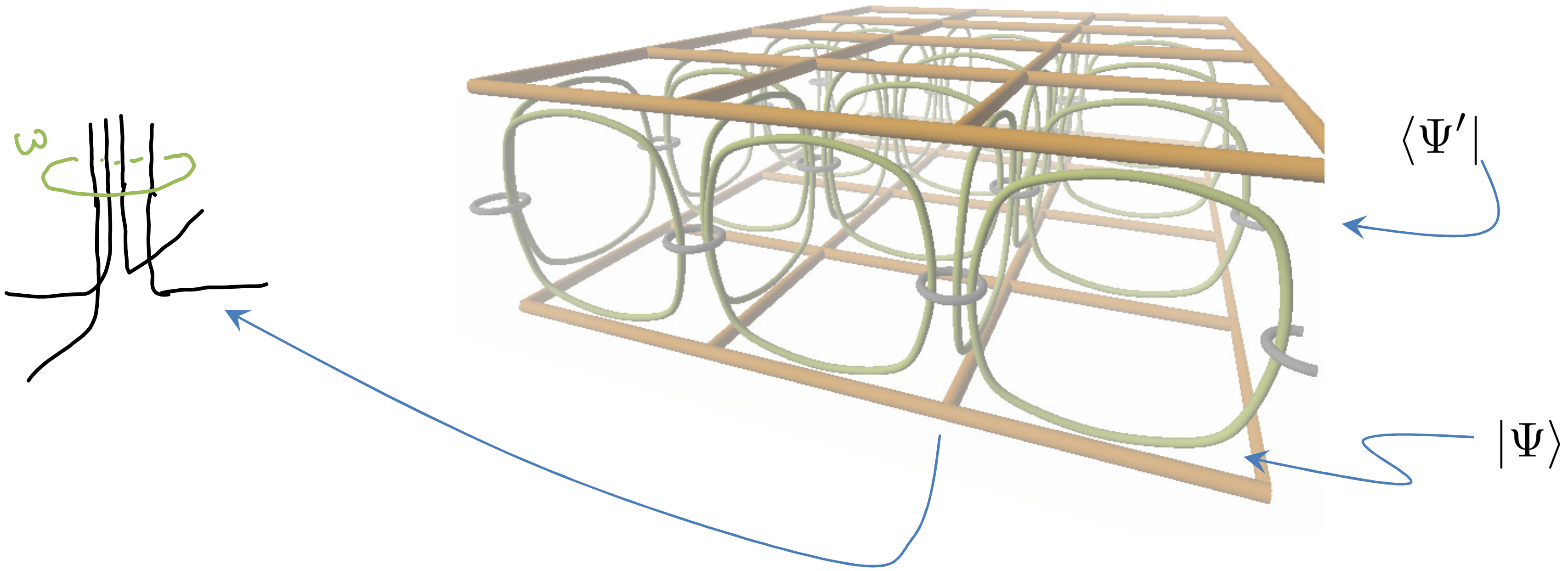
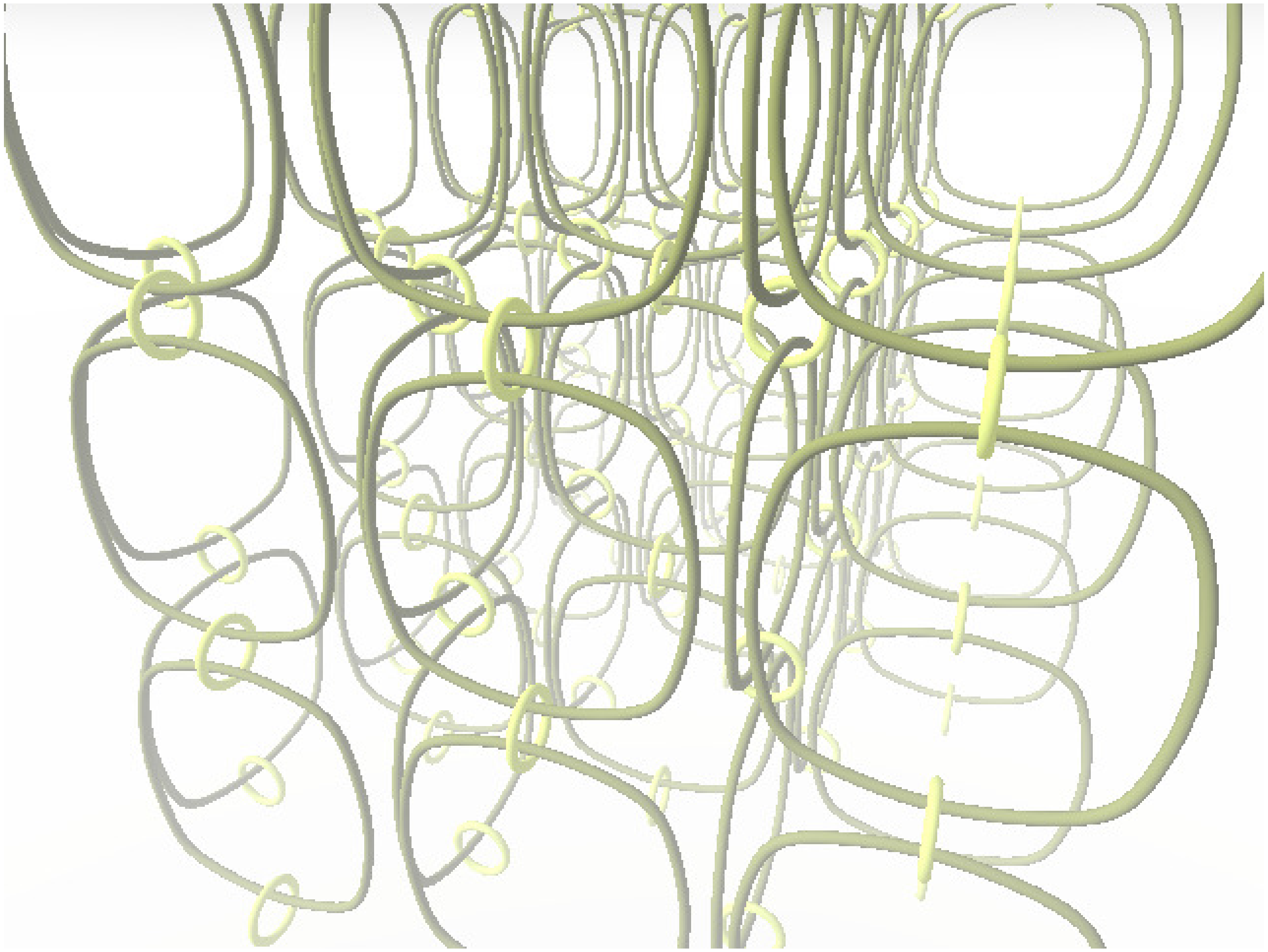


Killing property : Only nonzero if all strings coming into vertex fuse to identity

$$\langle \Psi' | V | \Psi \rangle$$

The terms of the Hamiltonian (1) The vertex term



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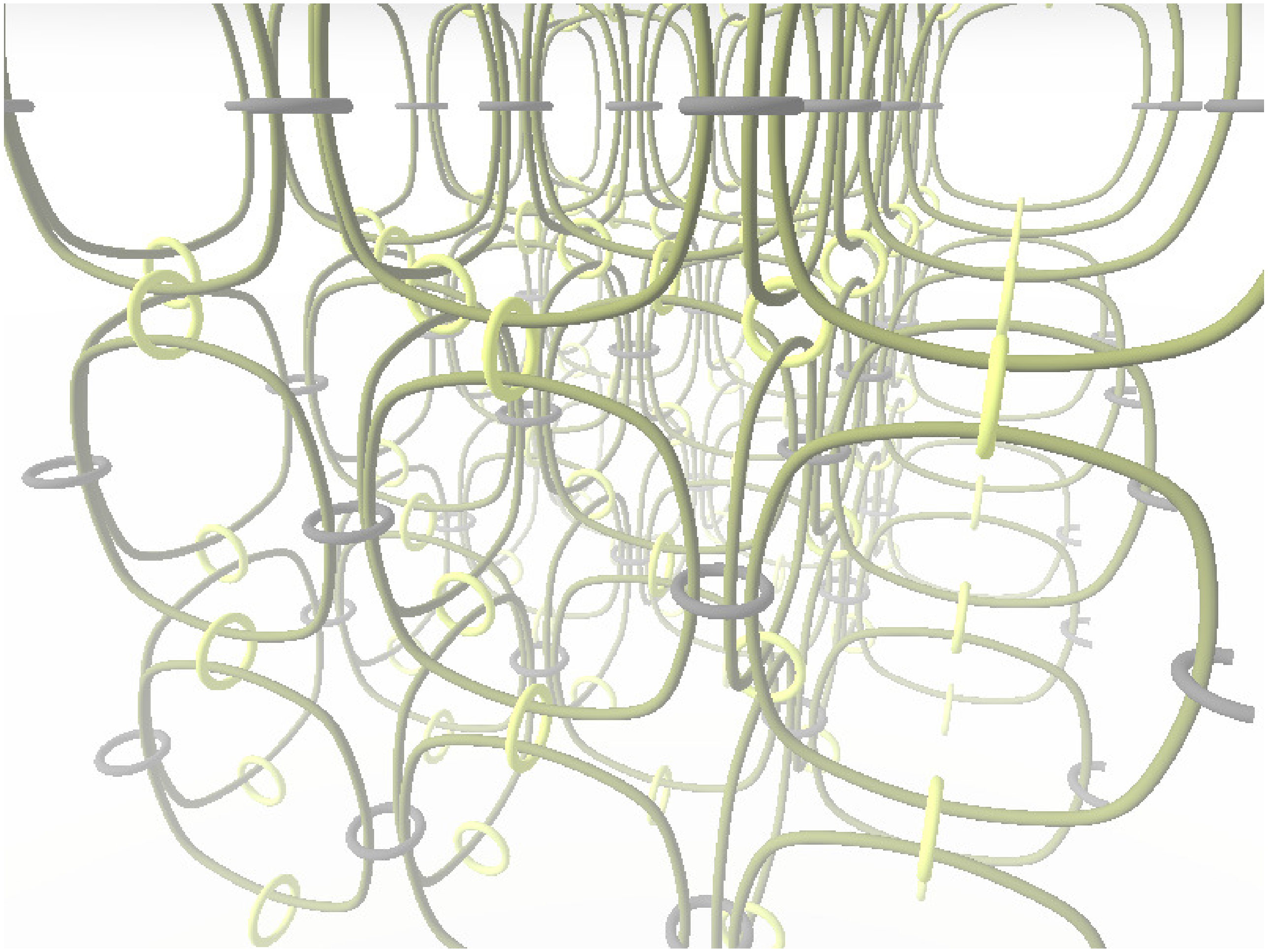


At every time slice project the vertex

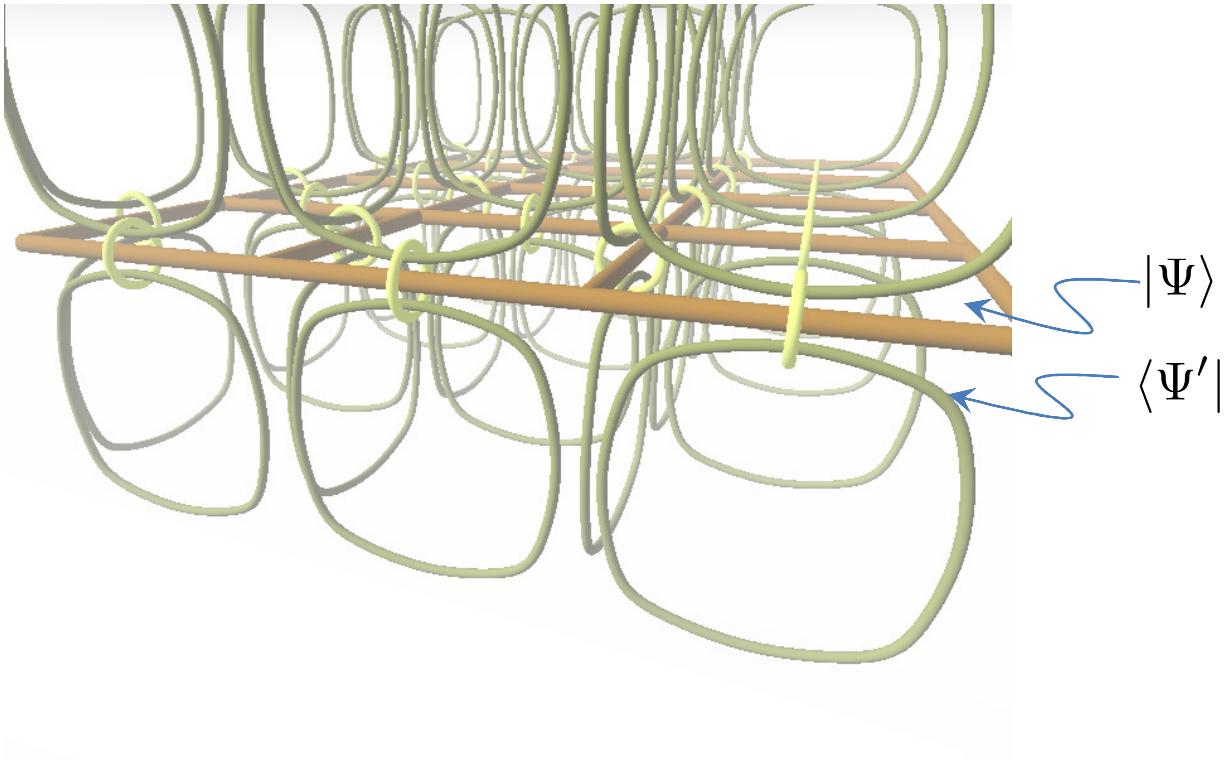
The terms of the Hamiltonian (2) The plaquette term

$$H = -\sum_{vertices=i} V_i - \sum_{plaquettes=j} P_j$$

No particle “flux” through any plaquette (duality)

Yellow ω loops are insertion of complete set (free propagation)

$$\sum_{\Psi_i} |\Psi_i\rangle\langle\Psi_i|$$



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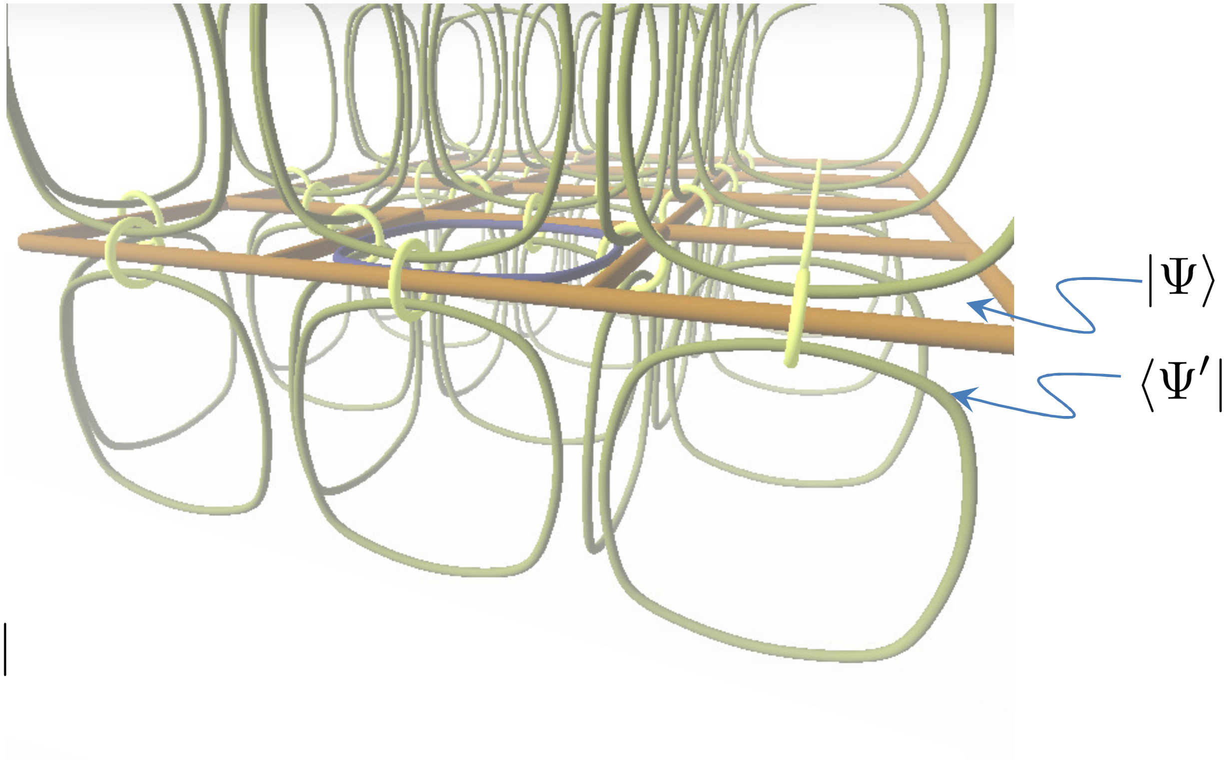
$$\sum_{\Psi_i} |\Psi_i\rangle\langle\Psi_i|$$

Becomes

$$\sum_{\Psi_i,\Psi'_j} |\Psi_i\rangle\langle\Psi_i|P|\Psi'_j\rangle\langle\Psi'_j|$$



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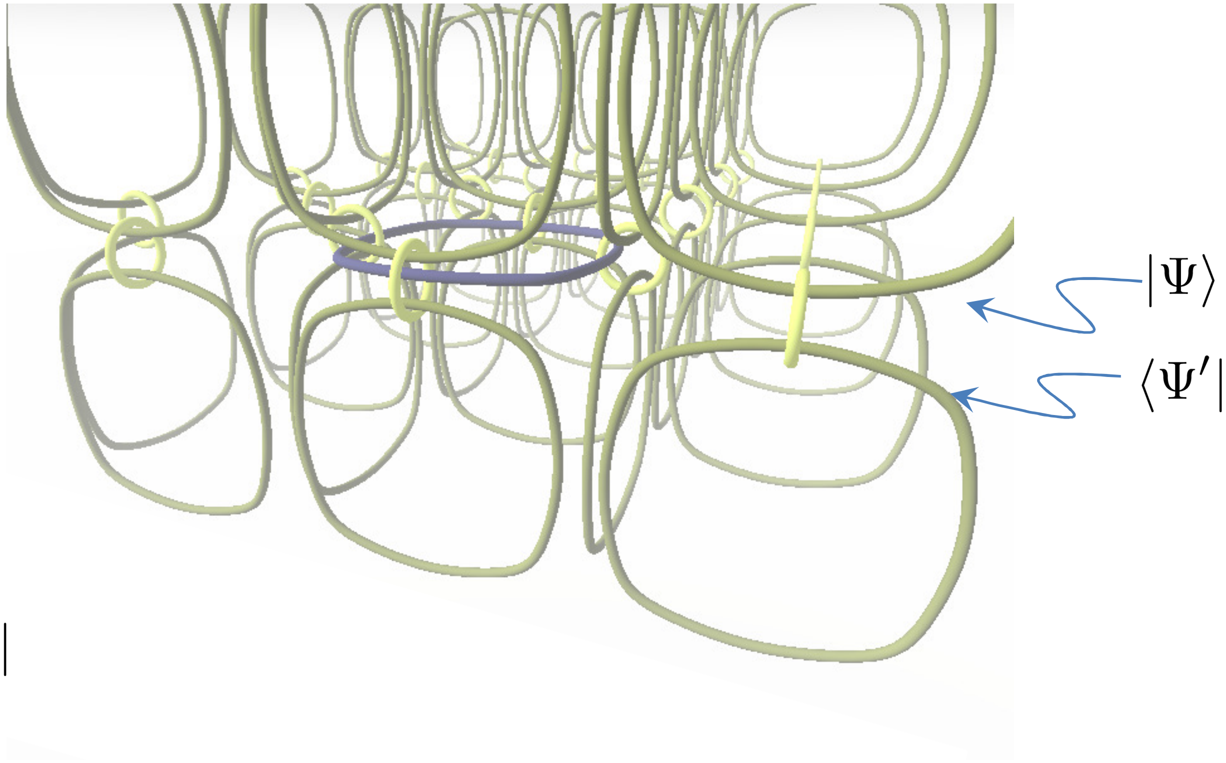
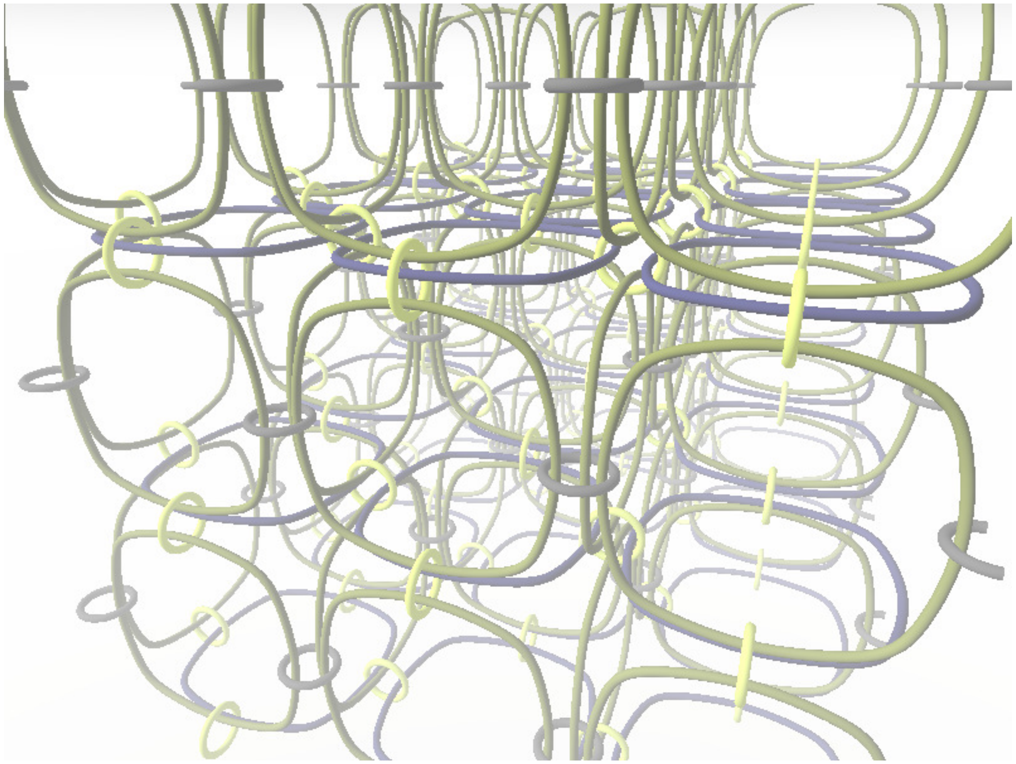


Putting it all together



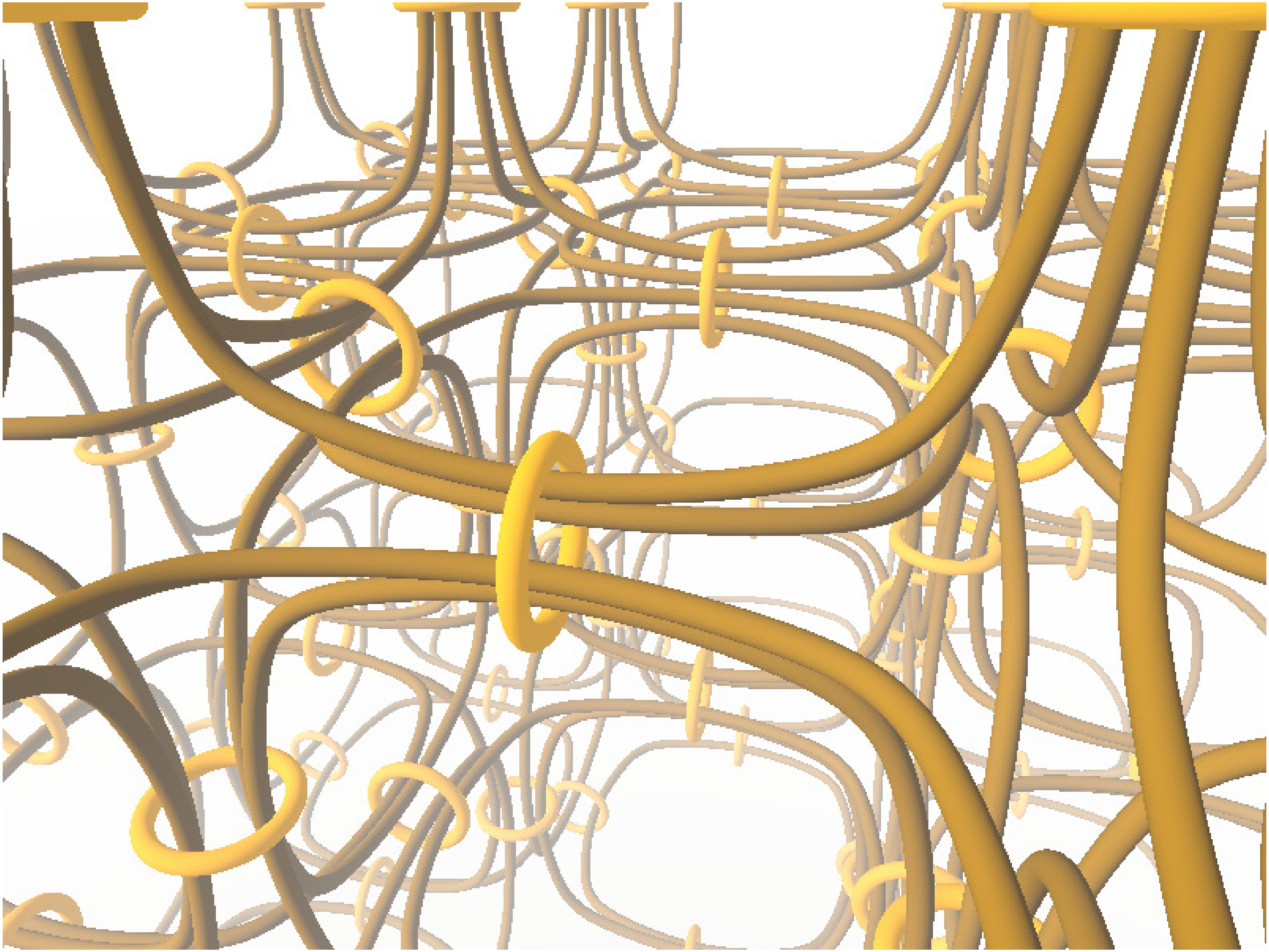
All of these strings are ω strings

Green = sum over all bond variables

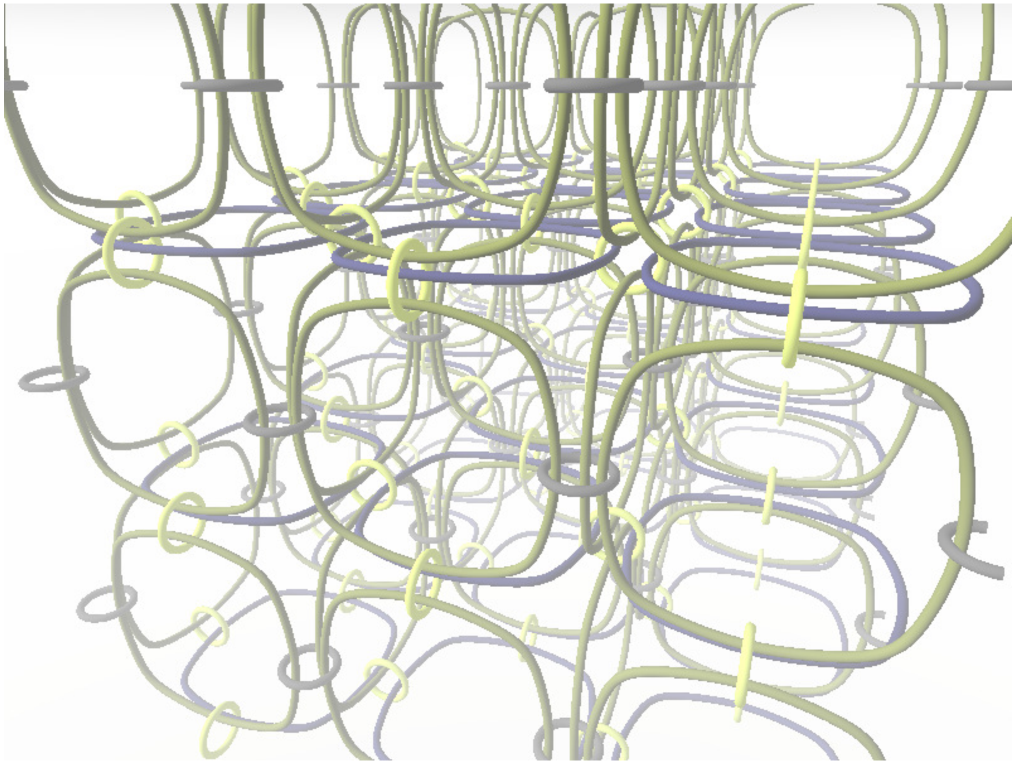
Yellow = Free propagation in absence of hamiltonian (inserting complete set)

Purple = Vertex projector (all bonds coming into a vertex must fuse to I)

Blue = Turns yellow complete set into plaquette projector: No flux through a plaquette (duality)



Putting it all together



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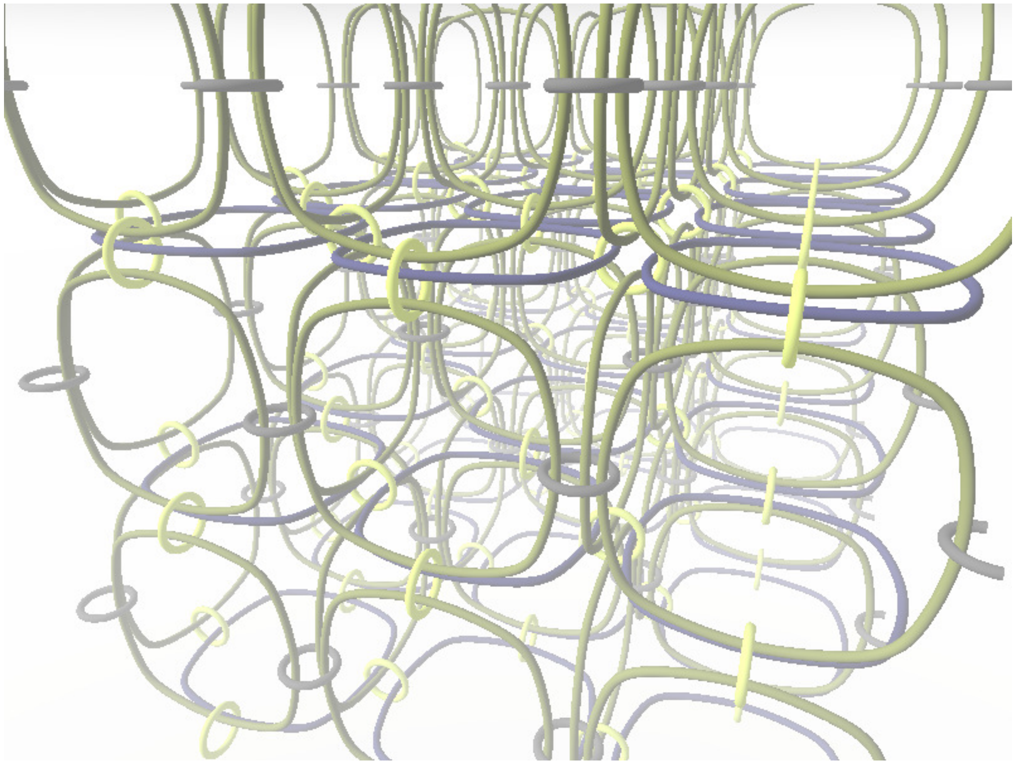
Evaluation of this link calculates

$$\sum_{\Psi_{i_1},\Psi_{i_2},\ldots} \ldots |\Psi_{i_n}\rangle\langle\Psi_{i_n}|P|\Psi_{i_{n-1}}\rangle\langle\Psi_{i_{n-1}}|V|\Psi_{i_{n-2}}\rangle\langle\Psi_{i_{n-2}}|P|\Psi_{i_{n-3}}\rangle\langle\Psi_{i_{n-3}}|V|\ldots$$

= Trotter decomposition of Levin-Wen partition function (of the ground state sector).



Chainmail

Roberts '95 for "SU(2)$_k$" models
=Turaev-Viro State Sum Invariant

Evaluation of this link calculates

$$\sum_{\Psi_{i_1},\Psi_{i_2},\ldots} \ldots |\Psi_{i_n}\rangle\langle\Psi_{i_n}|P|\Psi_{i_{n-1}}\rangle\langle\Psi_{i_{n-1}}|V|\Psi_{i_{n-2}}\rangle\langle\Psi_{i_{n-2}}|P|\Psi_{i_{n-3}}\rangle\langle\Psi_{i_{n-3}}|V|\ldots$$

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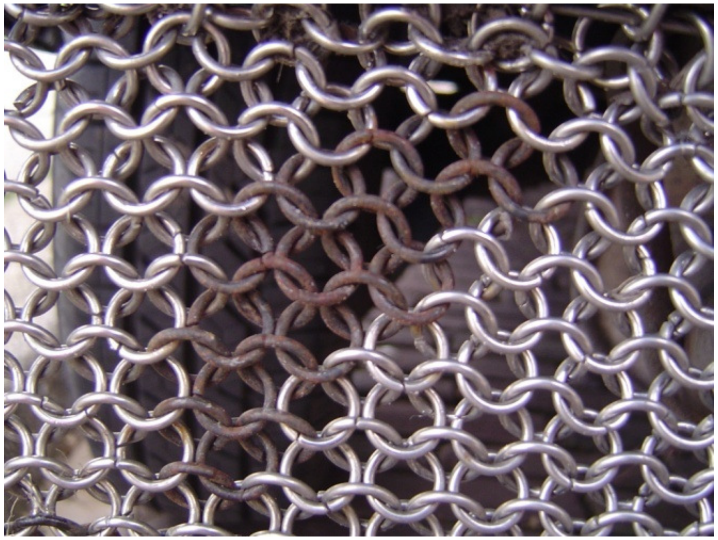
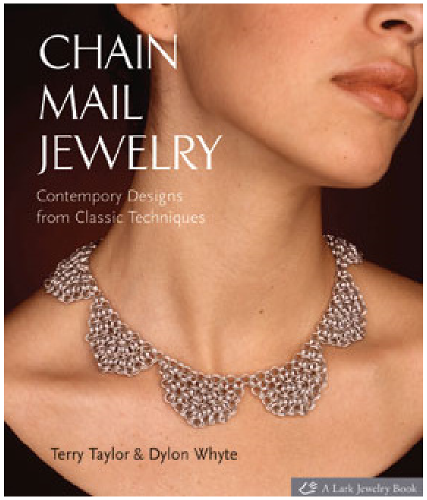
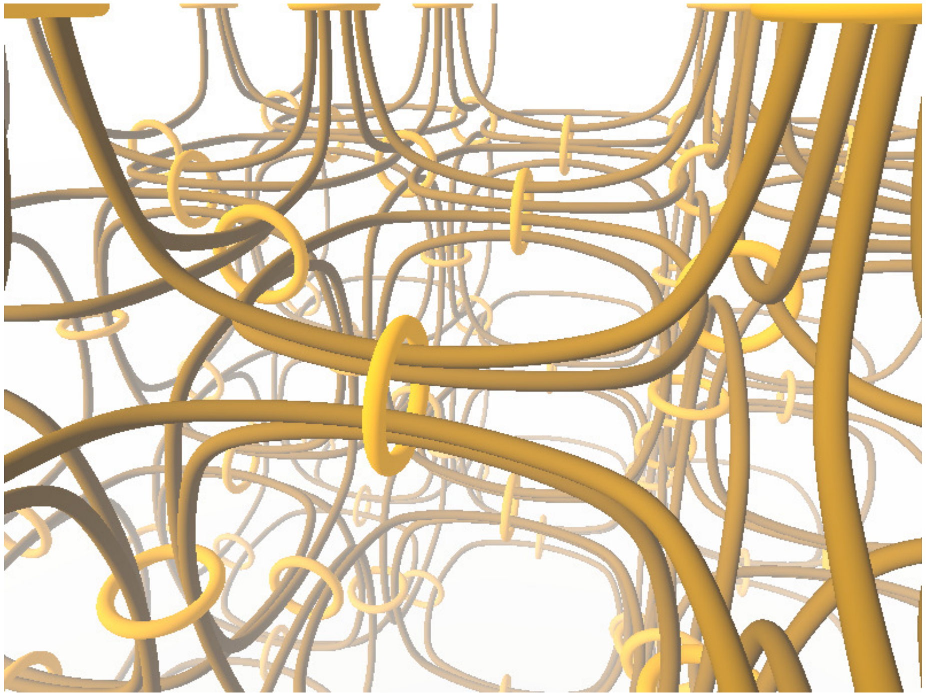
Chainmail





Chainmail in Modern Fashion

(Google will show you more which I can't)

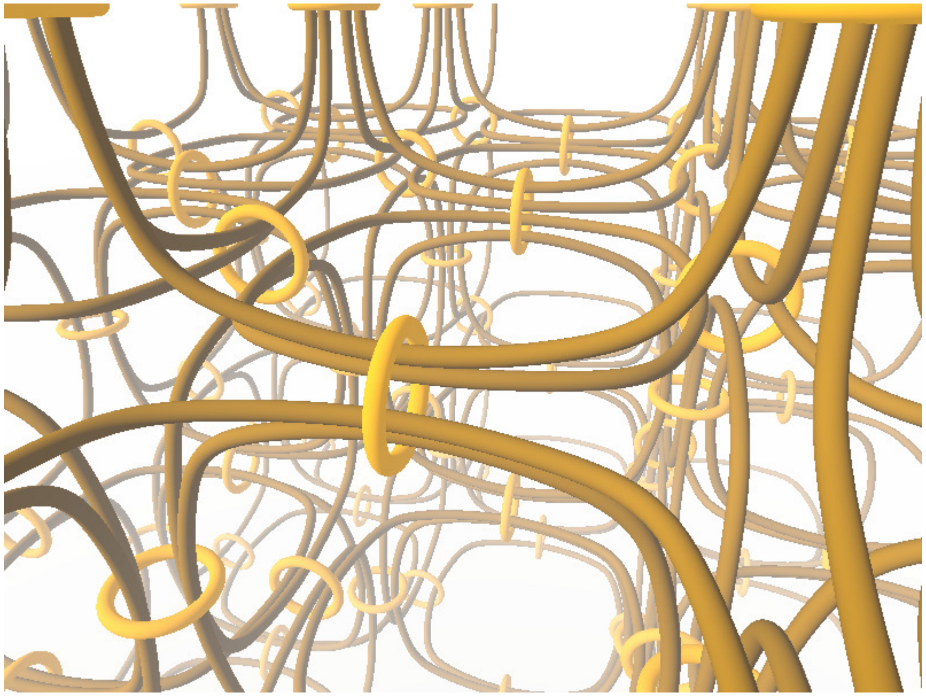


Chainmail

Roberts '95 for "$SU(2)_k$" models
=Turaev-Viro State Sum Invariant

$$\sum_{\Psi_{i_1},\Psi_{i_2},\ldots} \ldots|\Psi_{i_n}\rangle\langle\Psi_{i_n}|P|\Psi_{i_{n-1}}\rangle\langle\Psi_{i_{n-1}}|V|\Psi_{i_{n-2}}\rangle\langle\Psi_{i_{n-2}}|P|\Psi_{i_{n-3}}\rangle\langle\Psi_{i_{n-3}}|V|\ldots$$

= Trotter decomposition of Levin-Wen partition function (of the ground state sector).



Chainmail

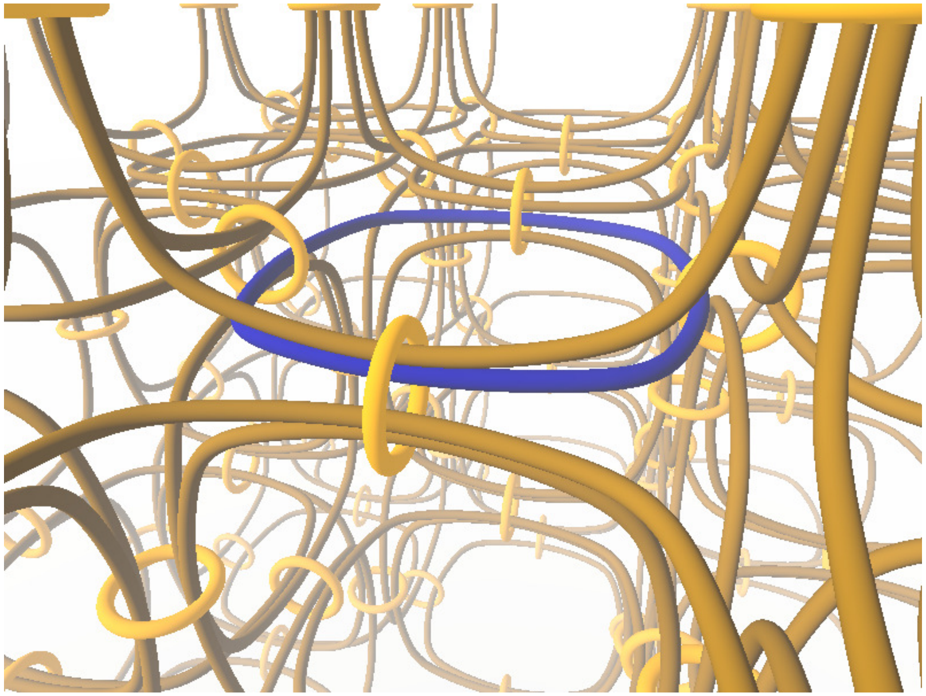
Roberts '95 for "$SU(2)_k$" models
=Turaev-Viro State Sum Invariant

$$\boxed{Z_{CS}[M, ChainMailLink] = Z_{CS}[M] Z_{CS}[\overline{M}]}$$ = Turaev-Viro Invariant

original mirror

Note : Result is achiral

Even though the Chainmail Link is evaluated within a Chiral Chern-Simons theory.



Chainmail

Roberts '95 for "$SU(2)_k$" models
=Turaev-Viro State Sum Invariant

Independent of lattice and
decomposition of manifold

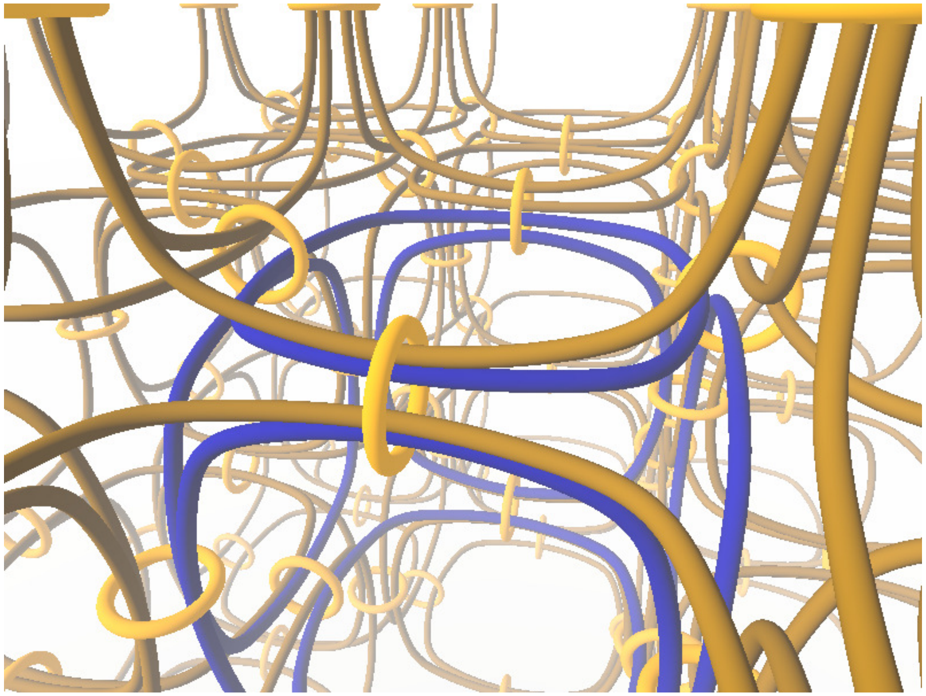
$$\boxed{Z_{CS}[M, ChainMailLink] = Z_{CS}[M] Z_{CS}[\overline{M}]}$$

= Turaev-Viro Invariant

original mirror

Note : Result is achiral

Even though the Chainmail Link is evaluated within a Chiral Chern-Simons theory.



Chainmail

Roberts '95 for "$SU(2)_k$" models
=Turaev-Viro State Sum Invariant

Independent of lattice and decomposition of manifold

$$\boxed{Z_{CS}[M, ChainMailLink] = \underset{\nearrow}{Z_{CS}[M]}Z_{CS}[\overline{M}]}$$

= Turaev-Viro Invariant

original mirror

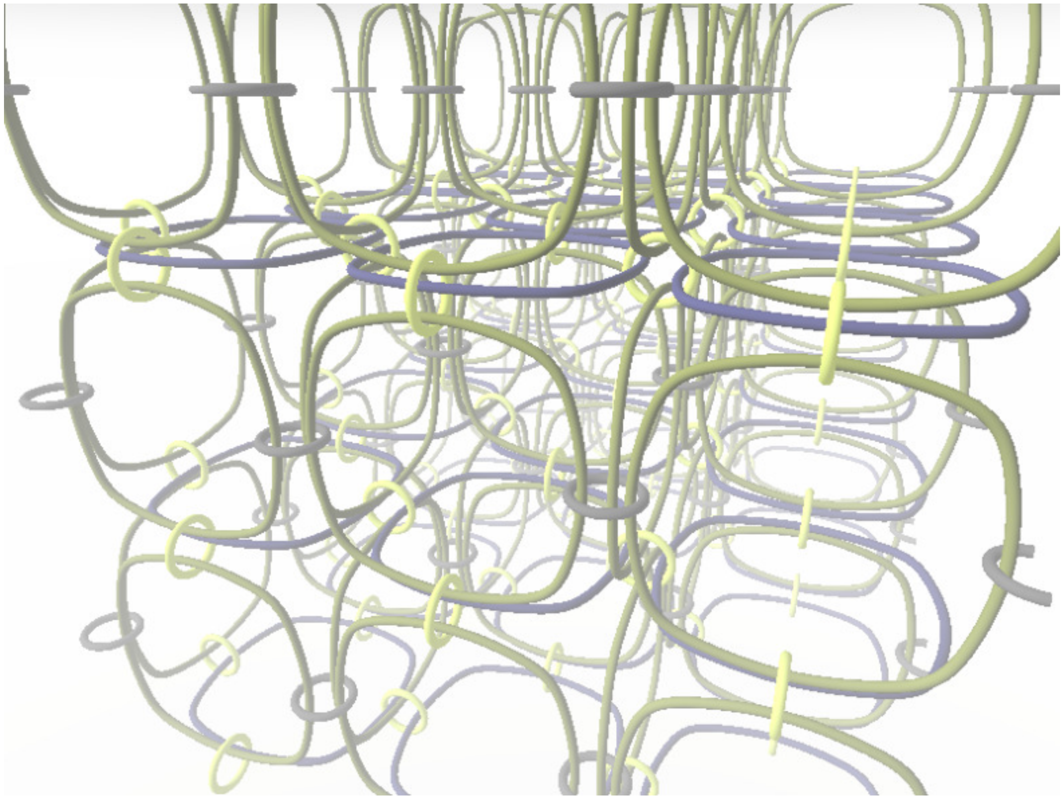
Note : Result is achiral

Even though the Chainmail Link is evaluated within a Chiral Chern-Simons theory.

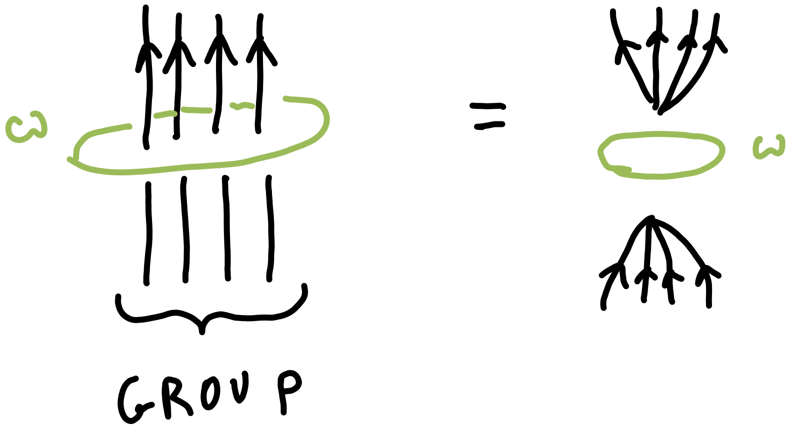
Quasiparticles

Quasiparticles are a violation of the vertex term, or the plaquette term (or both).

Want partition function in presence of violation

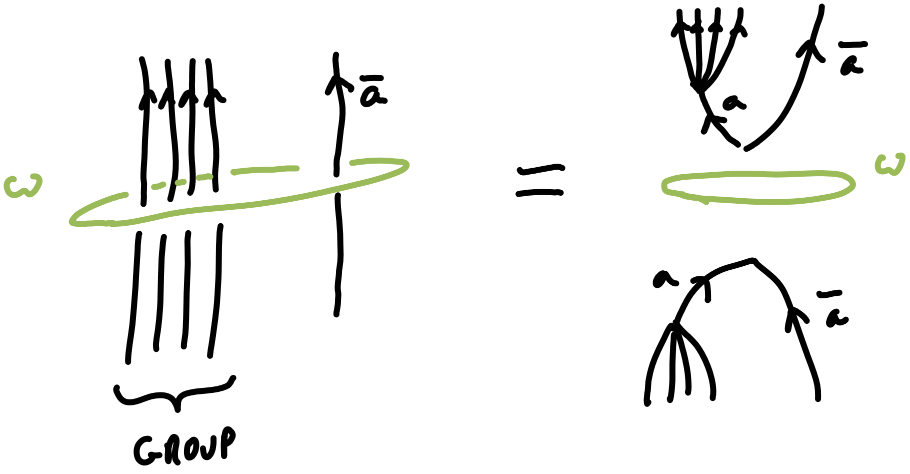


How to “force flux” through a vertex or plaquette



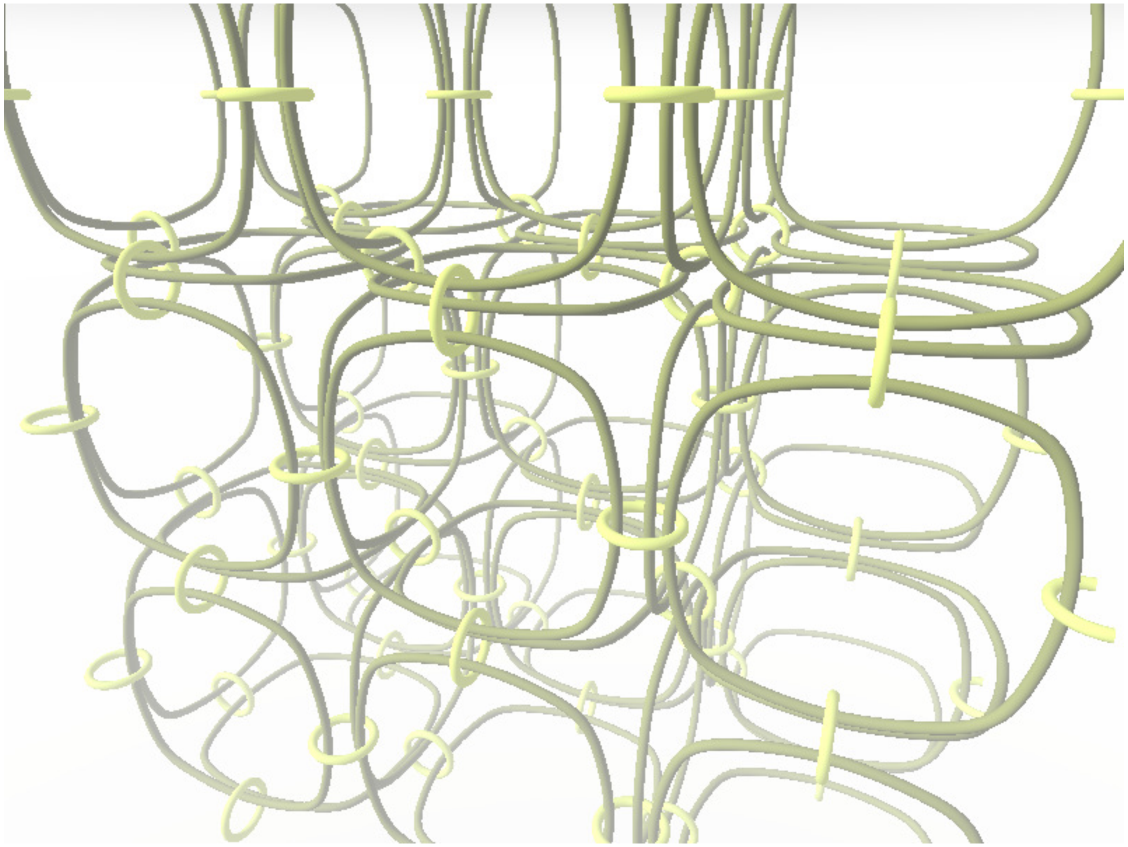
ω Killing property :

Group must fuse to vacuum



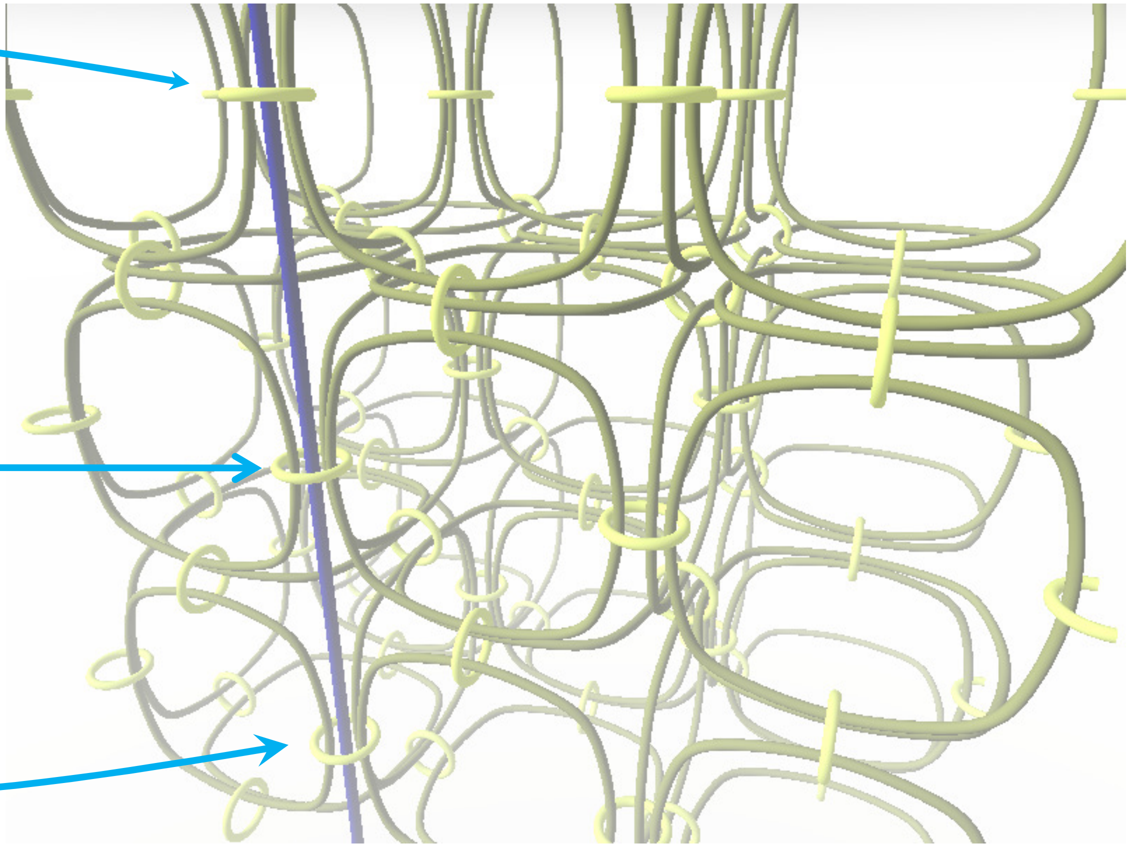
Group must fuse to quantum
Number “a” to cancel “ $\overline{\text{a}}$ ”
and create vacuum

“Vertex” Quasiparticles



“Vertex” Quasiparticles

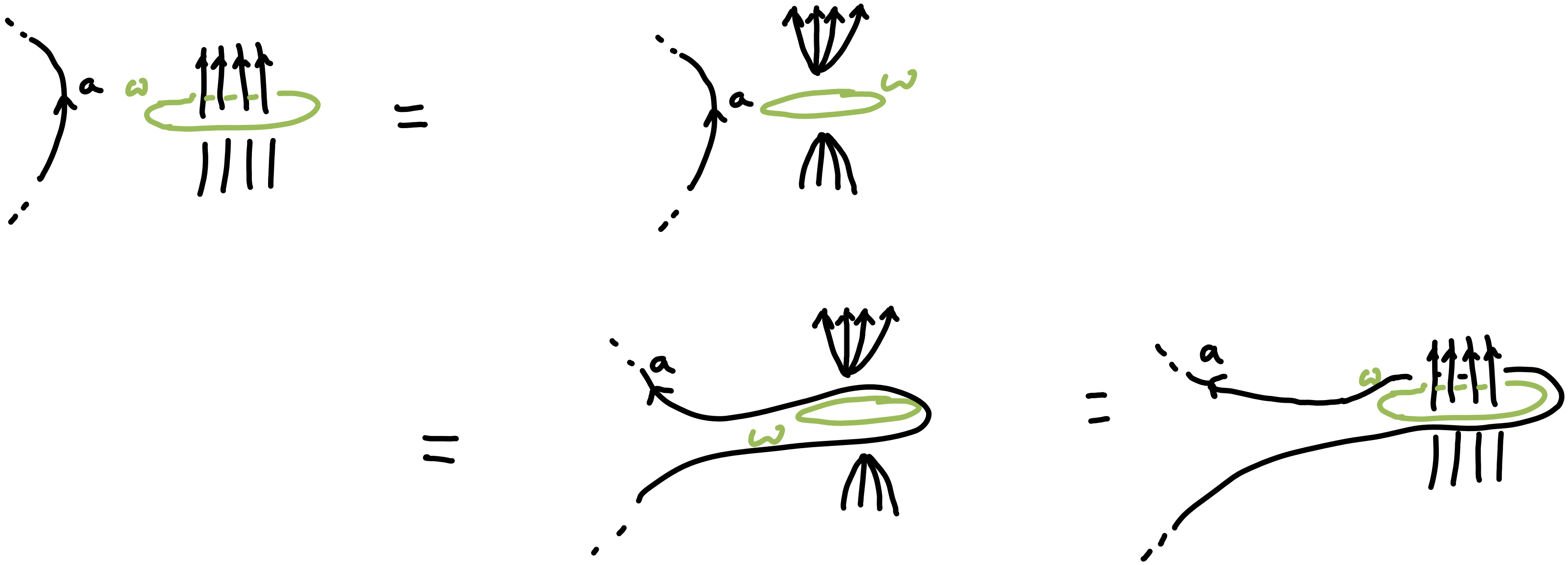
Blue string marked with quantum number $\bar{a}$



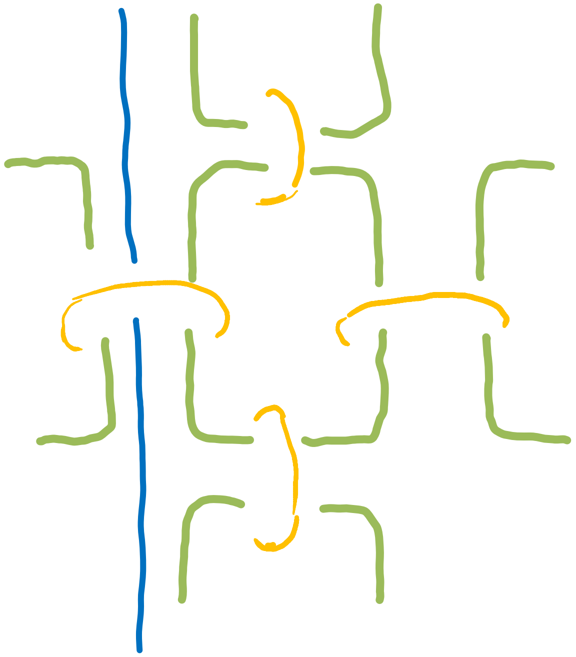
Forces the bonds entering this vertex to fuse to quantum number “a” at three consecutive time steps.

Knot evaluation give ground state partition function in the presence of a “vertex” quasiparticle whose world line follows the specified path

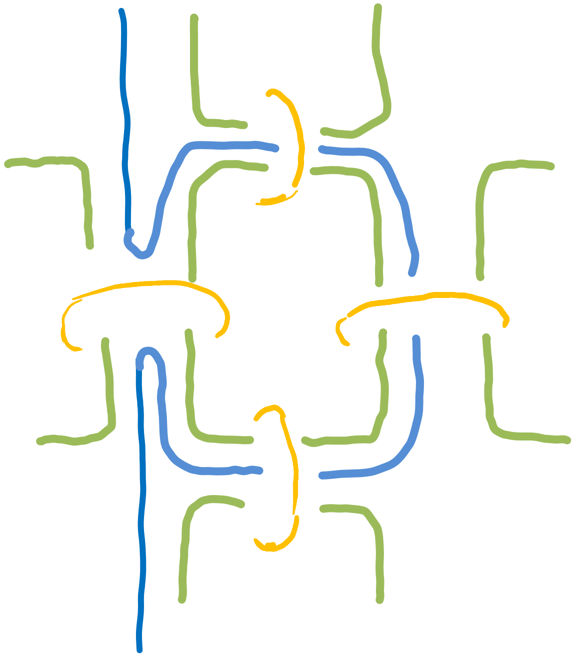
Handleslide From Killing



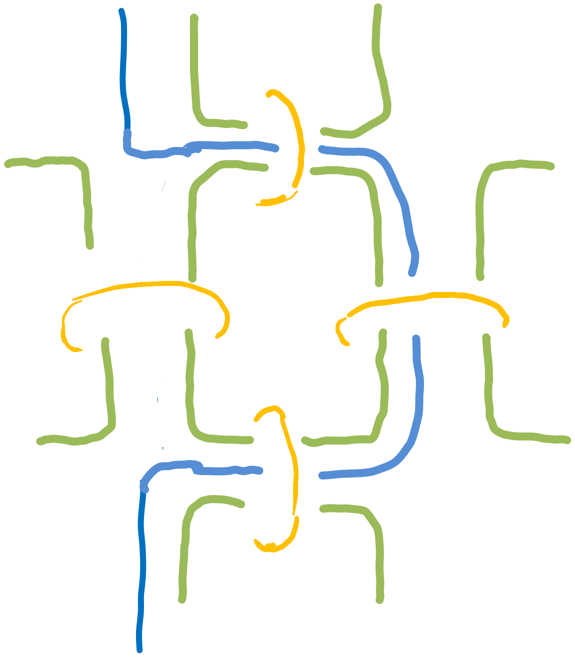
Handleslide



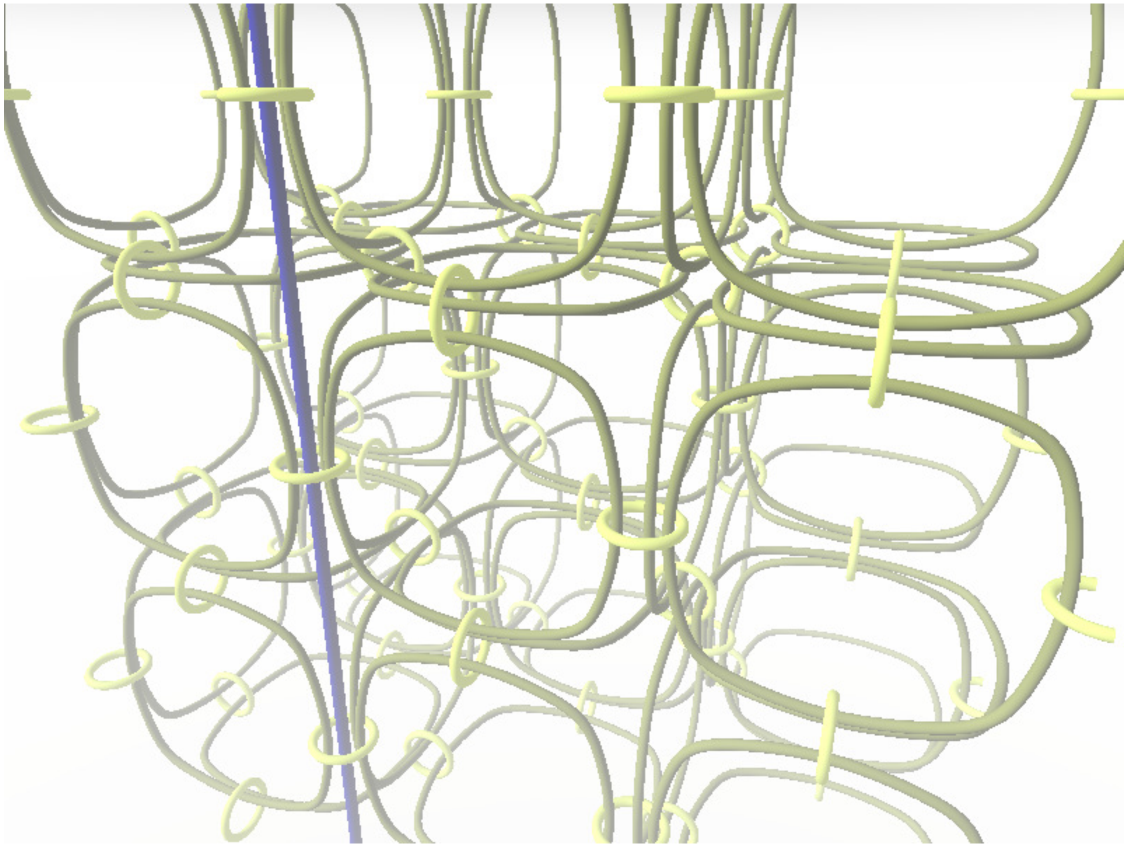
Handleslide



Handleslide



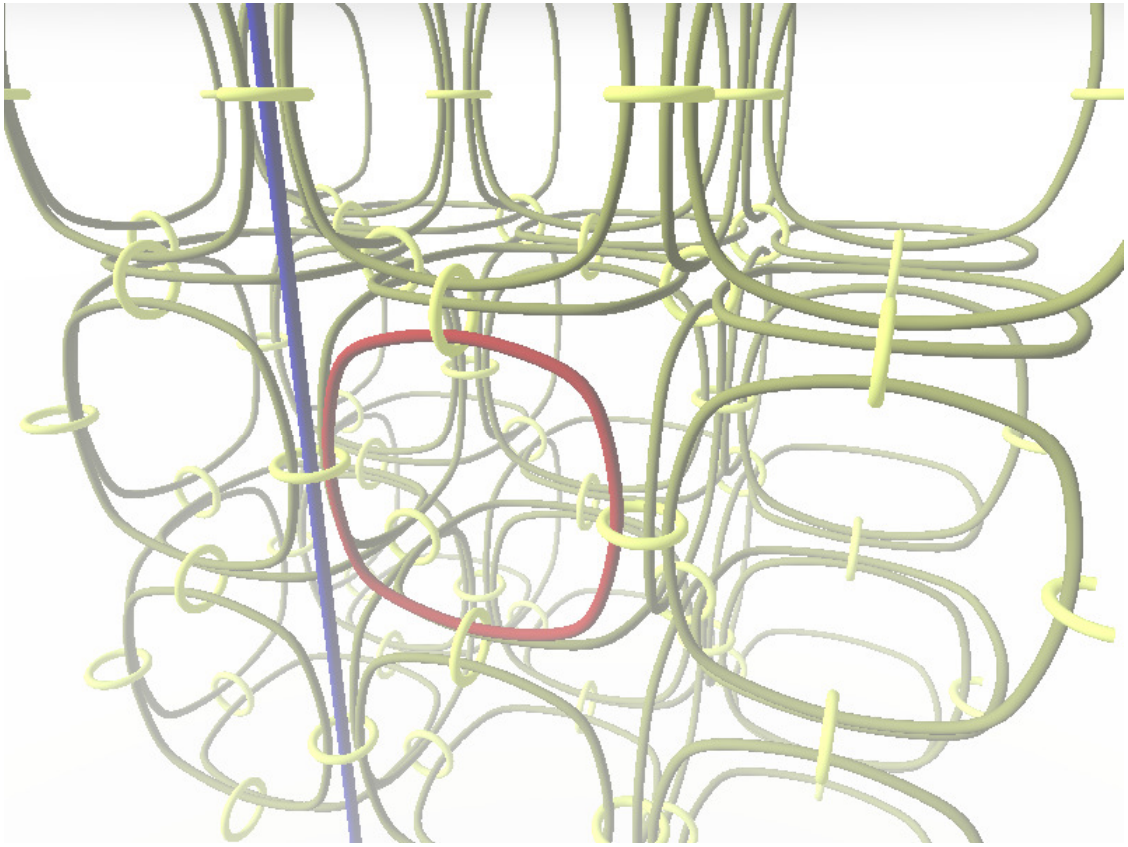
“Vertex” Quasiparticles



Blue string marked with quantum number $\bar{a}$

This gives the ground state partition function in the presence of a quasiparticle at this vertex site.

“Vertex” Quasiparticles



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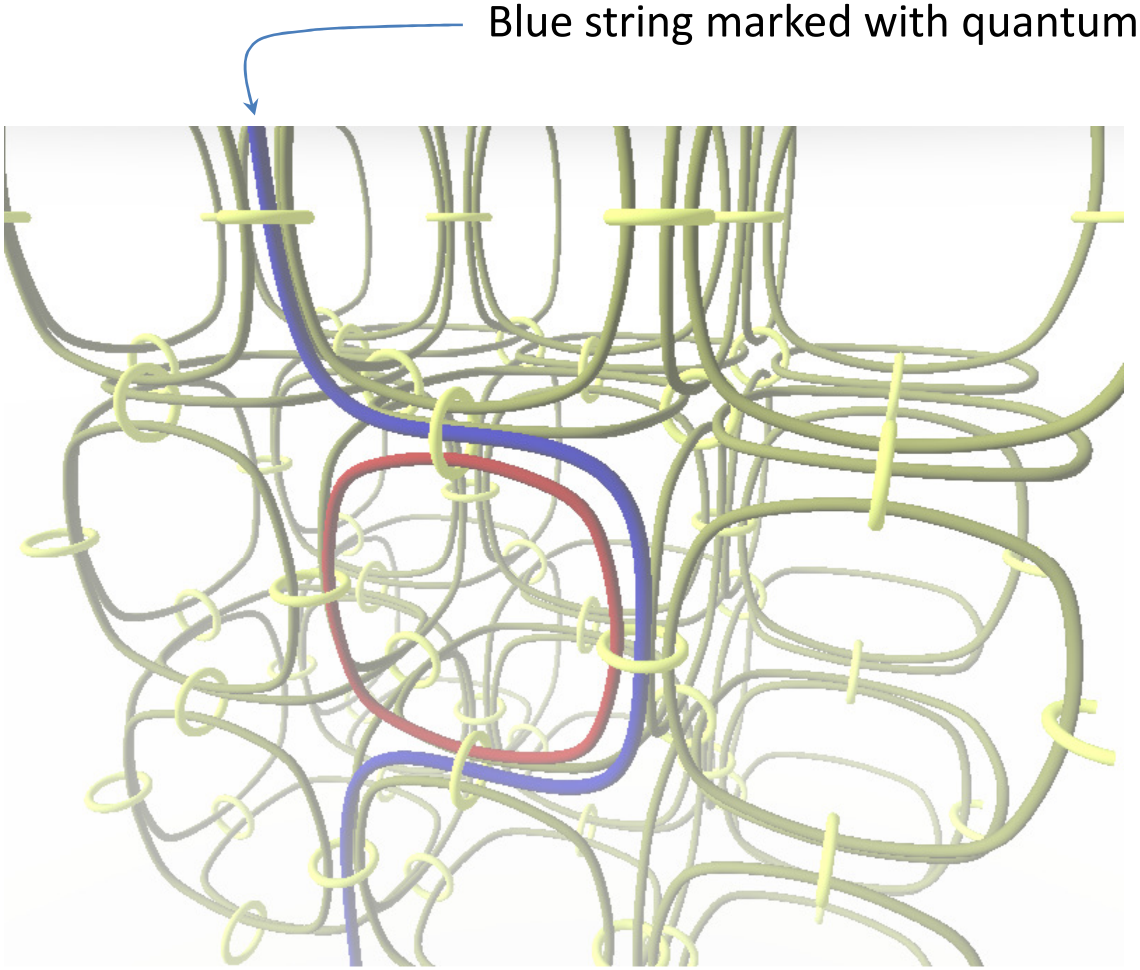
"Vertex" Quasiparticles



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“Vertex” Quasiparticles

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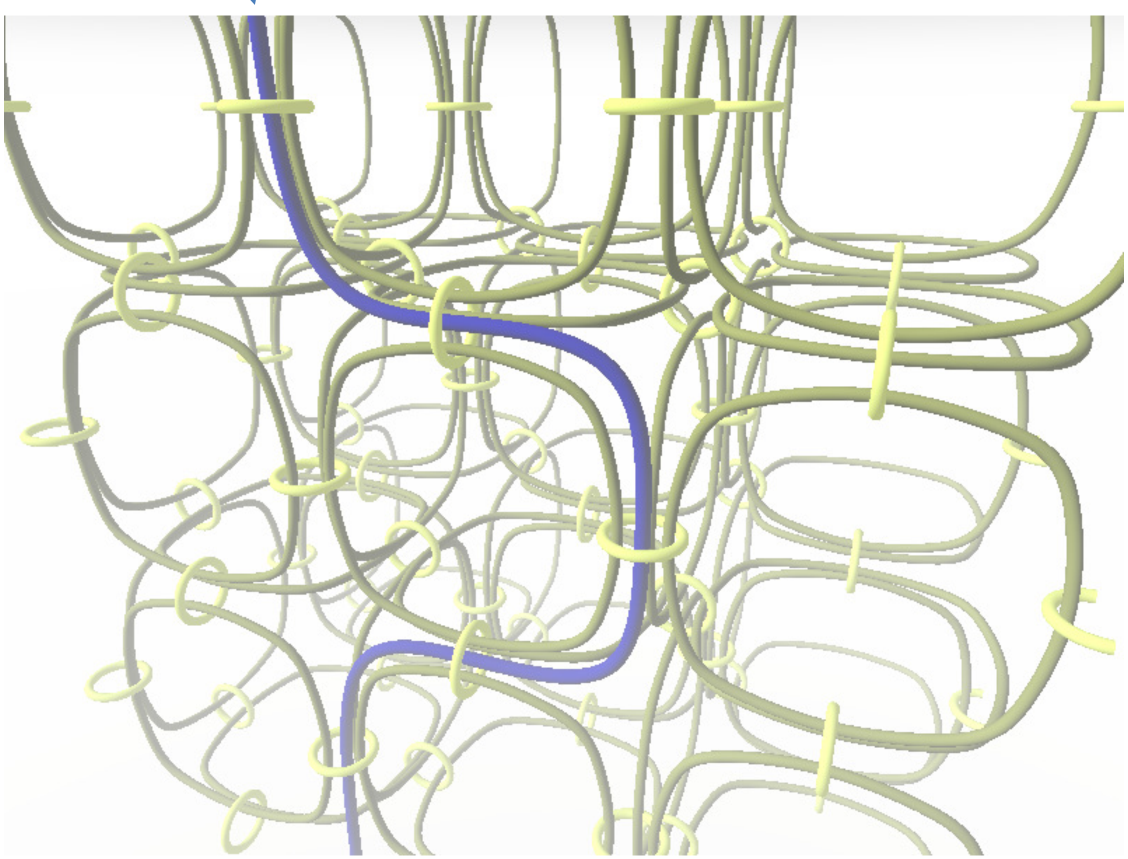
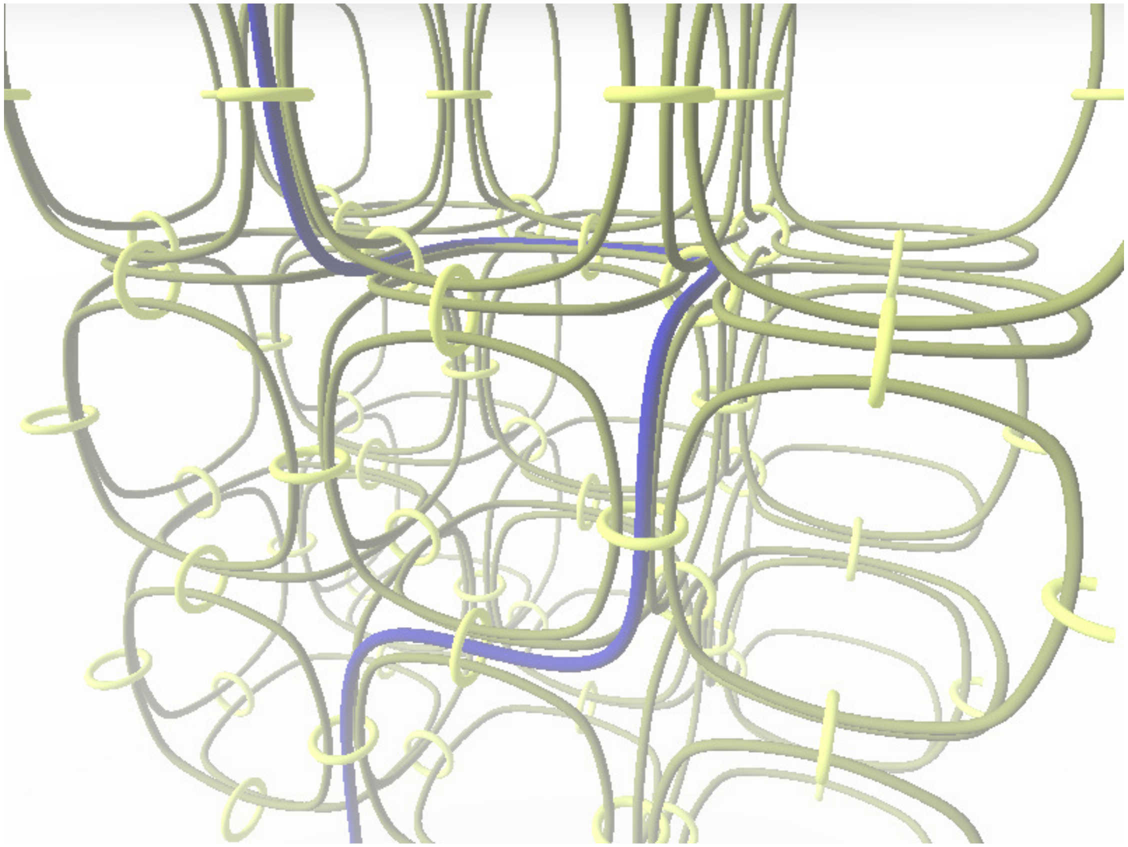


This gives the ground state partition function in the presence of a quasiparticle at this vertex site.

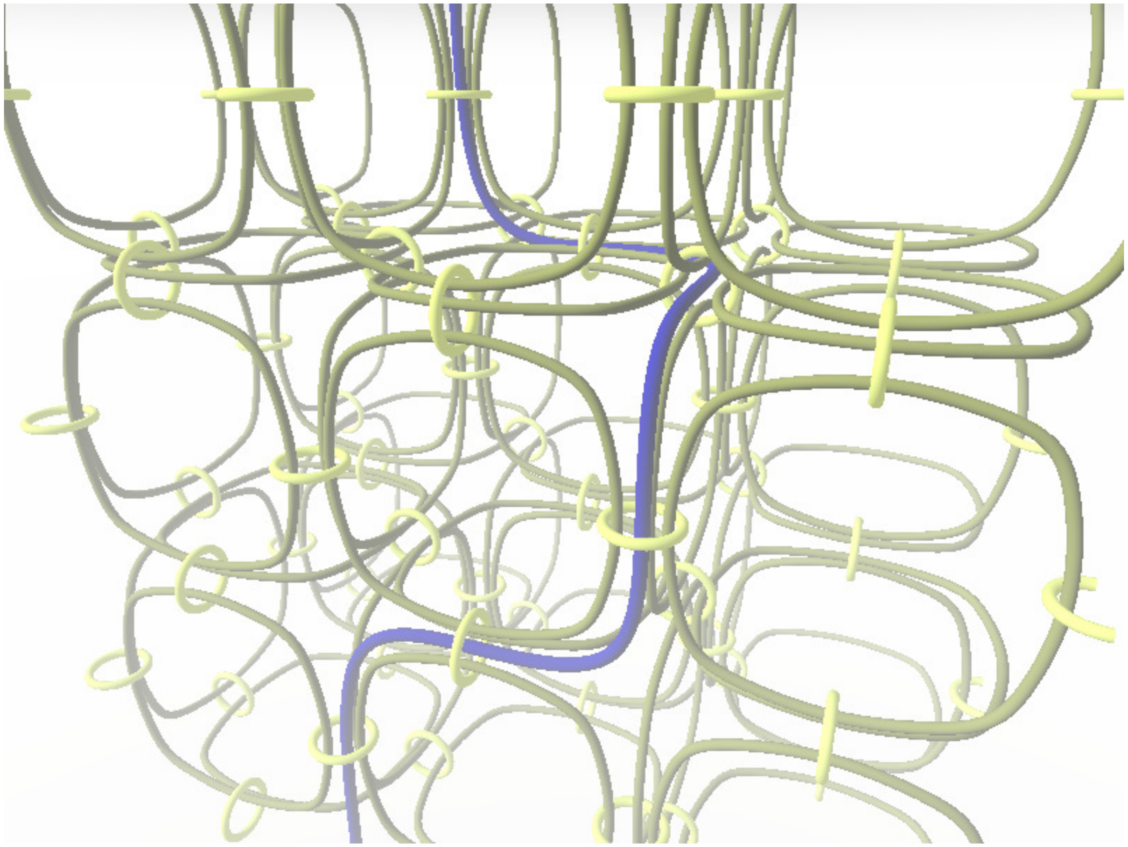
“Vertex” Quasiparticles

Blue string marked with quantum number $\overline{a}$



This gives the ground state partition function in the presence of a quasiparticle at this vertex site.

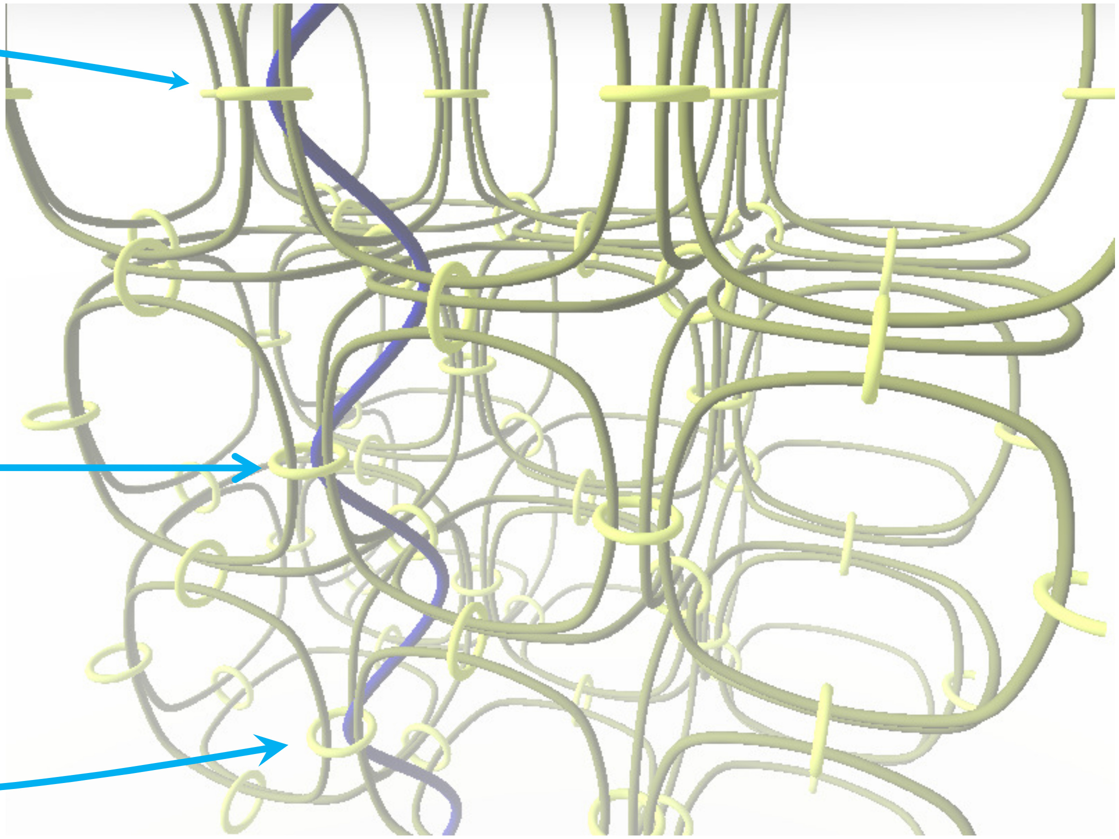
“Vertex” Quasiparticles



Blue string marked with quantum number $\overline{a}$

This gives the ground state partition function in the presence of a quasiparticle at this vertex site.

“Plaquette” or “Mirror” Quasiparticles

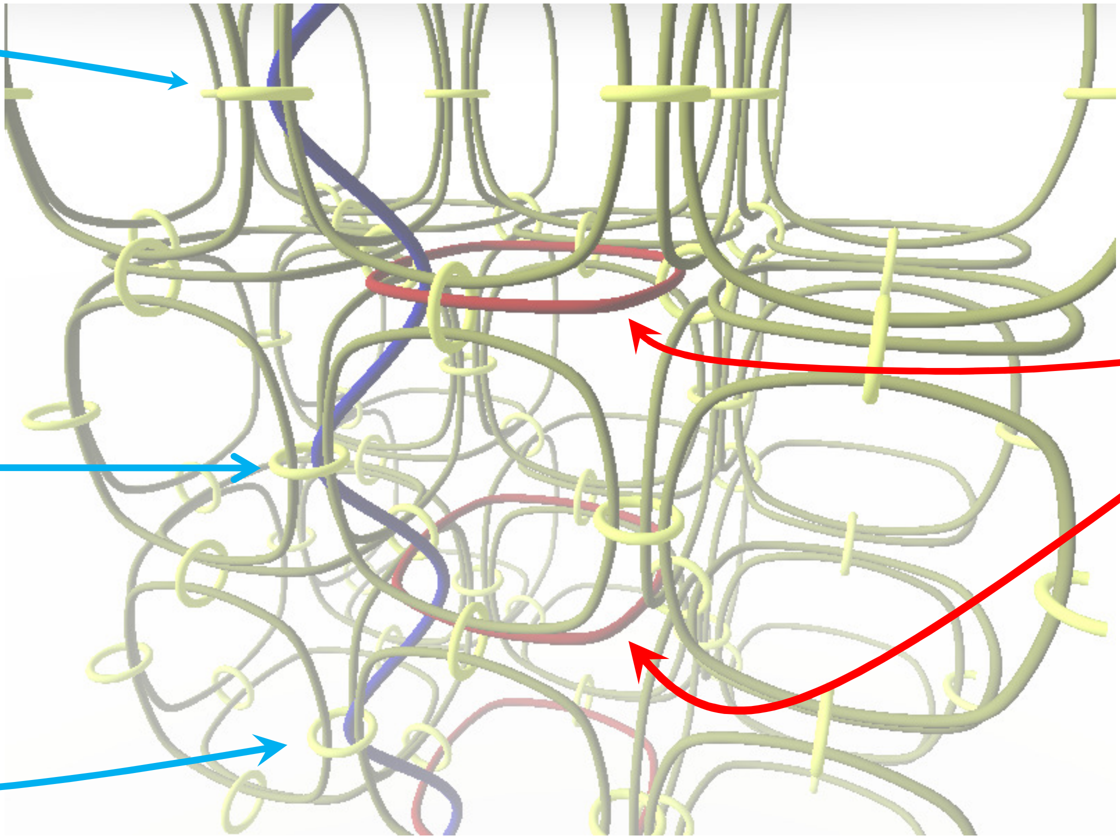


Blue string marked with quantum number $\overline{a}$

Forces the bonds entering this vertex to fuse to quantum number “a” at three consecutive time steps.

“Plaquette” or “Mirror” Quasiparticles

Blue string marked with quantum number $\bar{a}$



Forces the bonds entering this vertex to fuse to quantum number “a” at three consecutive time steps.

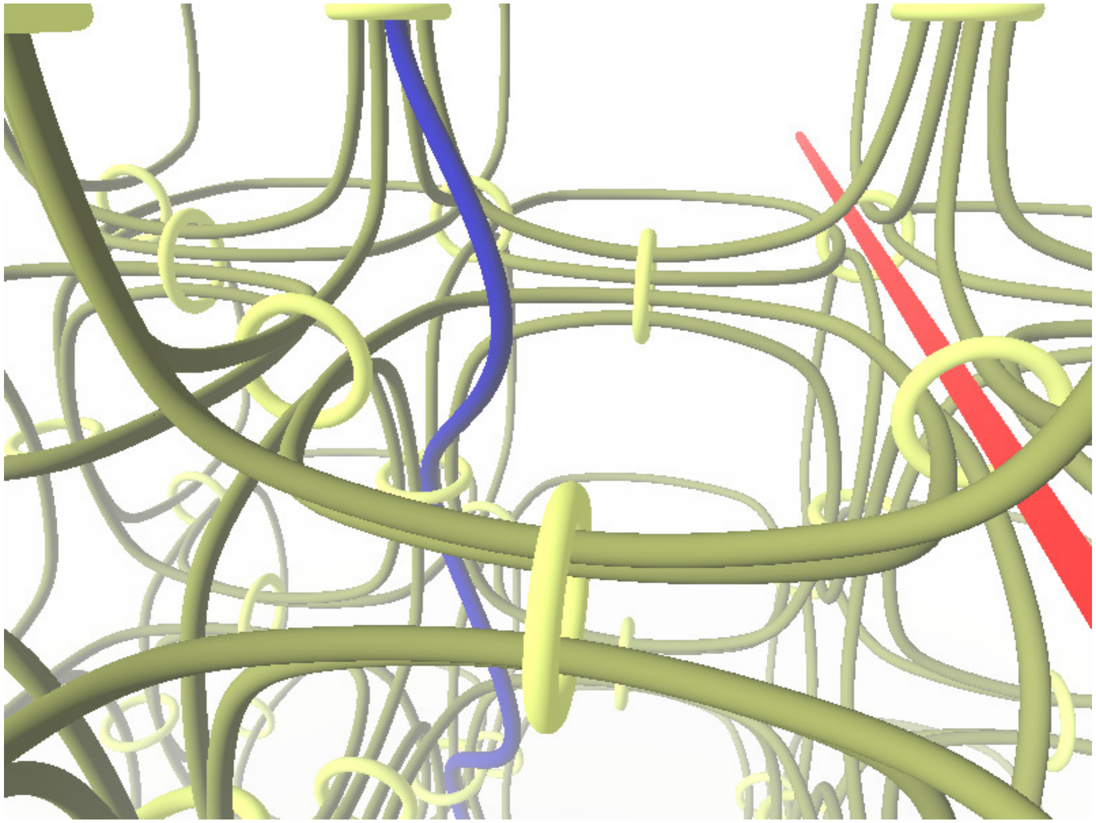
Also forces Flux “a” Through These plaquettes

Mirror quasiparticles must go *through* plaquettes when they cross between cells.

Proofs By Handlesliding

- The Vertex and Mirror particles are two independent sectors
- Vertex particles have the statistics of the original Chern-Simons model that defines our link evaluation
- Mirror particles have the opposite chirality

Mirror Quasiparticles

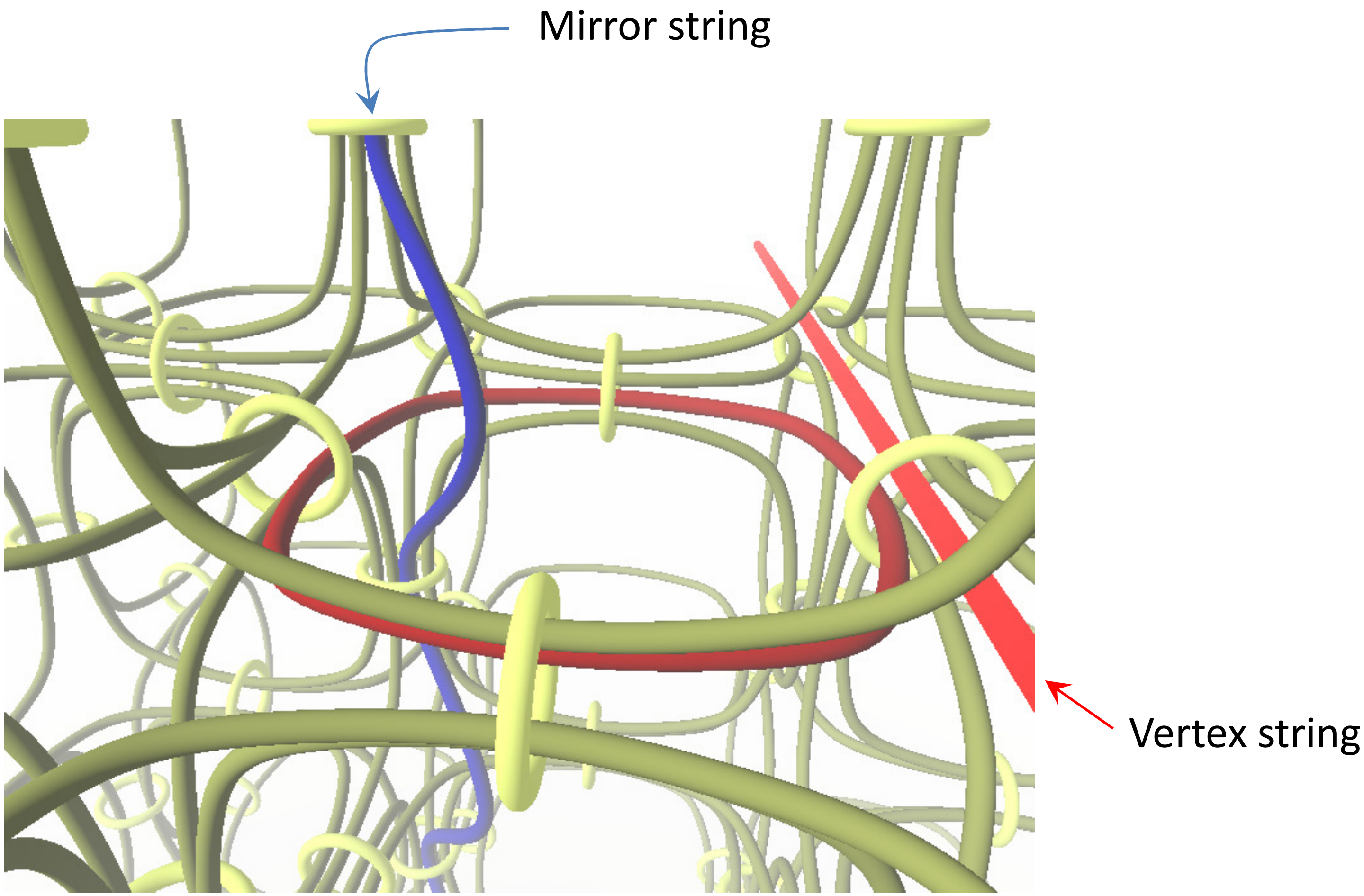


Mirror string

Vertex string

Vertex and Mirror can pass through each other freely

Mirror Quasiparticles



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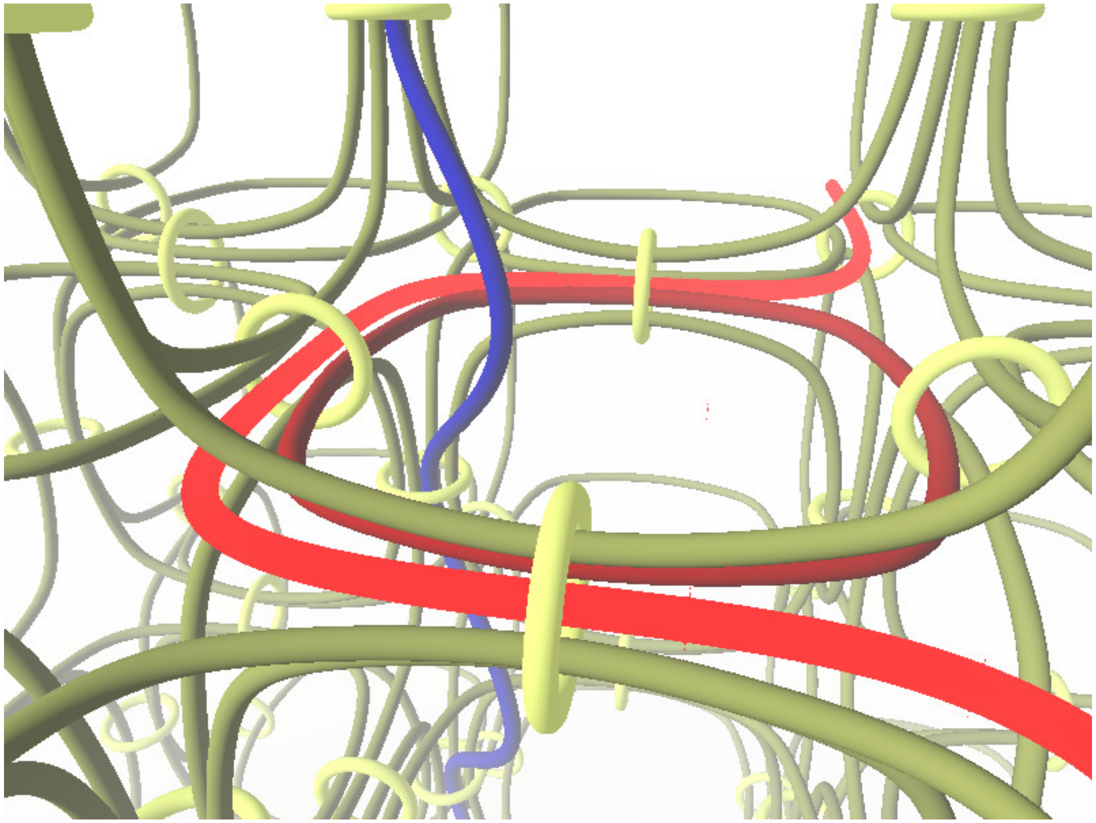

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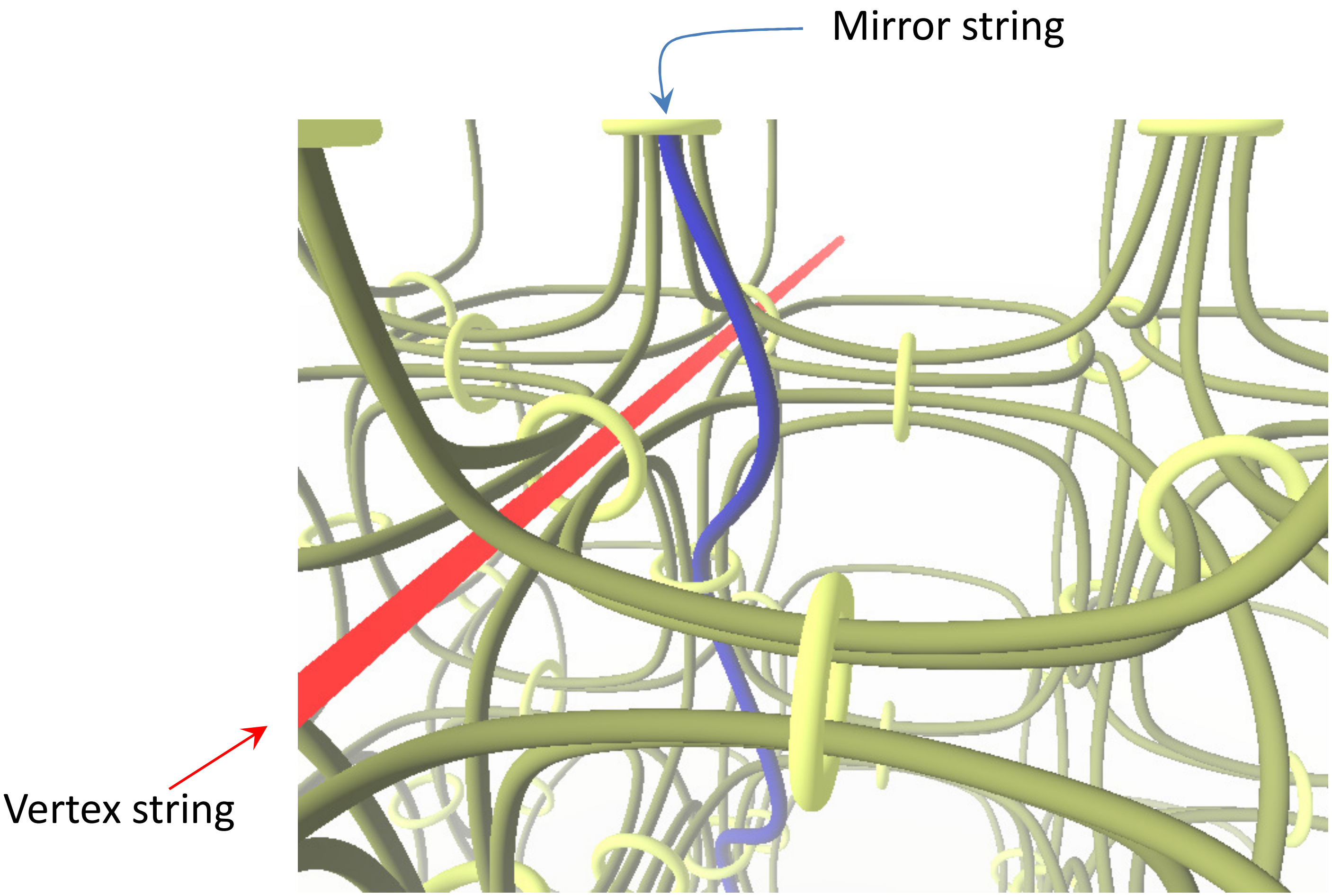

Vertex and Mirror can pass through each other freely

They are independent sectors !

Proofs By Handlesliding

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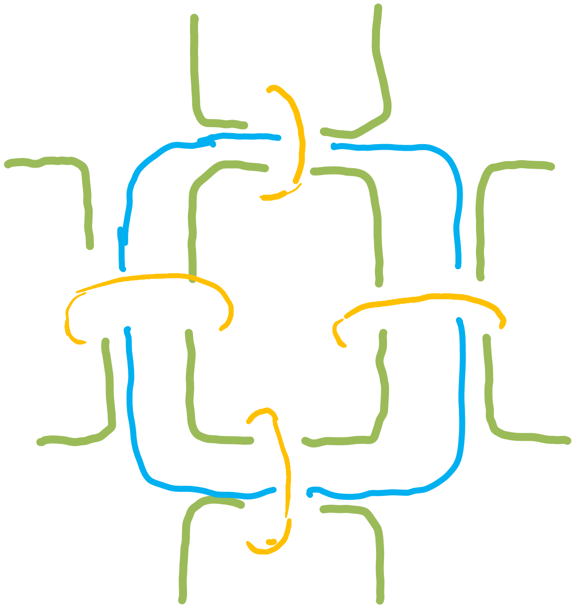

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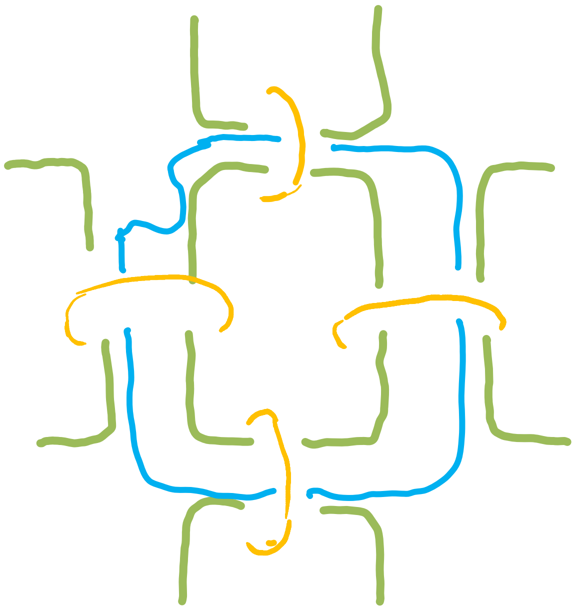

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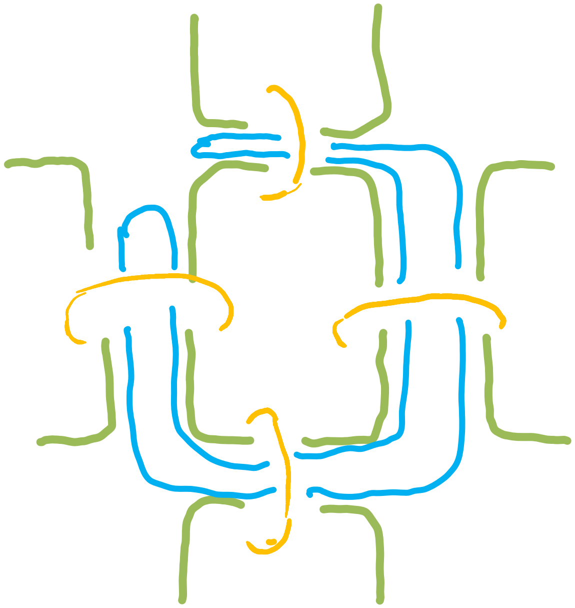

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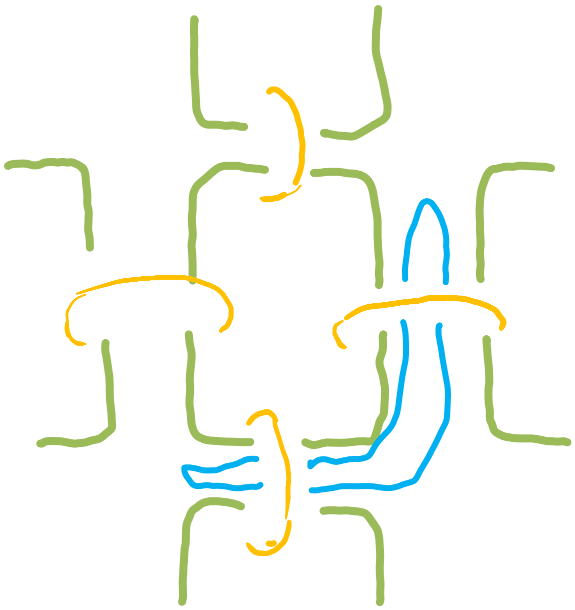

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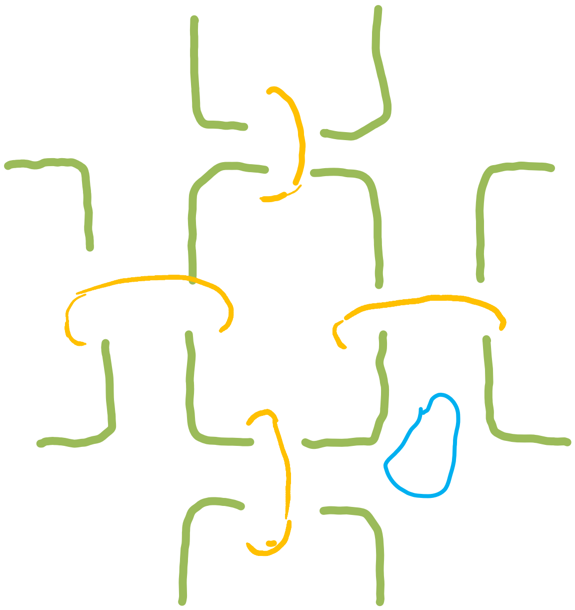
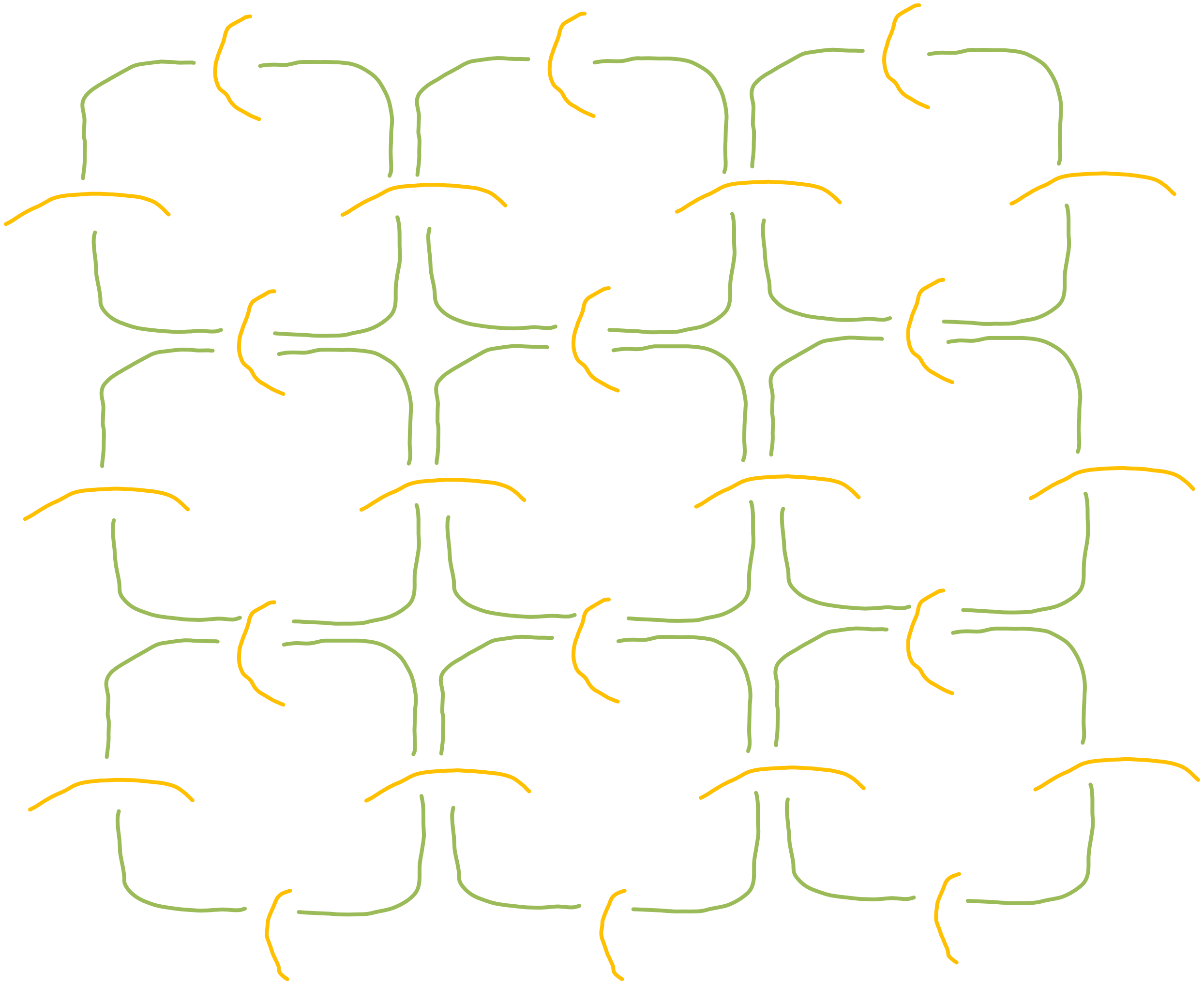


“Vertex” Quasiparticles

Can handleslide everything to a single plane – but must keep track of over and undercrossings



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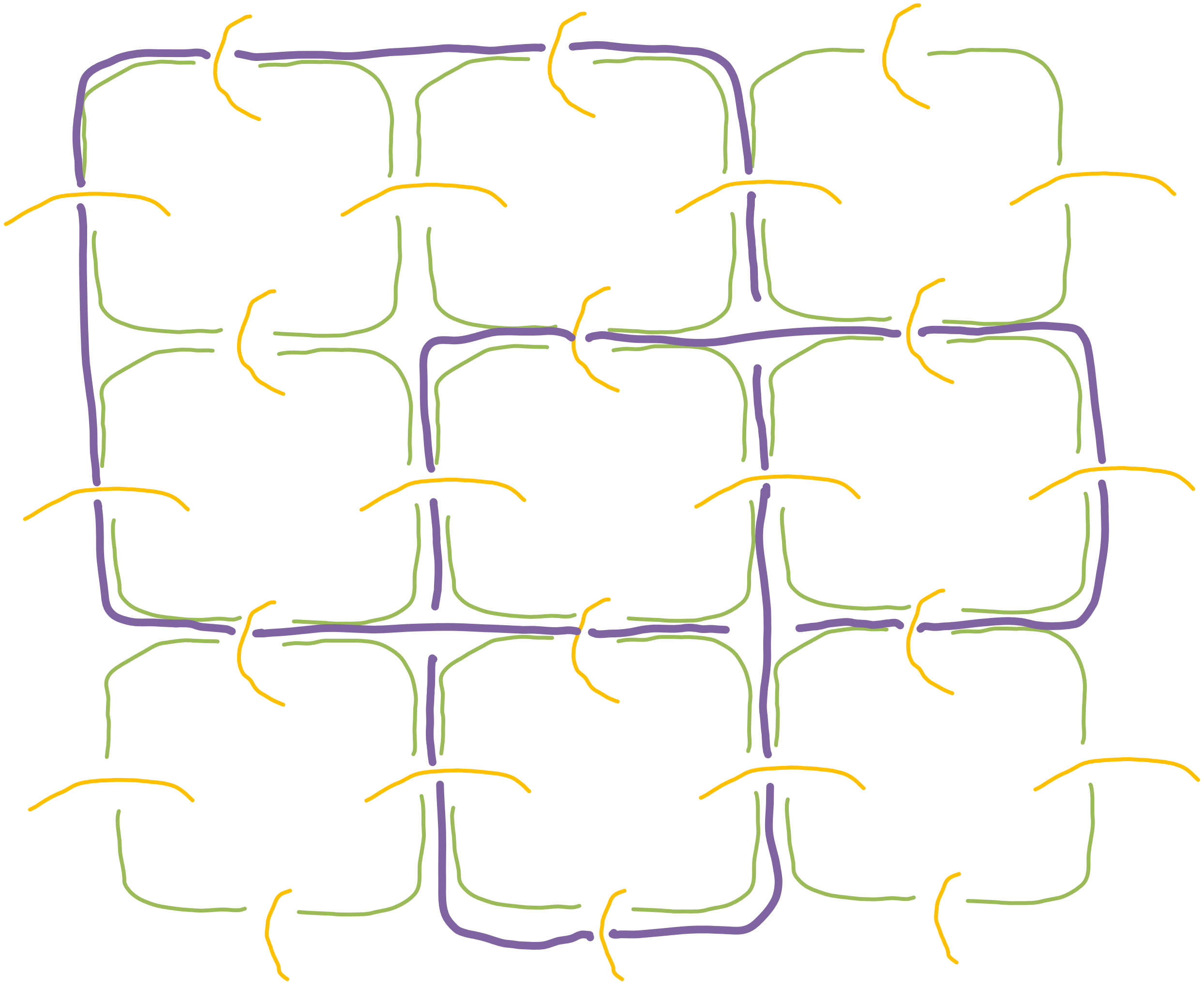

“vertex” particle Trefoil tied to chainmail

“Vertex” Quasiparticles

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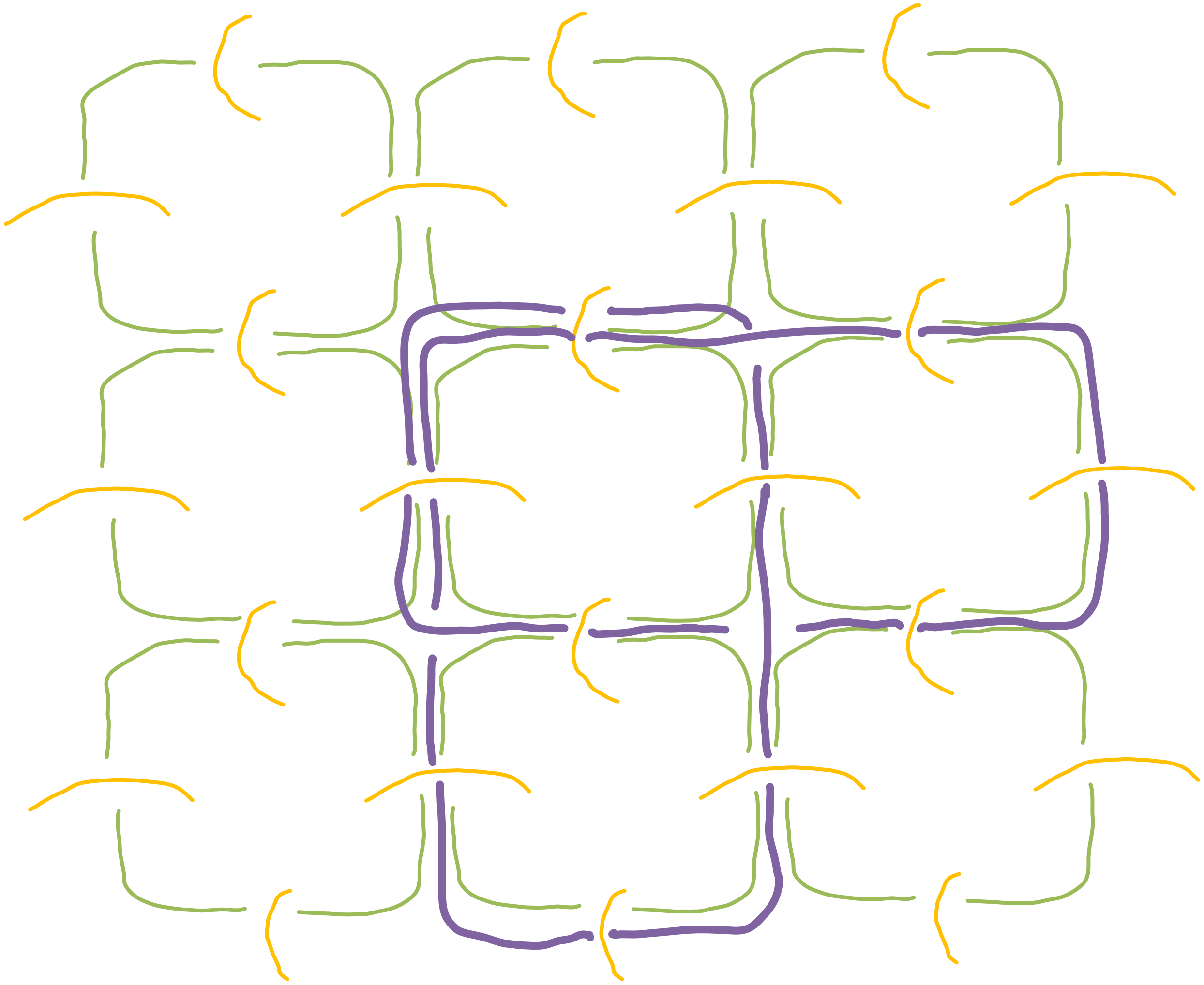

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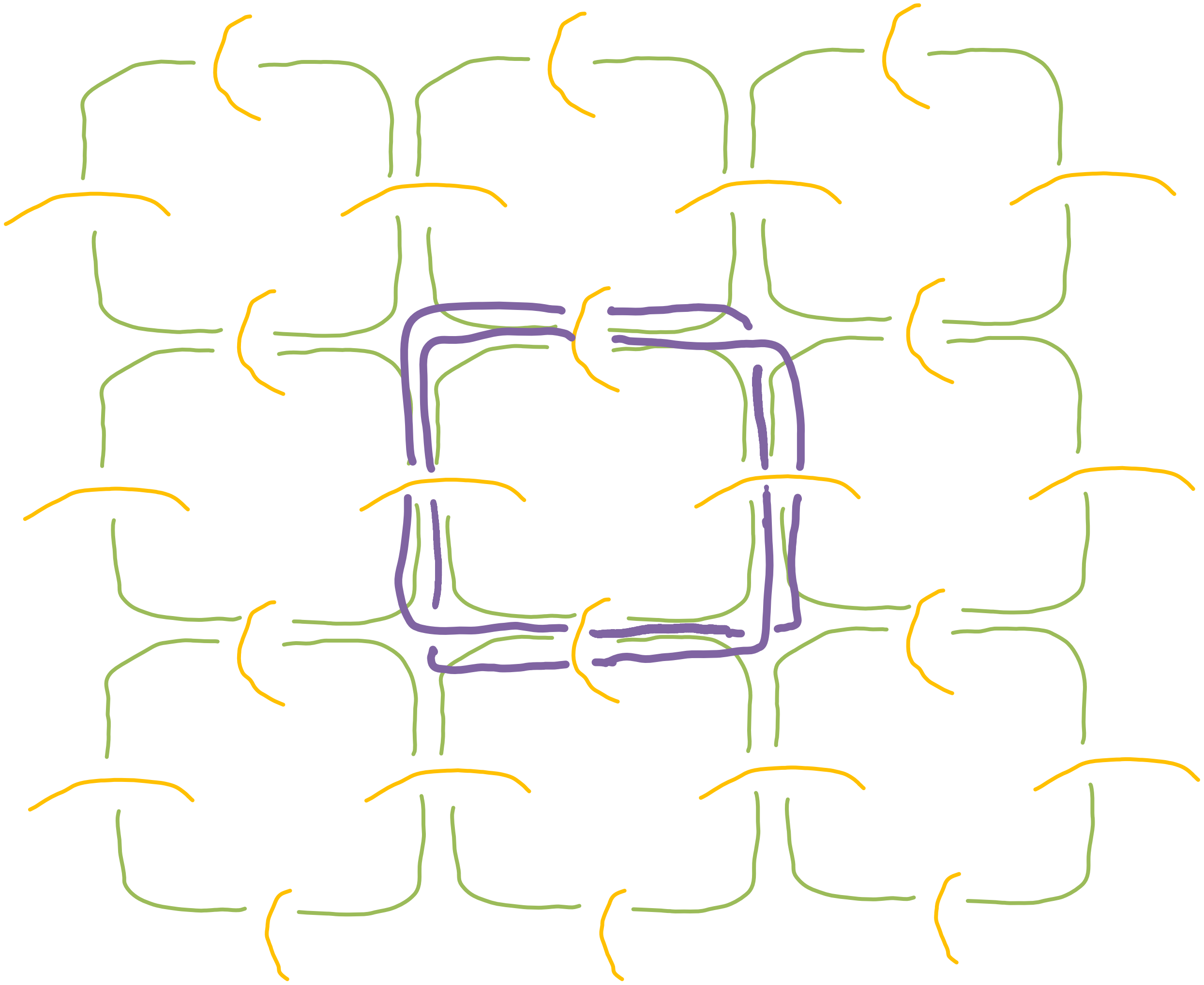

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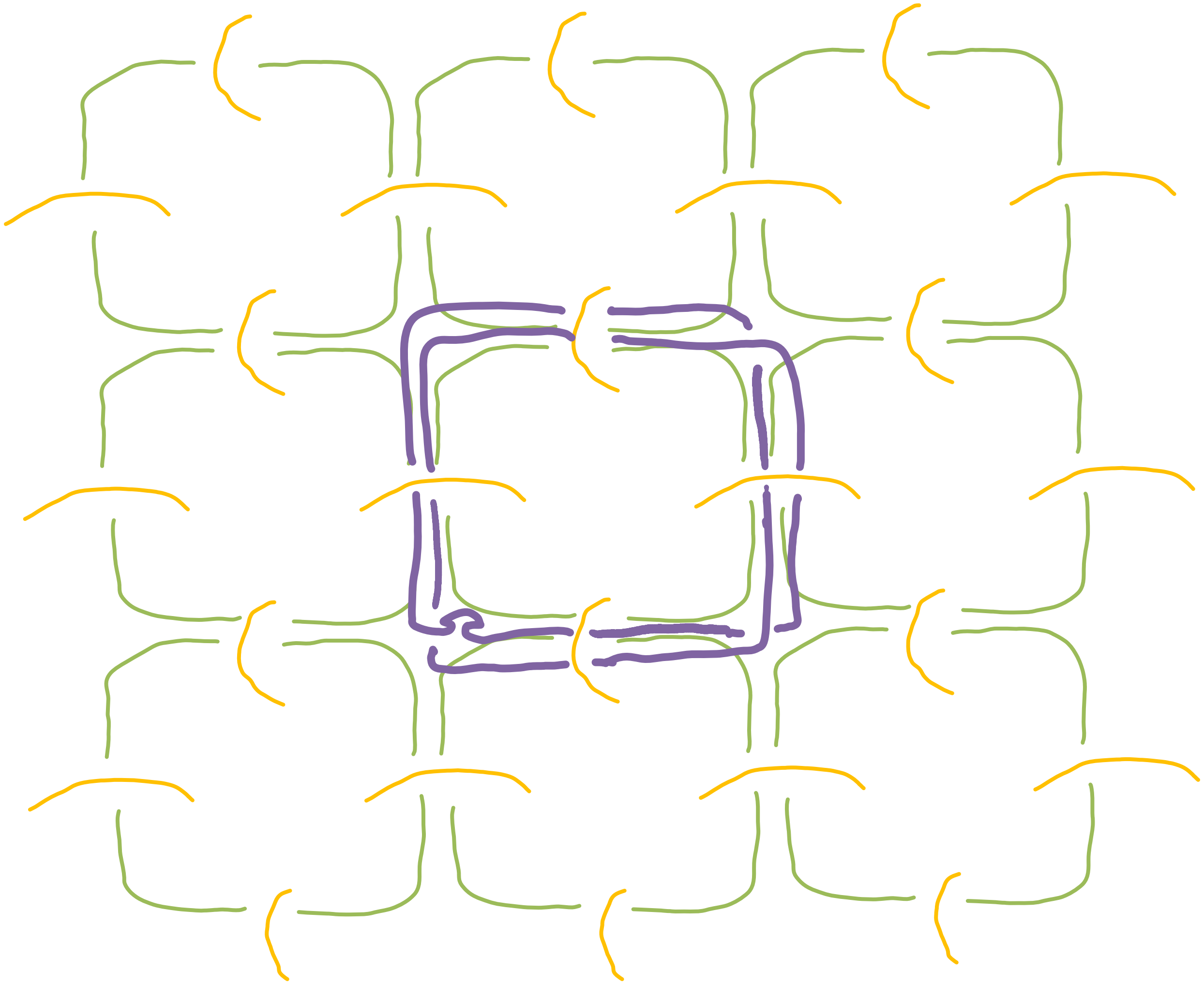

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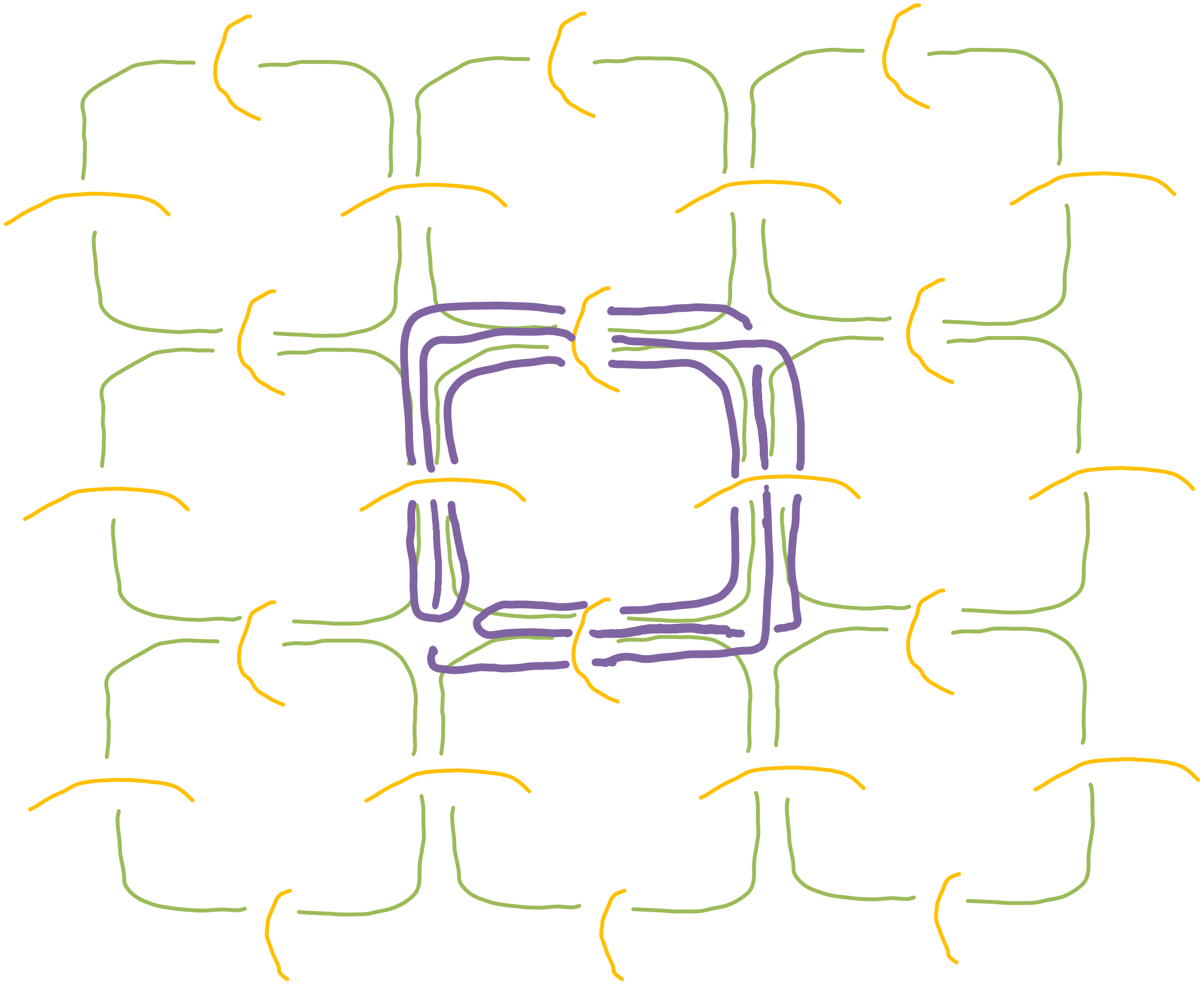

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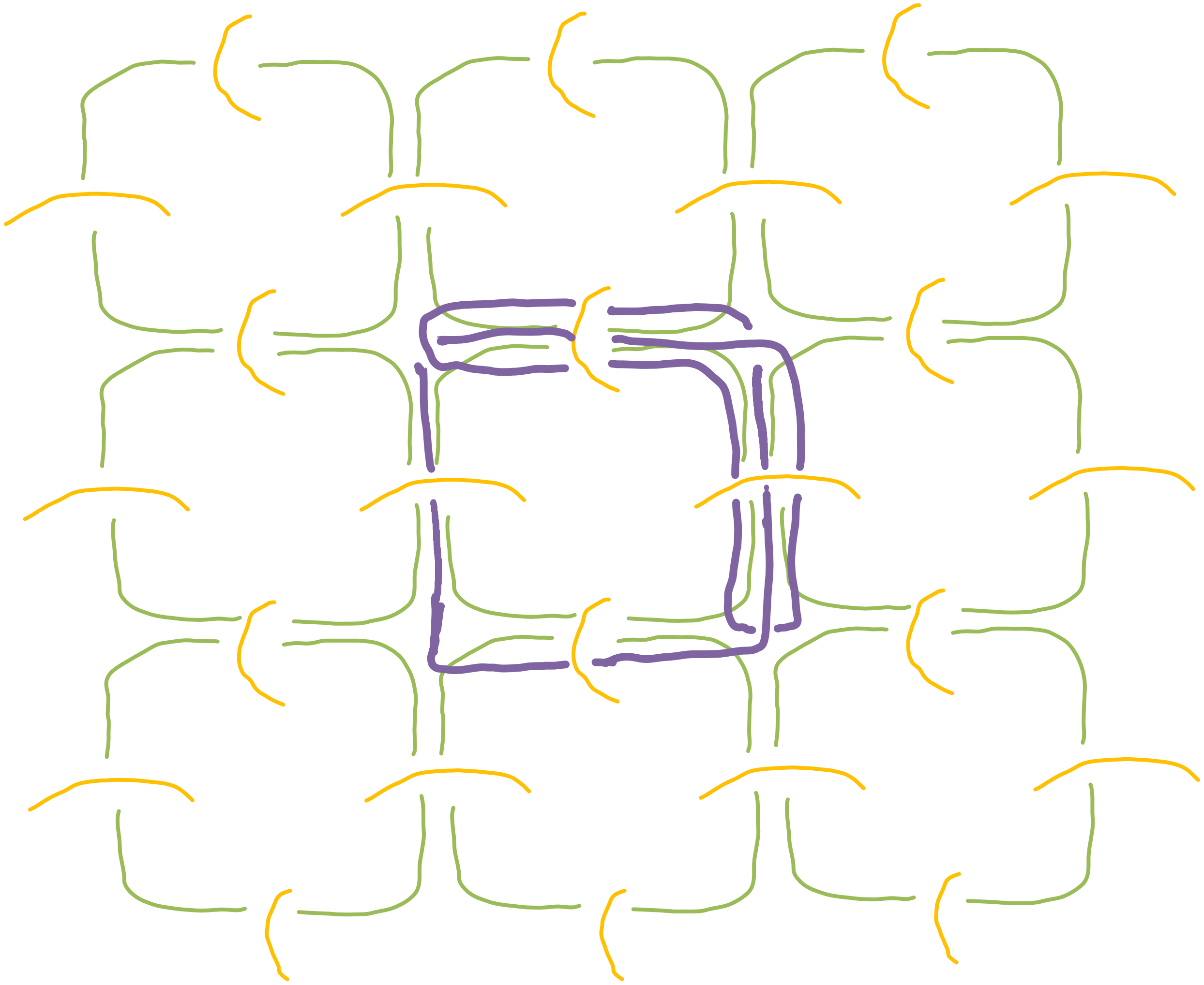

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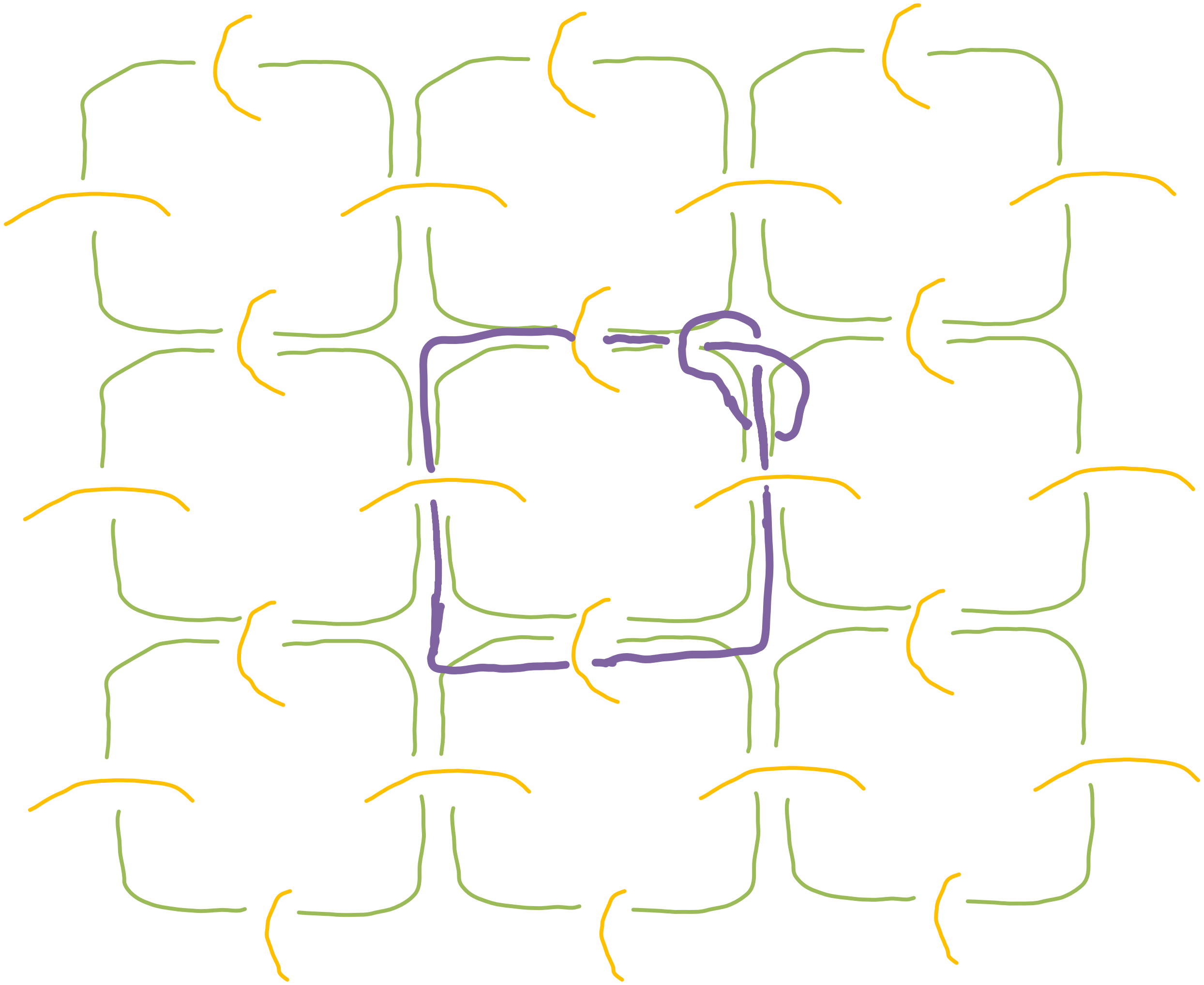

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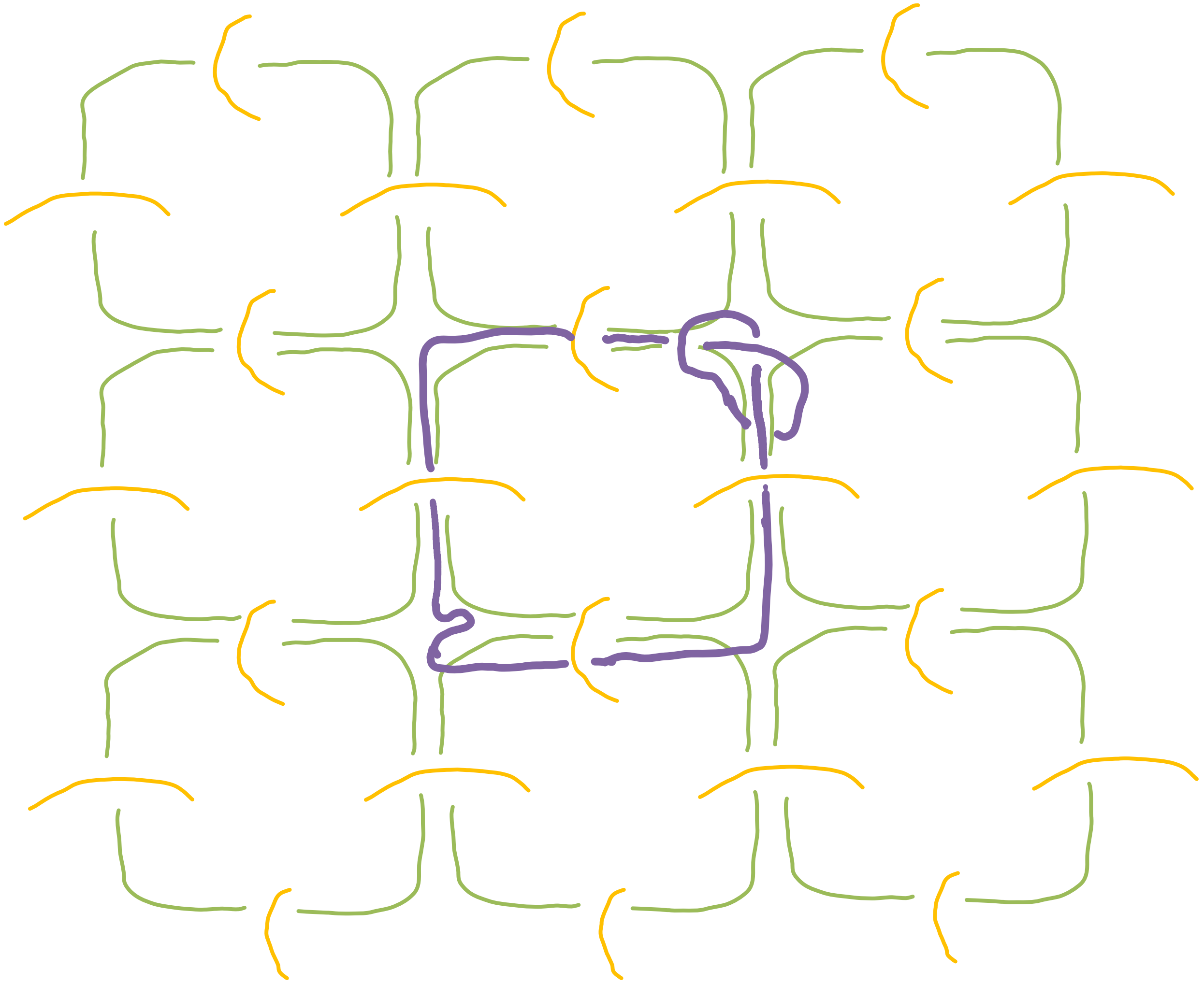

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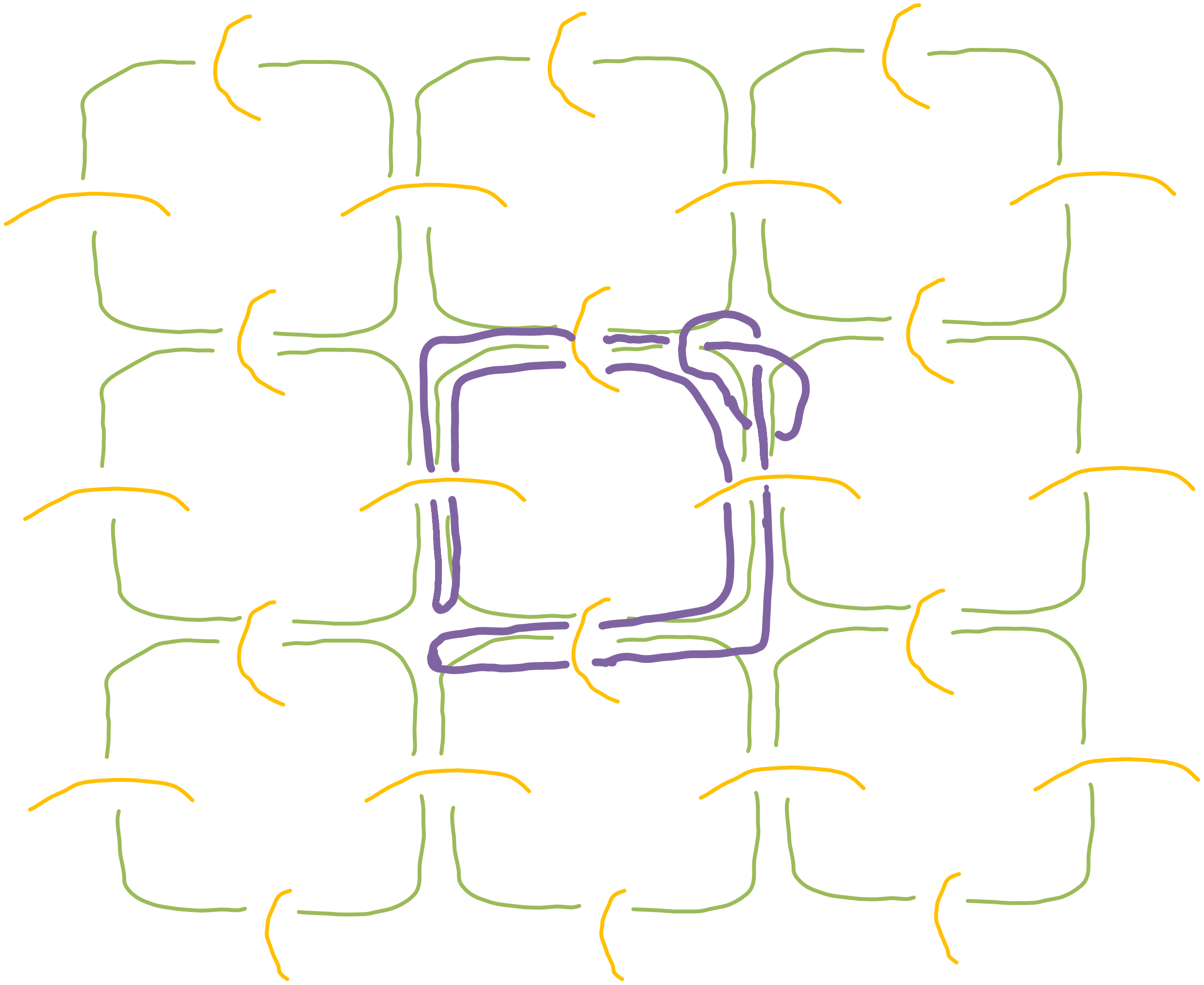

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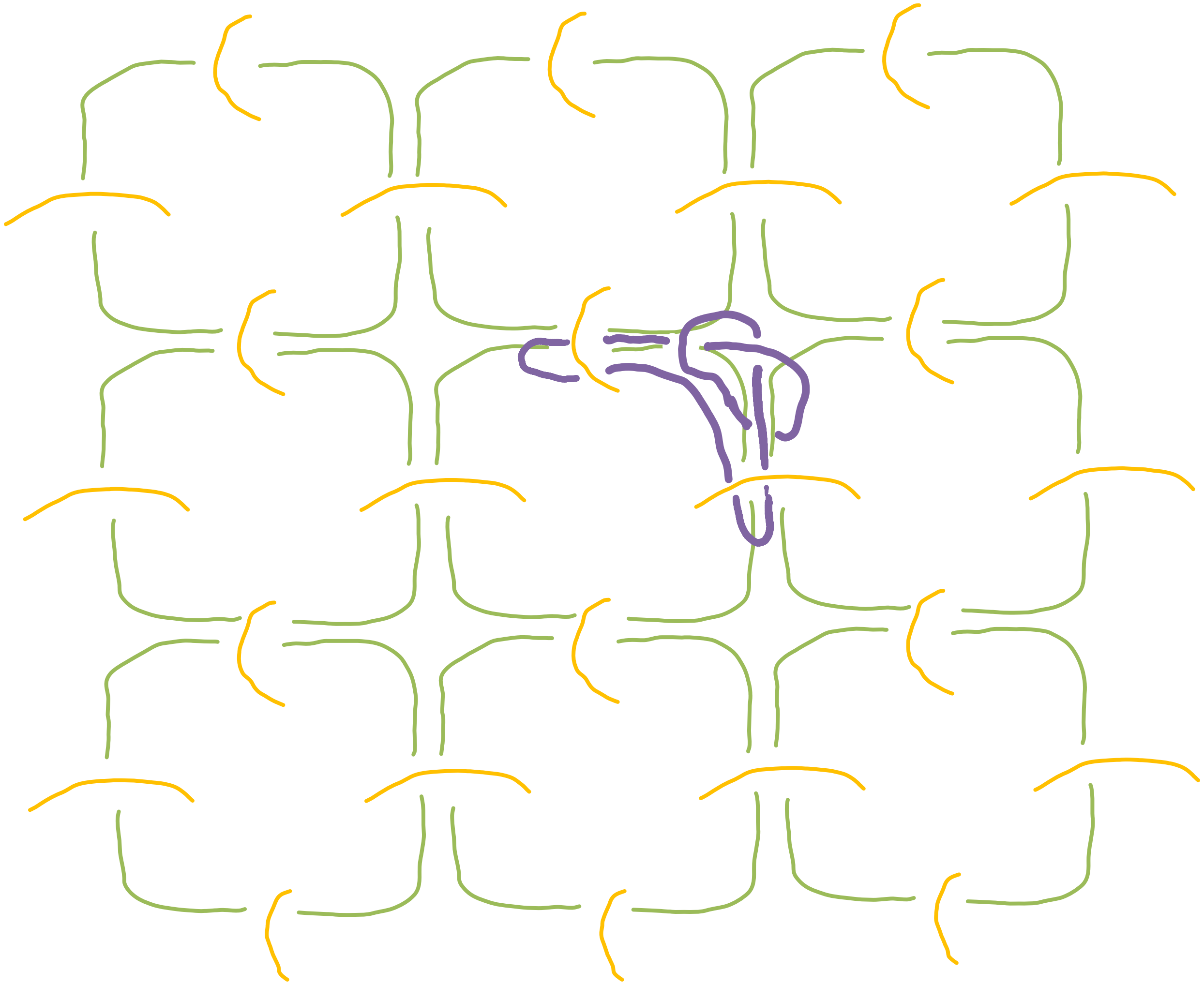

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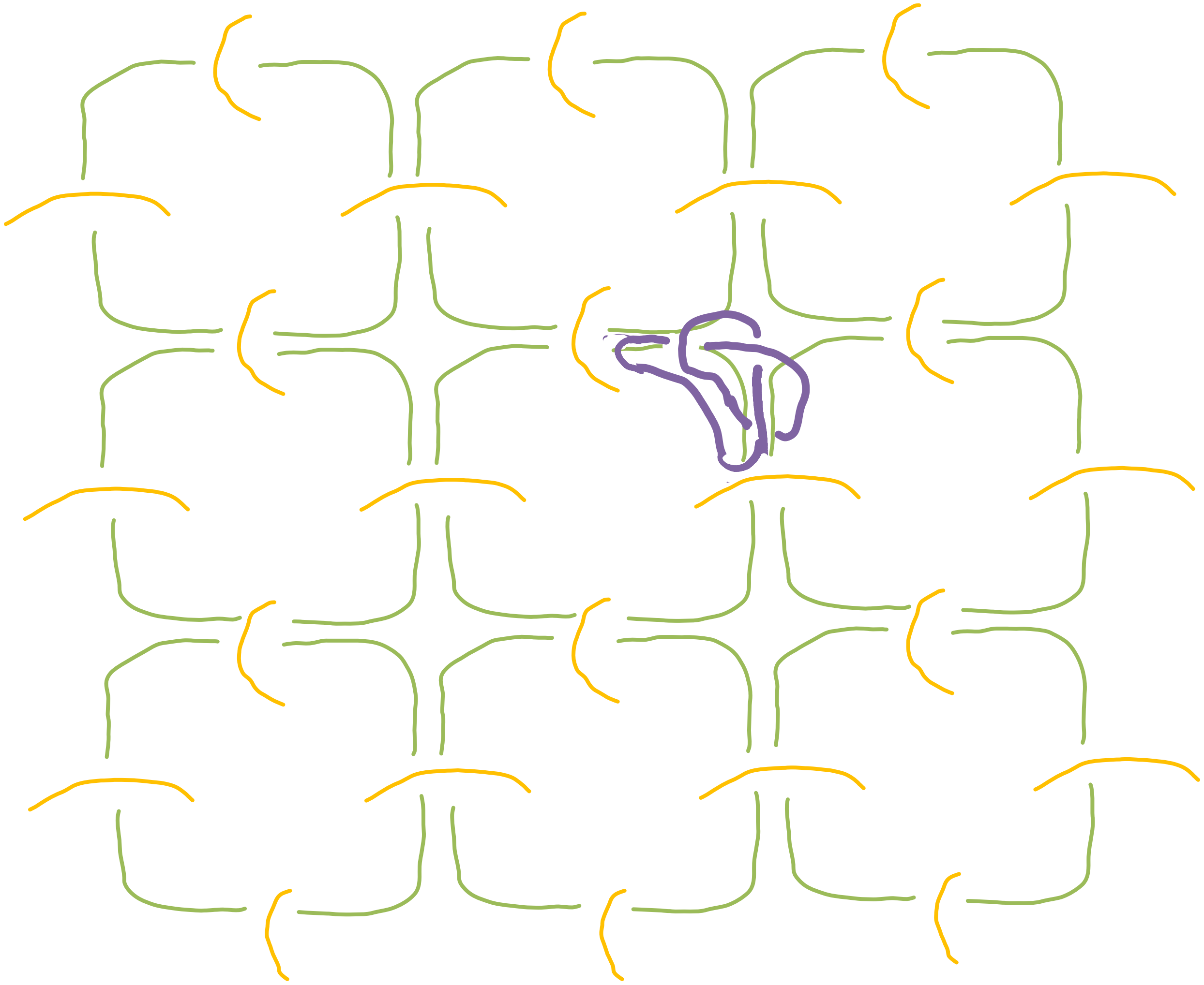

“vertex” particle Trefoil tied to chainmail

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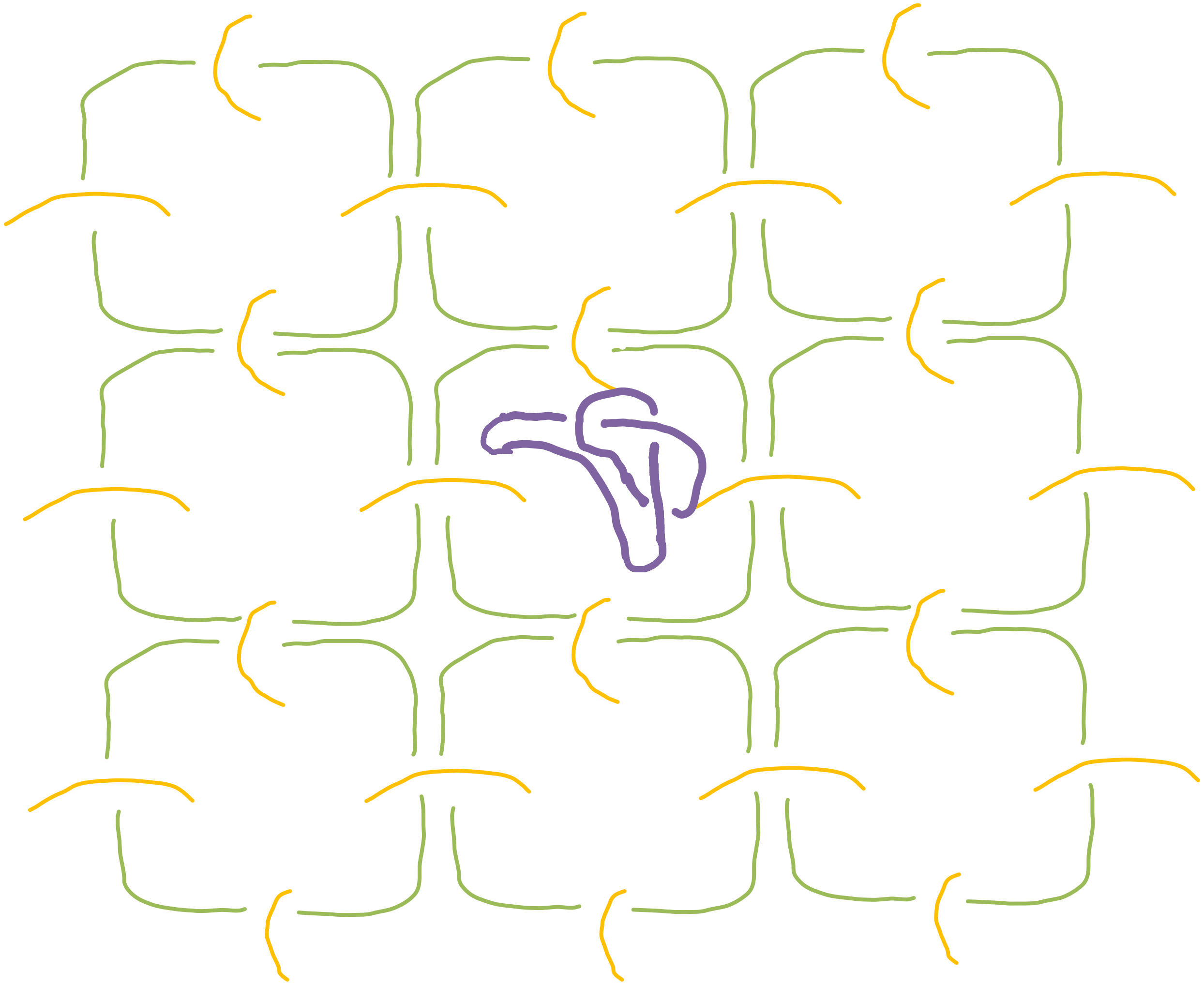

“vertex” particle Trefoil tied to chainmail

Knot can be handleslid off the chainmail scaffolding

The vertex particles have the same statistics as the Chern-Simons theory we used to define the link evaluation !

Proofs By Handlesliding

- The Vertex and Mirror particles are two independent sectors
- Vertex particles have the statistics of the original Chern-Simons model that defines our link evaluation
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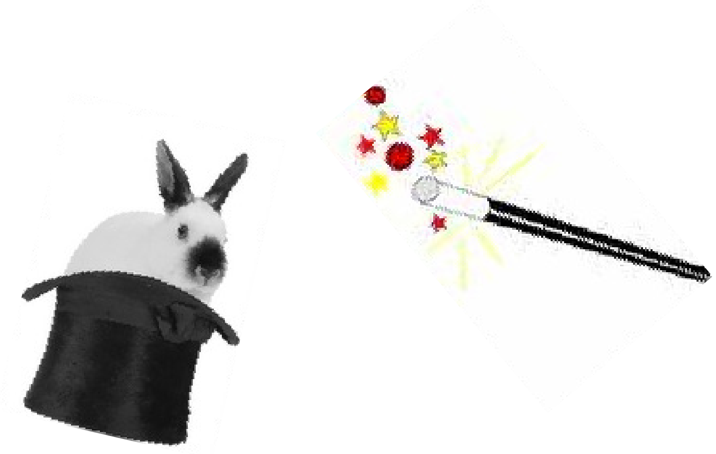
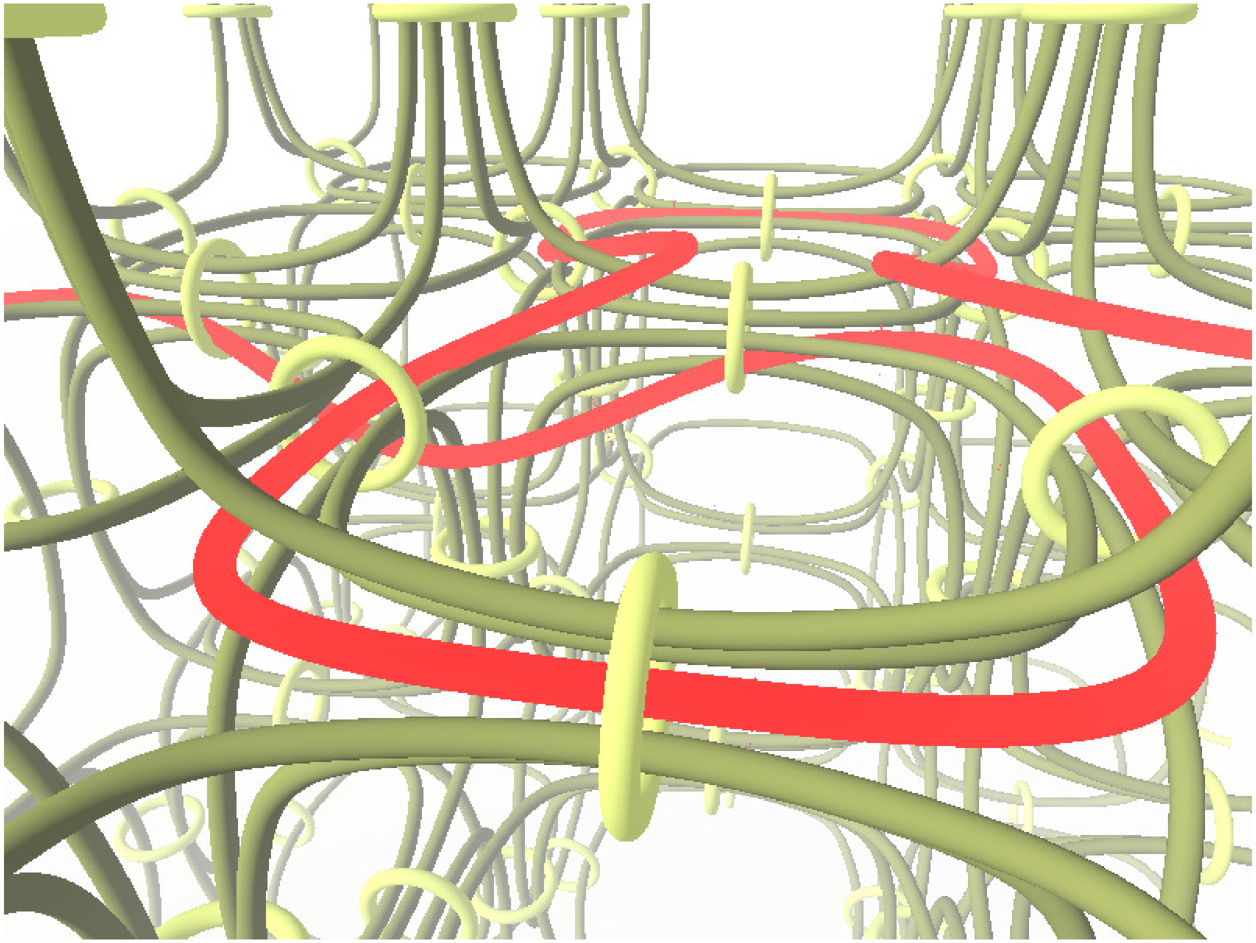
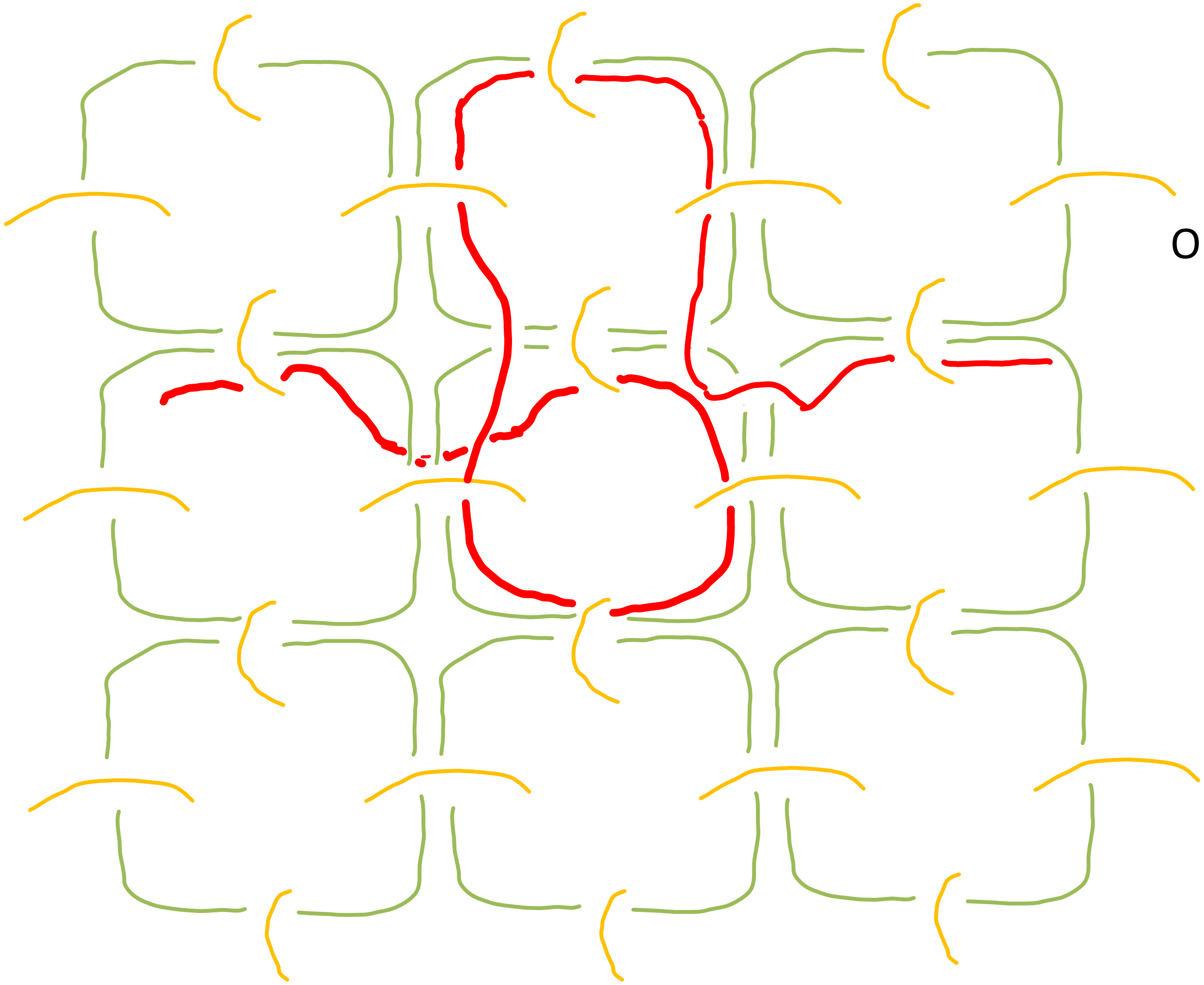


Mirror Quasiparticles



On scaffolding

Mirror Quasiparticles



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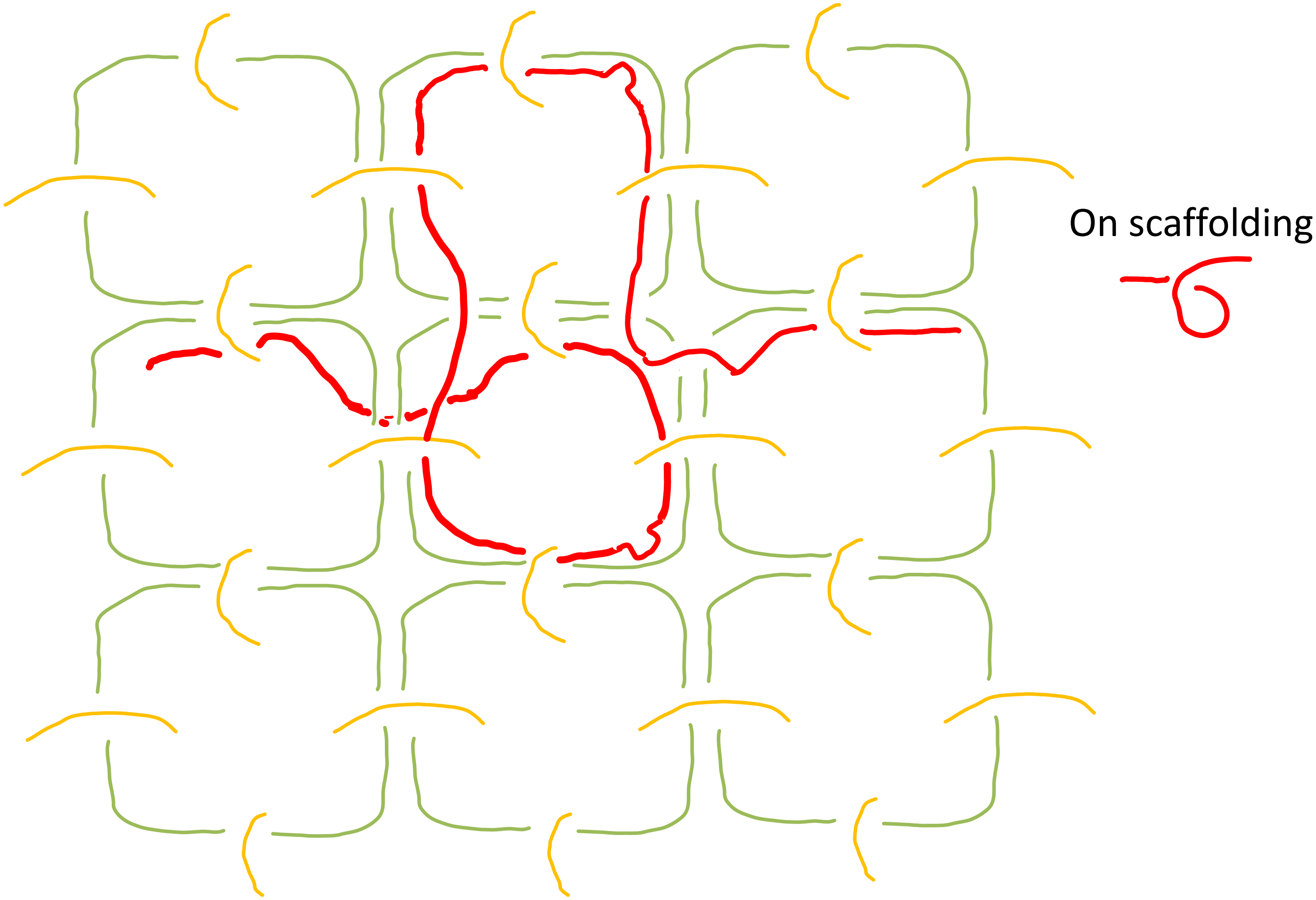

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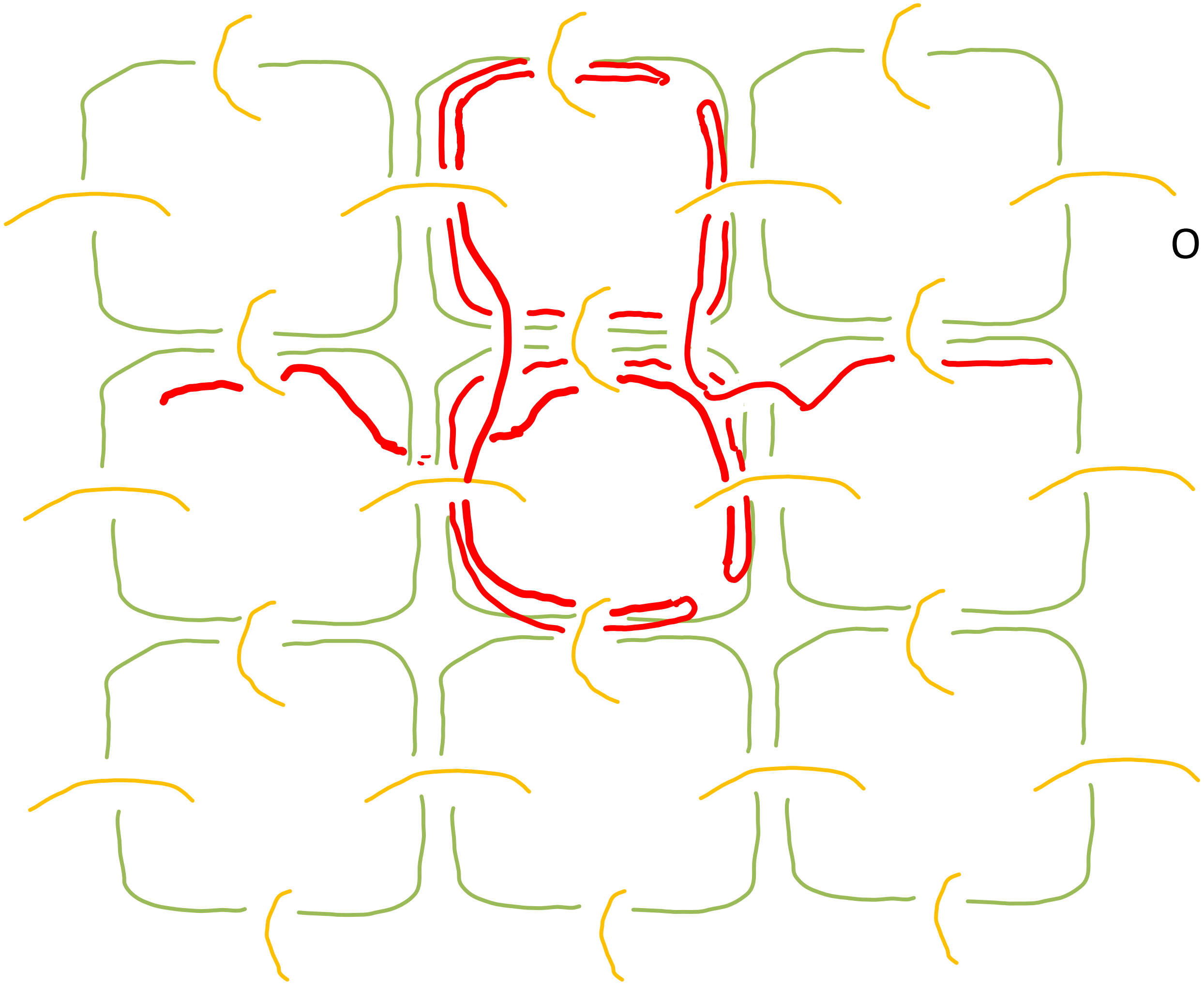

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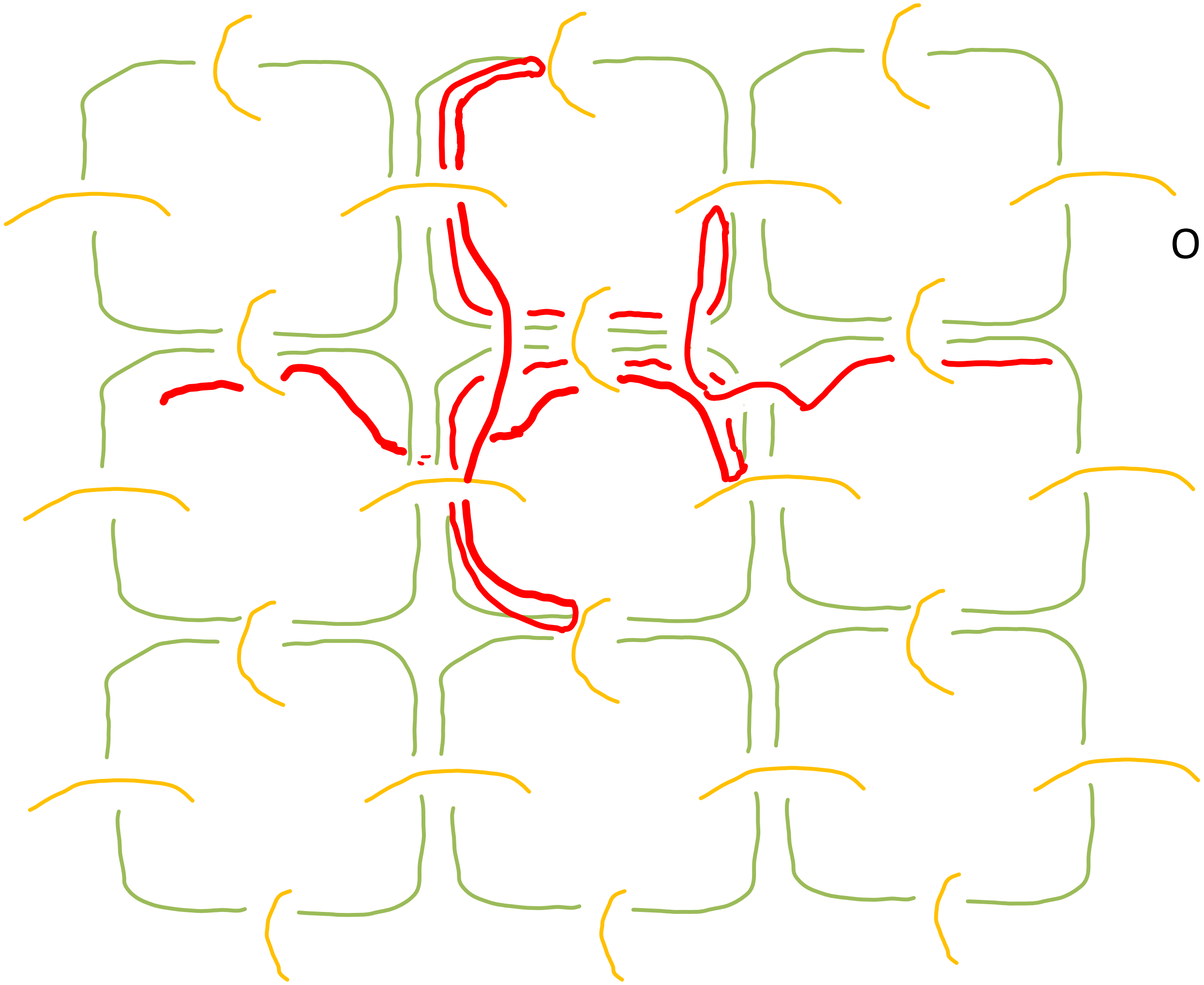

On scaffolding

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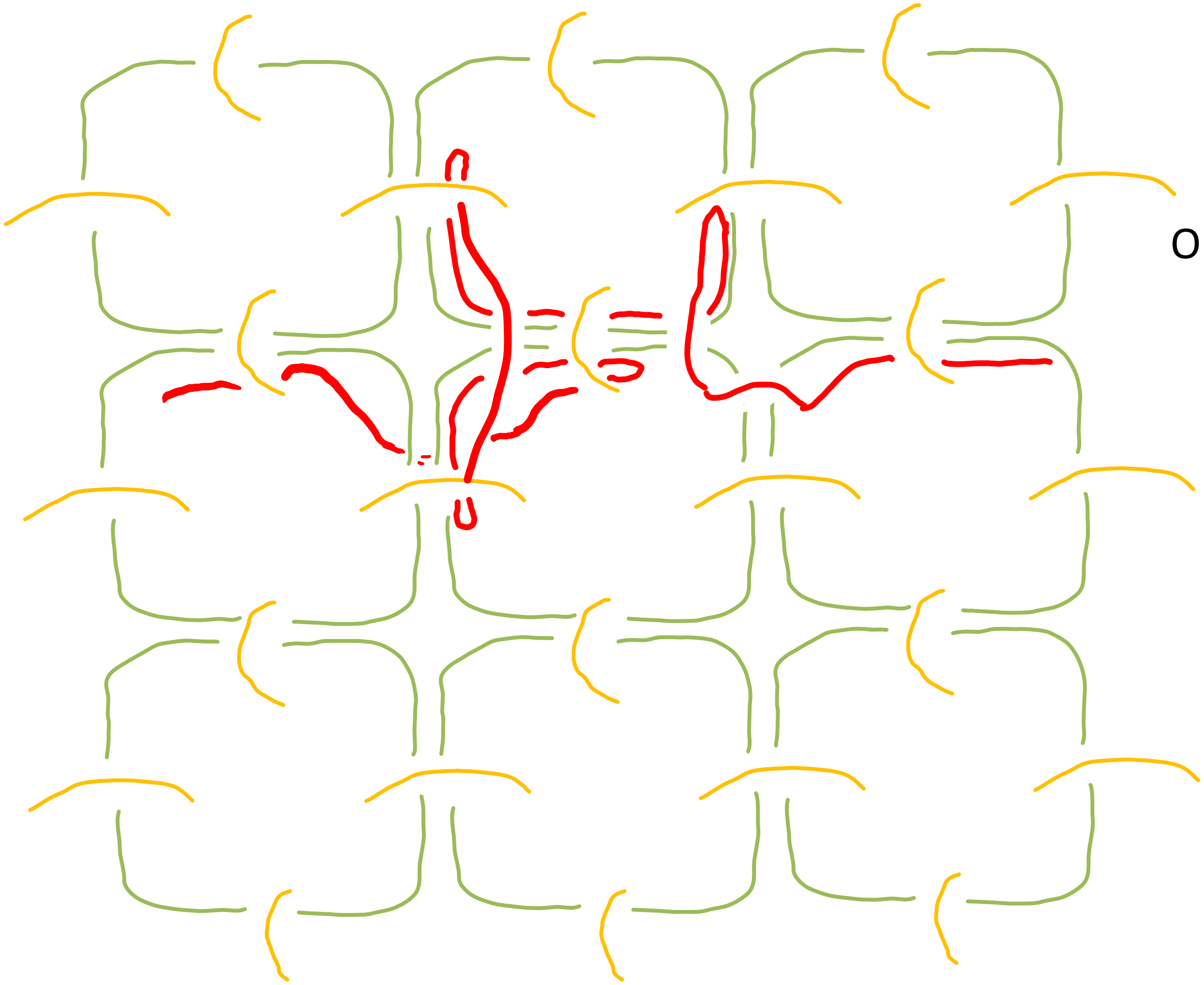
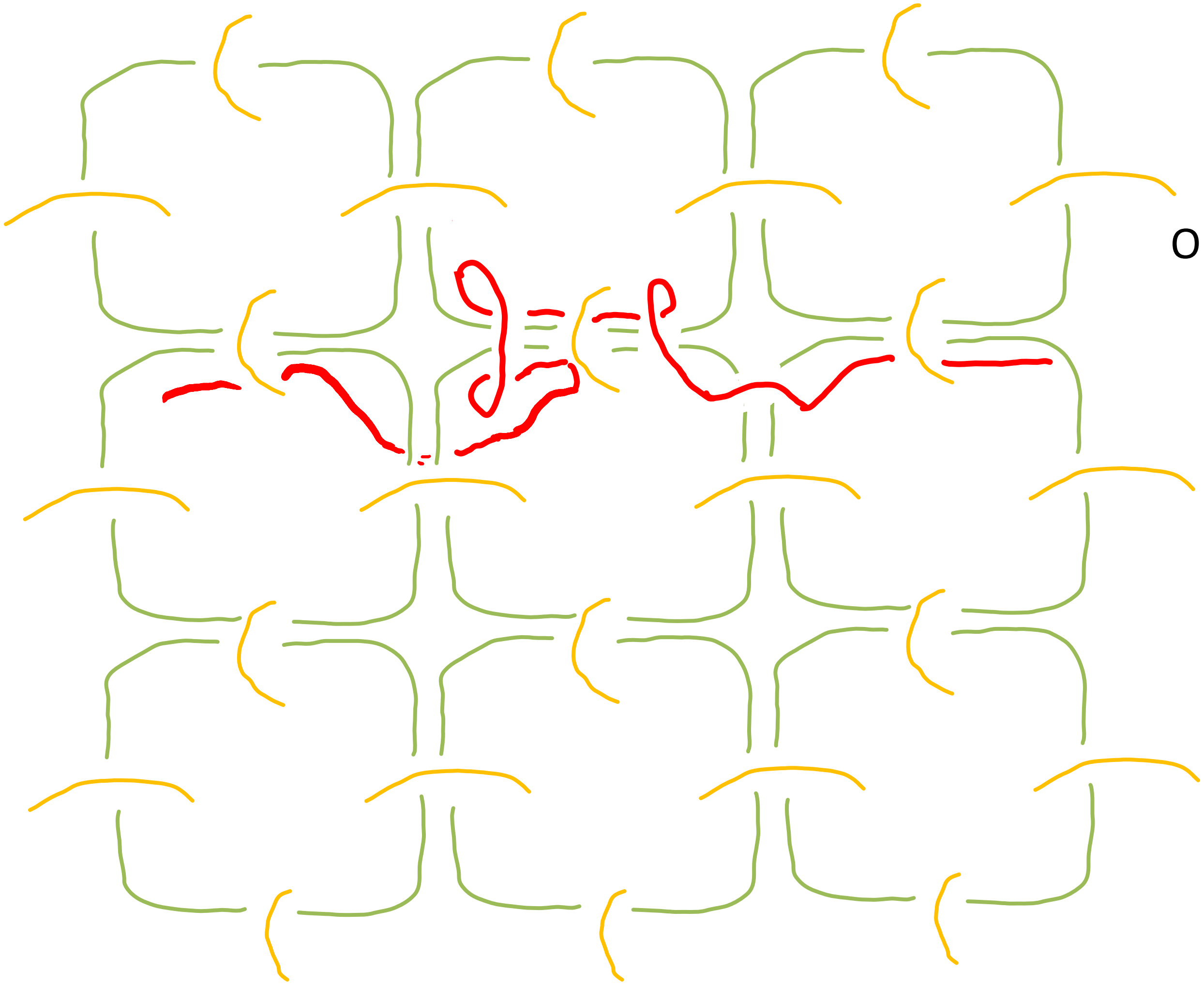


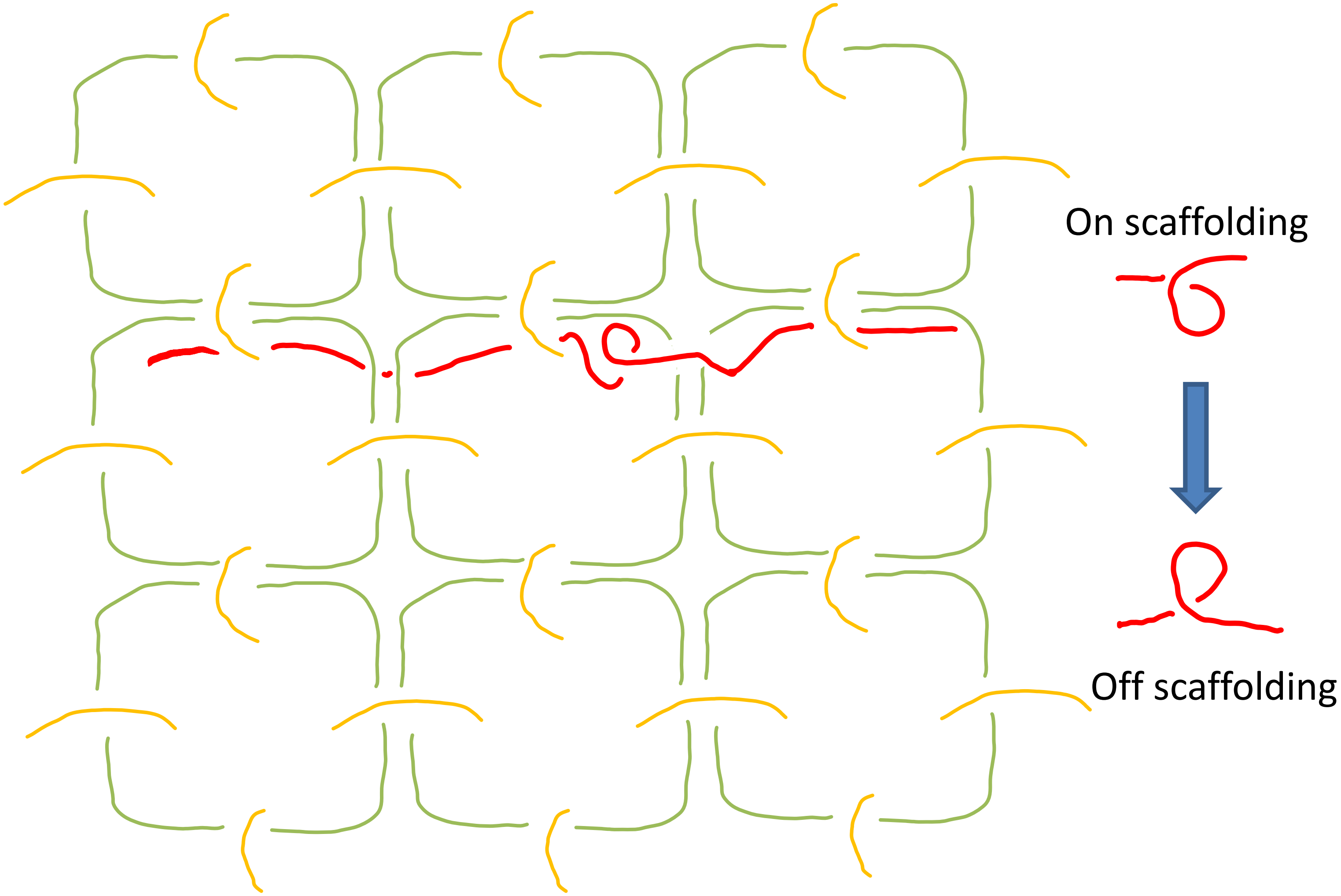
On scaffolding

Mirror Quasiparticles



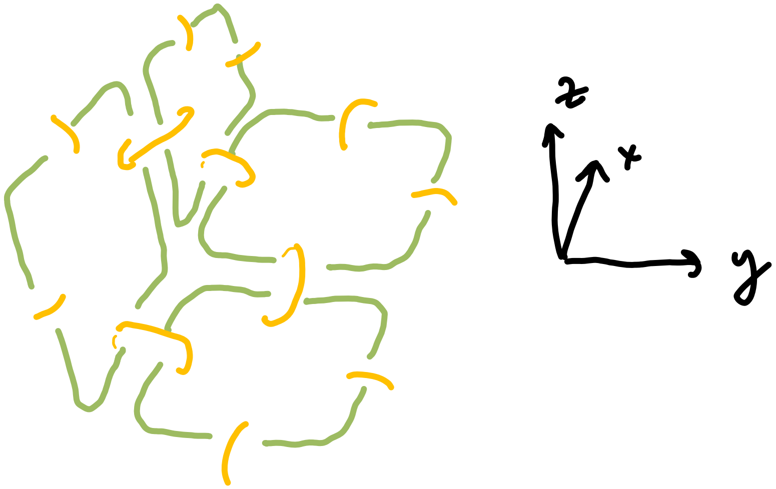
On scaffolding

Mirror Quasiparticles

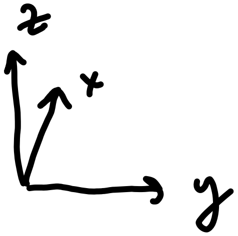


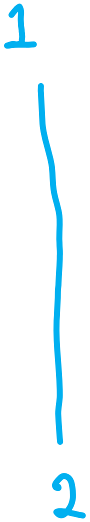
Pulling knot off scaffolding flips chirality!

Mirror Quasiparticles

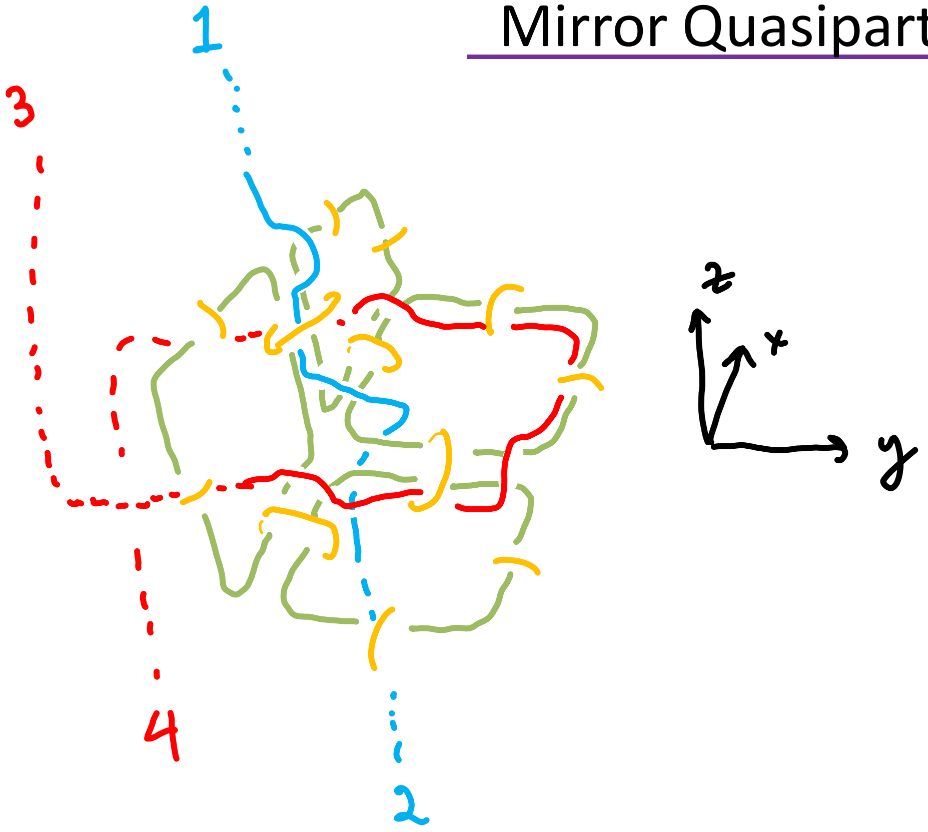


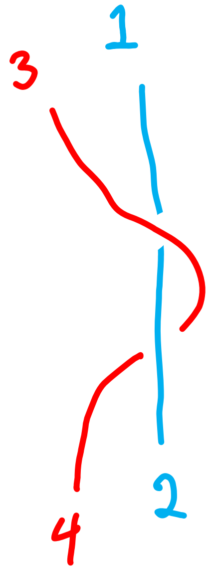
Mirror Quasiparticles





Mirror Quasiparticles





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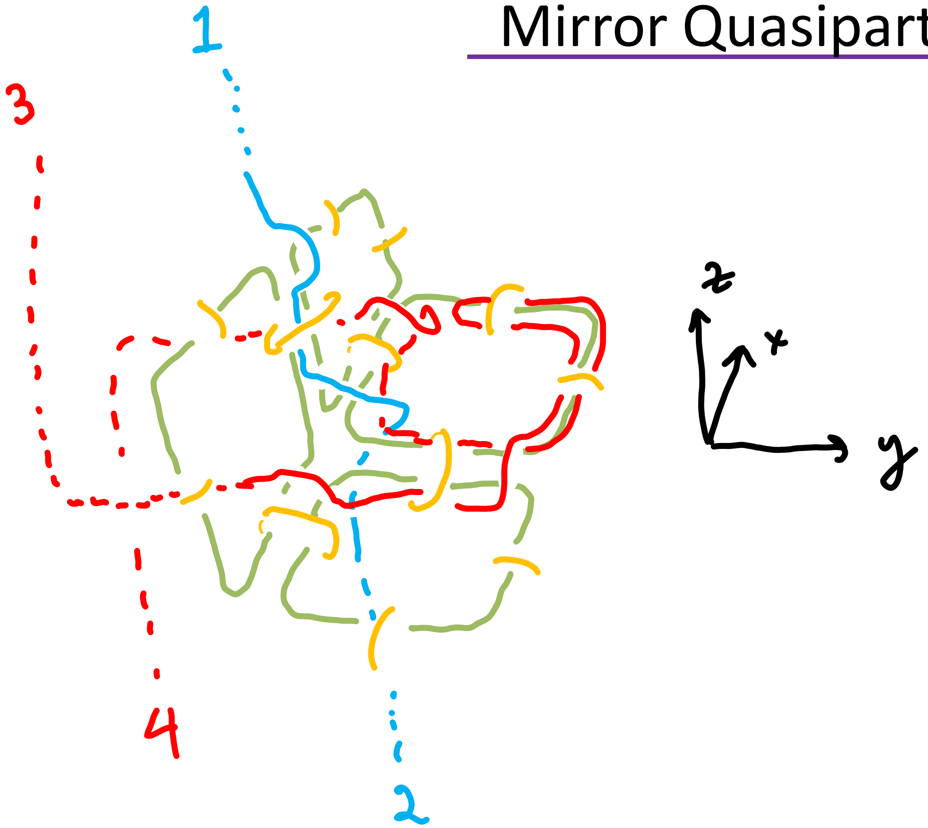
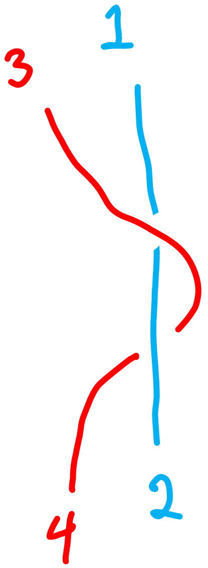
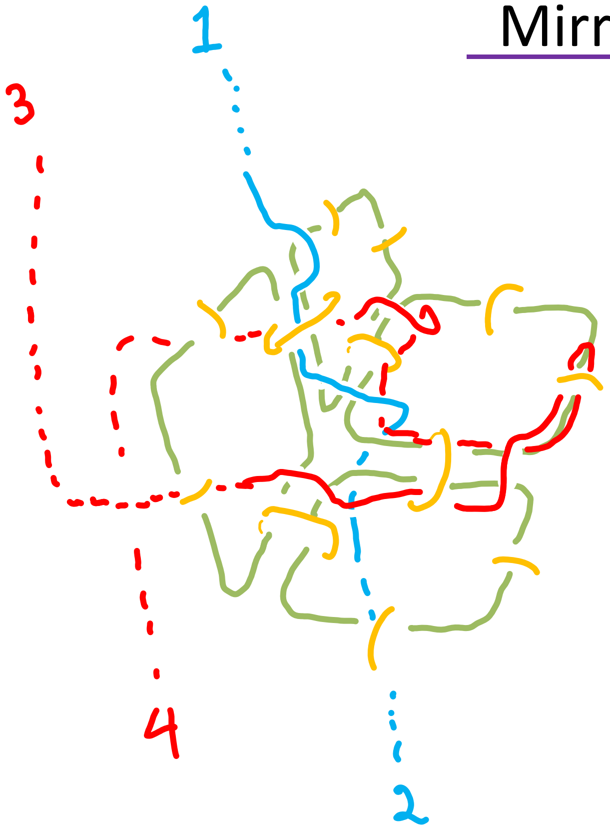
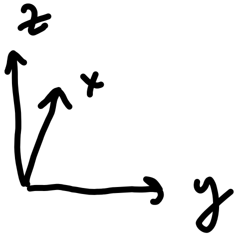
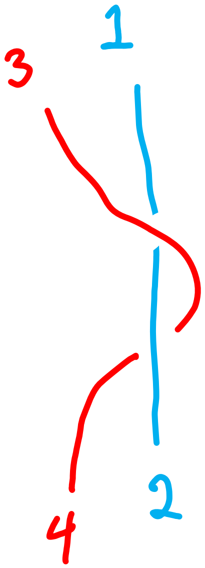




Mirror Quasiparticles







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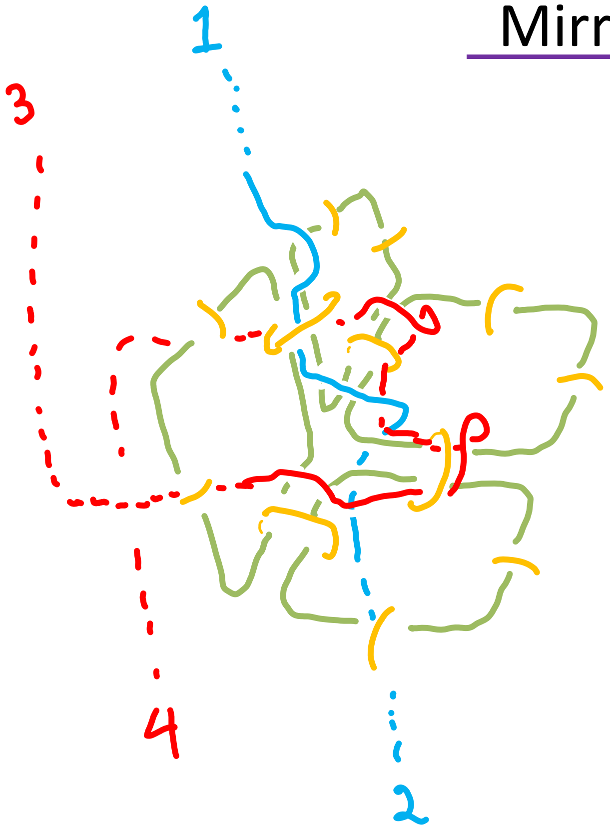

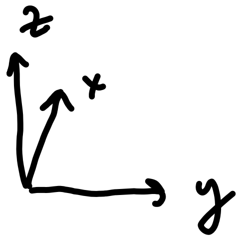

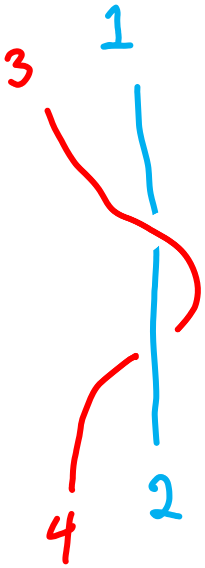

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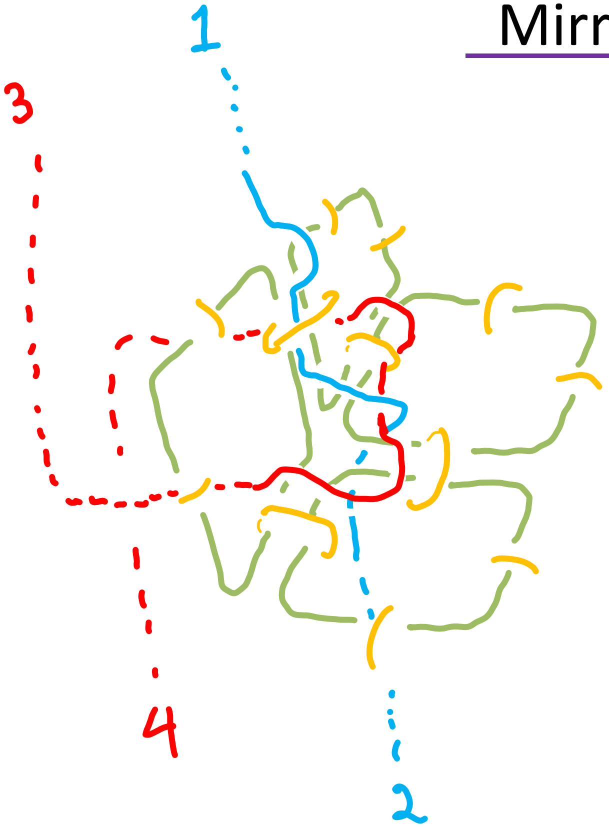

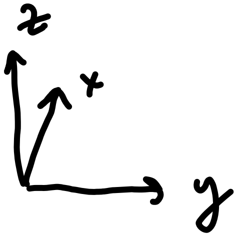

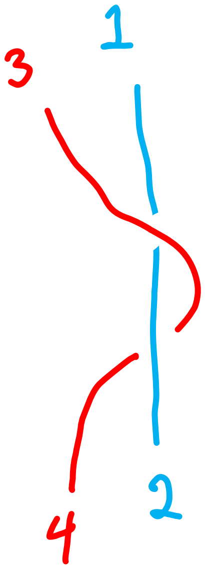

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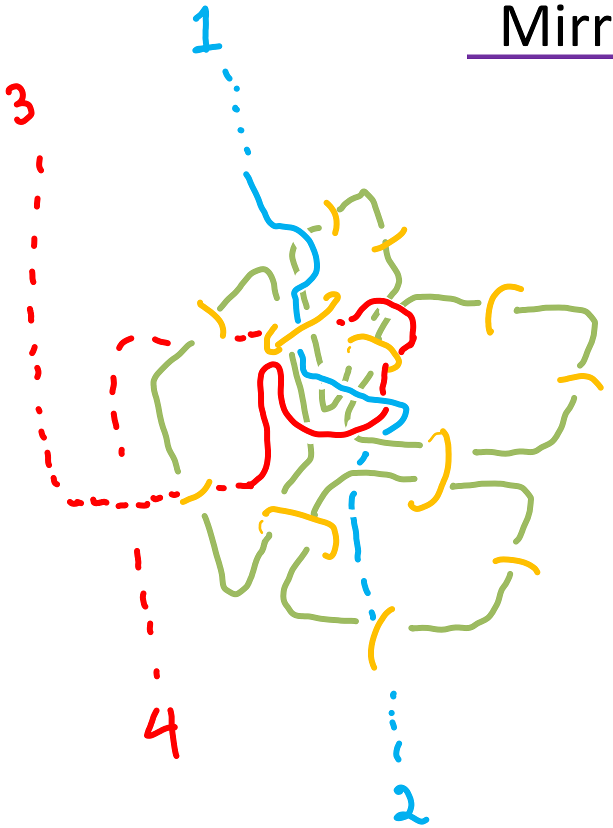
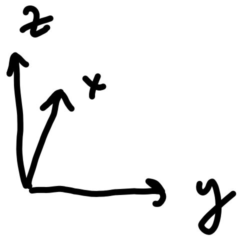
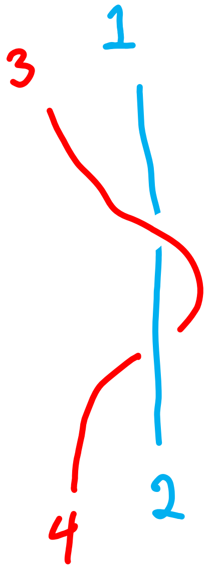
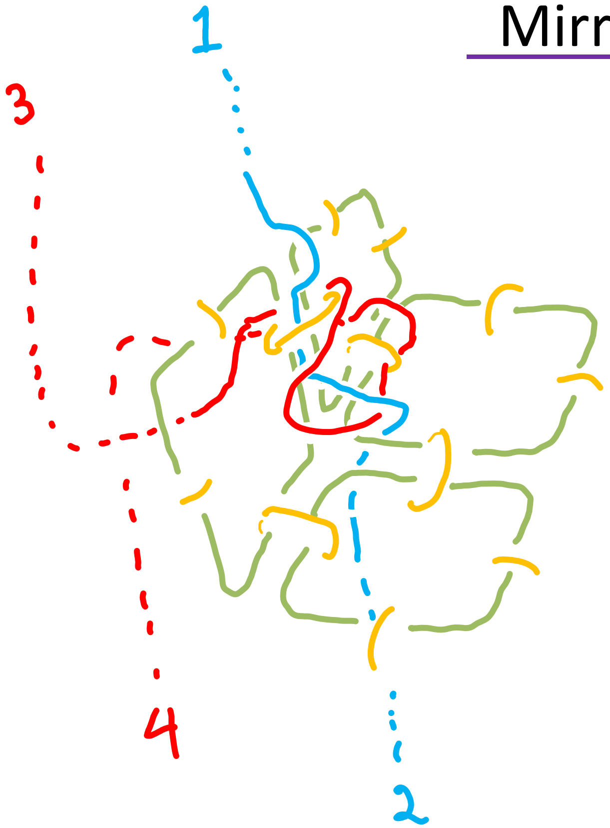
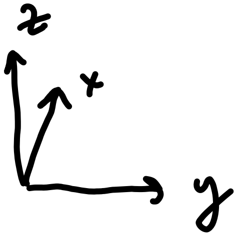


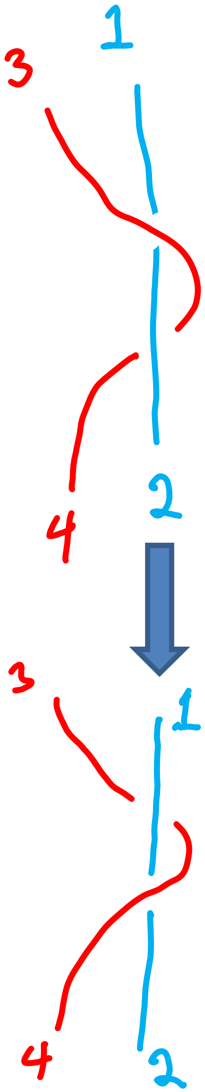




Mirror Quasiparticles



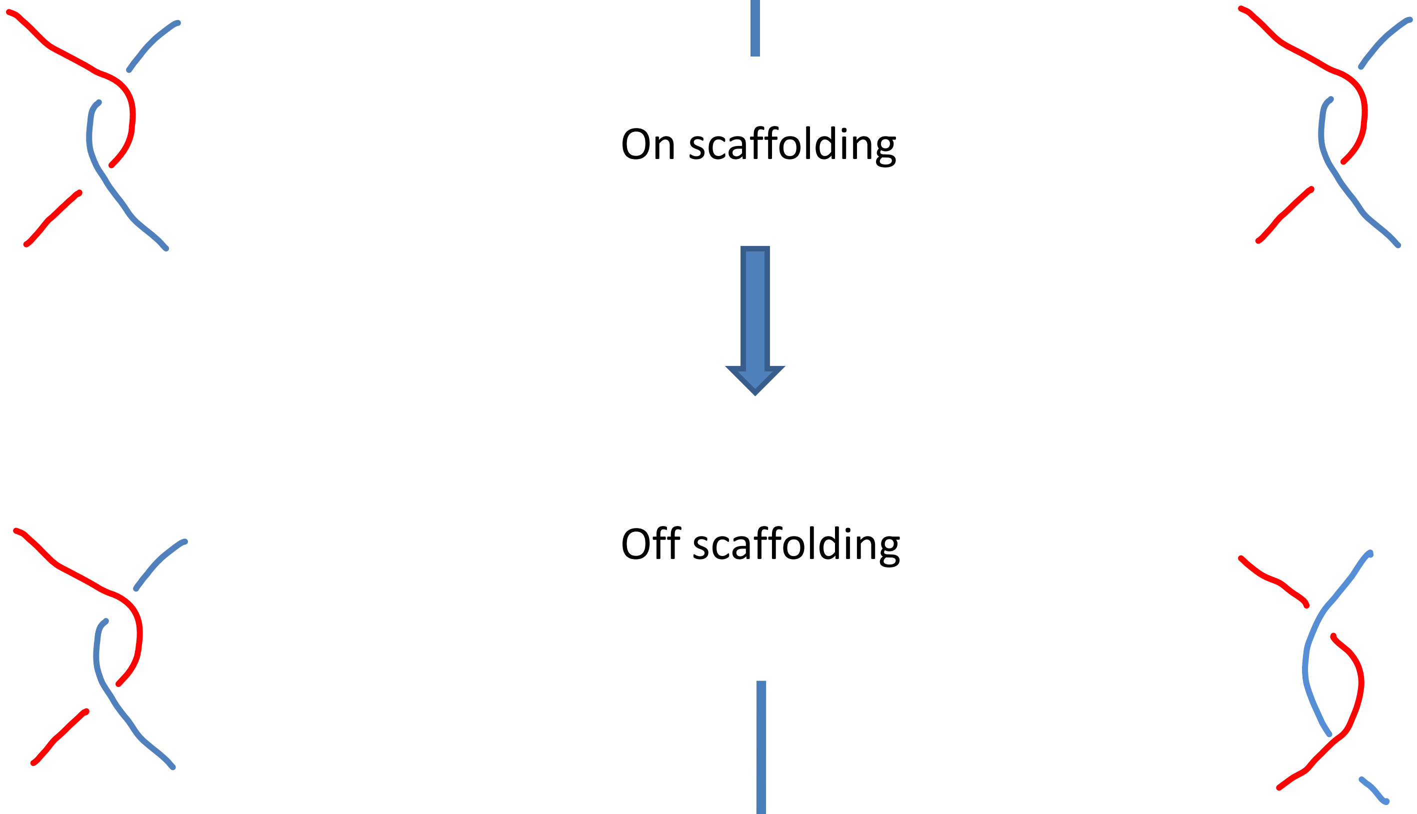




Pulling knot off scaffolding flips chirality!

“Vertex” Quasiparticles

“Mirror” Quasiparticles



On scaffolding

Off scaffolding

Pulling “Vertex” knot off scaffolding leaves knot unchanged

Pulling mirror knot off scaffolding flips chirality!

“Vertex Quasiparticles

$$Z_{CS}[M; ChainMail + L_{vertex}] = Z_{CS}[M; L_{vertex}]Z_{CS}[\overline{M}]$$

Vertex particle world lines
Runs along edges

Same world lines
No underlying chainmail

“Mirror” Quasiparticles

$$Z_{CS}[M; ChainMail + L_{mirror}] = Z_{CS}[M]Z_{CS}[\overline{M}; \overline{L_{mirror}}]$$

Mirror particles world lines
snake between edge
and plaquettes

Mirror world lines
in mirror manifold
No underlying chainmail

Why Does this Work?: The Geometric Story

$$\boxed{Z_{CS}[M, \mathrm{Link} \cup \omega] = Z_{CS}[M', \mathrm{Link}]}$$

An ω can be removed from a Link at the price of changing the manifold by *SURGERY*

A link made entirely of ω's (ex, chainmail) living in manifold M
is equivalent to the vacuum partition function of some other manifold $\tilde{M}$

Roberts: For the Chainmail Link $\tilde{M} = M \# \overline{M}$

$$\boxed{Z_{CS}[M; ChainMail] = Z_{CS}[M \# \overline{M}] = Z_{CS}[M] Z_{CS}[\overline{M}]}$$

The Mirror World is Real

If there are quasiparticle world lines inserted in the Chainmail, one only needs to figure out where they end up in $M \# \overline{M}$

Results

Put Levin-Wen Topological Lattice Models in a New and (Hopefully) Clearer Context (No Tensor Algebra)

Lattice Independent Framework
Topological Invariance is Manifest

Clarify Connection to:

- Chern-Simons Theory
- ChainMail
- Turaev-Viro State Sums

Understand why/how we get left and right handed sectors

- By just handle-sliding we get:
  - How sectors decouple
  - How we get left and right handed particles

Real Geometry: Surgery on chainmail produces $M\#\overline{M}$

Thoughts About 3+1 D (in progress)

There is a generalization of Turaev-Viro to 4D known as Crane-Yetter, which can also be described with Chain Mail.

Can we construct a nontrivial topological theory in 3+1 this way?

Unfortunately, CY[M^4] is almost trivial - sensitive to only the “signature” of M^4

However, if M^4 has a boundary

$$CY[M^4] \sim Z_{CS}[\partial M^4]$$

This is a very nontrivial “topological insulator”

Two chiralities are separated on the opposite surfaces

Results

Put Levin-Wen Topological Lattice Models in a New and (Hopefully) Clearer Context (No Tensor Algebra)

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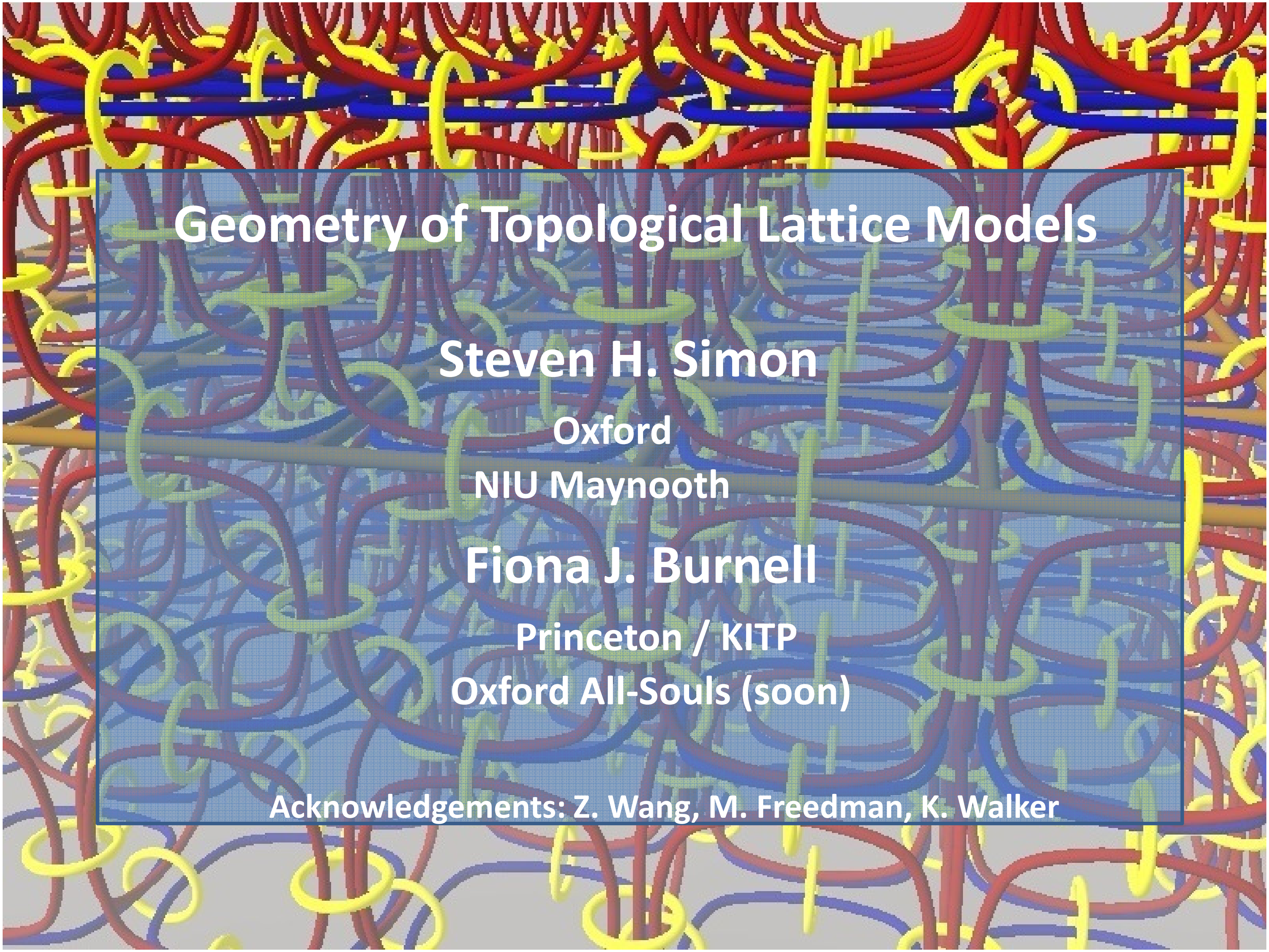
Thoughts About 3+1 D (in progress)



Geometry of Topological Lattice Models

Steven H. Simon

Oxford

NIU Maynooth

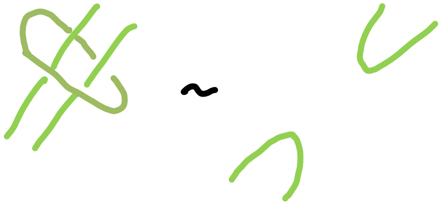
Fiona J. Burnell

Princeton / KITP

Oxford All-Souls (soon)

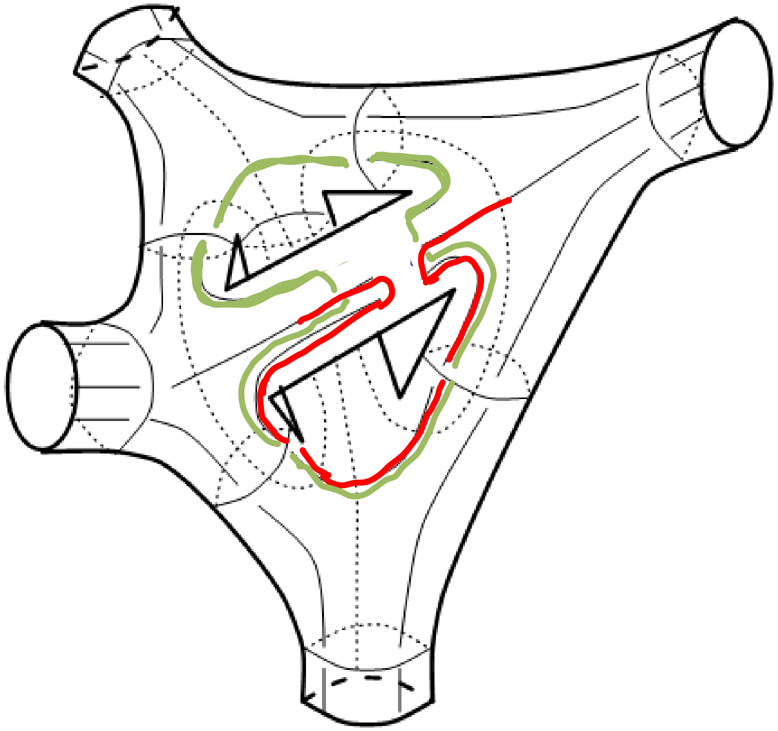
Acknowledgements: Z. Wang, M. Freedman, K. Walker

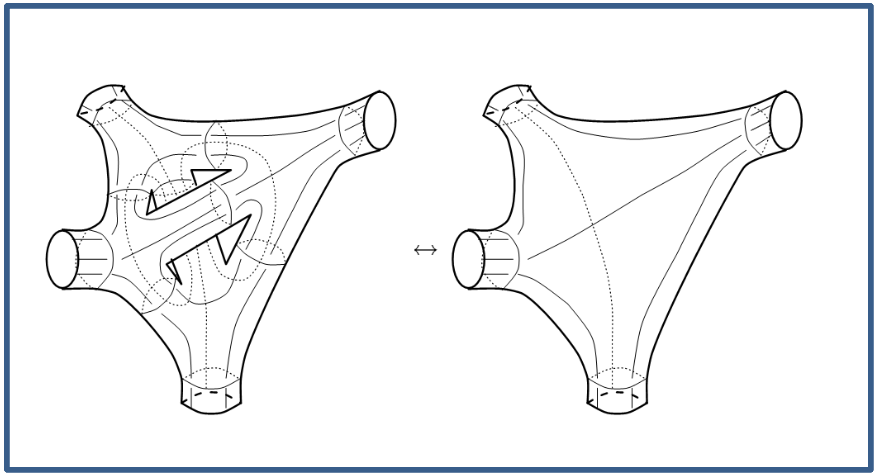
Killing



Handleslide

Why is chainmail independent of lattice geometry?





CAUTION

Physics Ends
Math Begins

I am going to try to make this comprehensible

CAUTION

Real Math

Why Does this Work?: The Geometric Story

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The Mirror World is Real

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CAUTION

Real Math

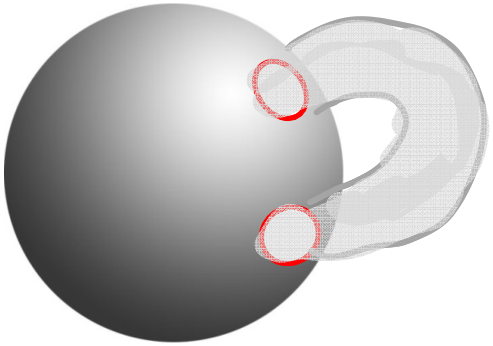
What is Surgery?

Start with a simple manifold (ex S^2)

Adds a *handle* to the solid ball.

Torus surface is boundary of solid torus.

Surgery in 2d is handle in 3d

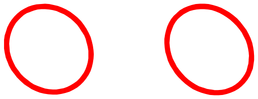


S^0 is two points

Mark a manifold with boundary to be removed

Two disks = $D^2 \times S^0$

Consider the boundary of the region to be removed (Two circles = $S^1 \times S^0$)



Replace the removed region with a manifold that has the same boundary as what we removed (Hollow Cylinder = S^1 x Interval)

For 3-manifolds: cut out a solid torus = $S^1 \times D^2$

The boundary of the solid torus is the torus surface $S^1 \times S^1$

Replace the removed solid torus with $D^2 \times S^1$ which has the same surface as what we removed

Switching which way we fill in the torus: Surgery in 3d is adding a handle in 4d

CAUTION

Real Math

Why Does this Work?: The Geometric Story

$$\boxed{Z_{CS}[M, \mathrm{Link} \cup \omega] = Z_{CS}[M', \mathrm{Link}]}$$

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WHY IS THIS TRUE?

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Roberts: For the Chainmail Link $\tilde{M} = M \# \overline{M}$

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The Mirror World is Real

The Mirror World is Real

If there are quasiparticle world lines inserted in the Chainmail, one only needs to figure out where they end up in $M \# \overline{M}$

CAUTION

Real Math

What is (Dehn) Surgery? (part 2)

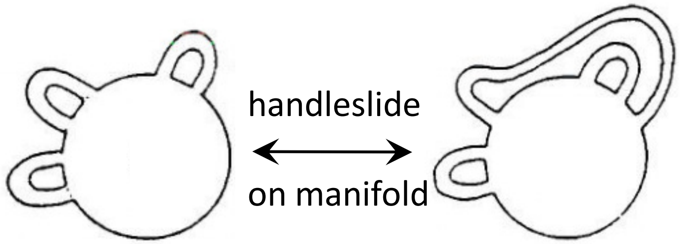
Can do surgery on a torus embedded in some nontrivial way

Can do multiple surgeries too

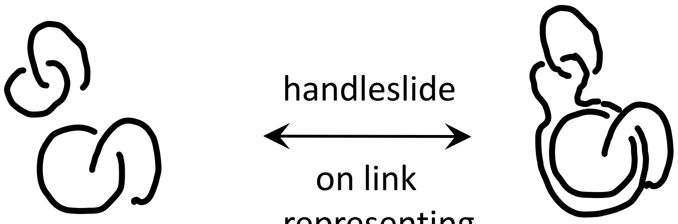
Lickorish-Wallace Theorem: Every Closed 3-Manifold can be obtained this way

* But surgery-presentation is not unique

Kirby: Two 3-manifolds are the same if their link presentations differ by handleslides and blow-ups



handleslide ⟷ on manifold



handleslide ⟷ on link representing a 3-manifold

And blow ups

CAUTION

Real Math

Defining Z_{CS} as a Link Invariant (Reshitikhin-Turaev/ Lickorish)

- First describe the 3-manifold M with Surgery presentation (a link in S^3)

- Take that link, and put ω's on it, then evaluate the link.

= T^3

Since it is made of ω's it is invariant under handleslides

(adding a normalizing prefactor to fix blowups)   This gives an invariant of the manifold

This invariant is known as $Z_{WRT}[M]$

It is believed this is the same as $Z_{CS}[M] \text{``}=\text{''} \int \mathcal{D}[A] e^{iS_{CS}[A,M]}$

In fact, path integral is only properly *defined by* surgery construction *!*

CAUTION

Real Math

Why Does this Work?: The Geometric Story

$$Z_{CS}[M, \mathrm{Link} \cup \omega] = Z_{CS}[M', \mathrm{Link}]$$

An ω can be removed from a Link at the price of changing the manifold by *SURGERY*

WHY IS THIS TRUE? (It is true *by definition* of Chern-Simons partition function)

A link made entirely of ω's (ex, chainmail) living in manifold M is equivalent to the vacuum partition function of some other manifold $\tilde{M}$

Roberts: For the Chainmail Link $\tilde{M} = M \# \overline{M}$

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The Mirror World is Real

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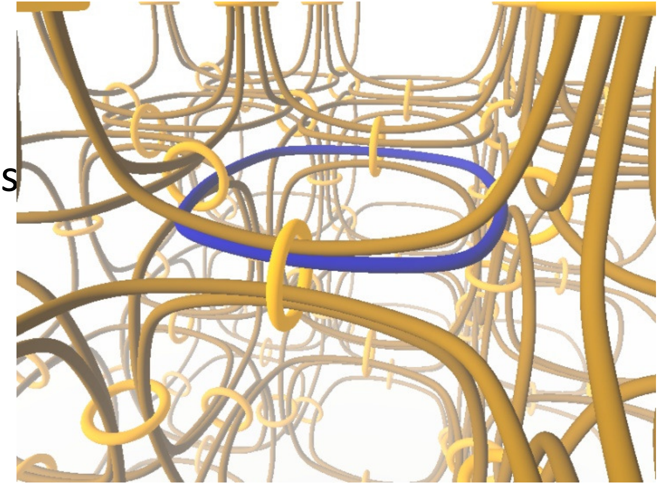
CAUTION
Real Math

Why is Chain Mail Link a Surgery Presentation of $M \# \overline{M}$ (Roberts)



- Chain Mail is Handle decomposition of skeleton of M

Plaquette loops are attaching curves for 2 handles (thick disks
Chain Mail loops are meridians of 1 handles (2-thick edges)
+3 handles (cells) not included.

Links in S^3 can be thought of as surgery construction of M^3
or as attaching curves for handles of M^4 where $M^3 = \partial M^4$

CAUTION

Real Math

What is Surgery?

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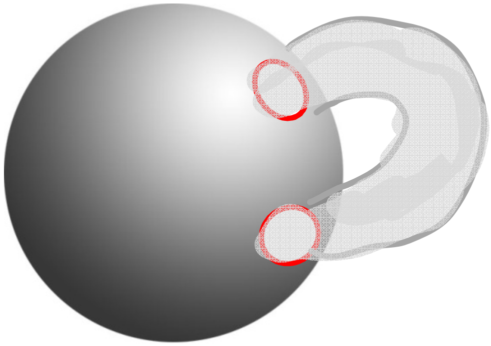

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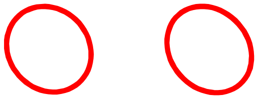

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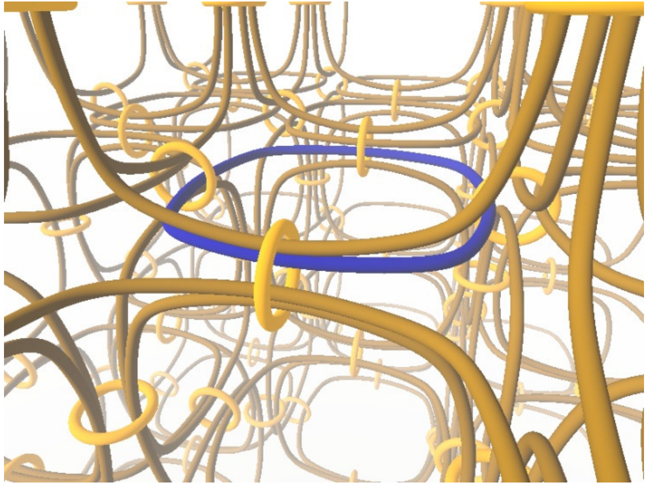
CAUTION

Real Math

Why is Chain Mail Link a Surgery Presentation of $M \# \bar{M}$ (Roberts)



Plaquette loops are attaching curves for 2 handles (thick disks)
Chain Mail loops are meridians of 1 handles (2-thick edges)
+3 handles (cells) this is a "Handle decomposition" of M

Links in S^3 can be thought of as surgery construction of M^3
or as attaching curves for handles of M^4 where $M^3 = \partial M^4$,

In M^4 :
Plaquette loops attach 2 handles = 2-thick disk
ChainMail loops attach 1 handles = 3-thick edge

The handle structure M^4 is identical to that of M^3 but thickened into one more dimension, so it "looks like" M^3 x Interval

$$\partial(M^3 \text{ x Interval}) = M^3 \cup \bar{M}^3.$$

But we did not add any 3-handles, so we get $M_{skeleton}$ x Interval instead.
∂ ($M_{skeleton}$ x Interval) = $M^3 \# \bar{M}^3$ … the connect sum because of the missing 3-handle.

CAUTION

Real Math

$M_{skeleton}$ = M without the “largest” handle filling in the center

∂ ($M_{skeleton}$ x Interval) = M # M … the connect sum because of the missing 3-handle.

Ex

0,1 handles of T^2 --- This is a skeleton, 2-handle is missing

Same handles up one dimension = 2 hole solid torus

Boundary of that is the two holed torus surface = $T^2 \# \overline{T^2}$

CAUTION
Real Math

Where are the quasiparticles after surgery?

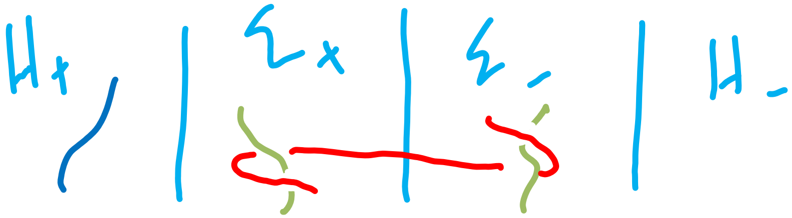
Related to…
Barrett et al, ’07
and Martins ‘06

Plaquette loops are attaching curves for 2 handles
Chain Mail loops are meridians of 1 handles

Define a Heegard splitting: H+ = (0 and 1)-handles = thickened edge lattice,
H- = (2 and 3) handles = plaquettes and 3-cells. (Attaching curves are same as above)

Thicken 2d Heegard surface (boundary between edge lattice and rest) into two layers Σ+ , Σ-

pull chain mail loops into Σ- , plaquette loops into Σ+



Consider decomposition:

$$S^3 = (H+) \cup (\Sigma+) \cup (\Sigma-) \cup (S^3 \setminus \mathrm{int}\, H+)$$

$$\qquad\qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow$$

Now perform surgery on the loops living in Σ+ and Σ- separately

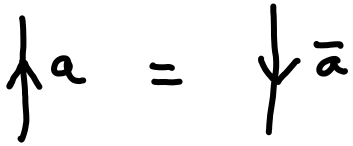
$$S^3 = \underbrace{(H+) \cup (H-}_{M} \;\#\; \underbrace{\overline{H-}) \cup (\overline{H+}}_{\overline{M}} \;\#\; H+) \cup (S^3 \setminus \mathrm{int}\, H+)$$

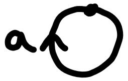
A chiral quasiparticle line (blue) lives in H+, lands unchanged in M after surgery
A mirror quasiparticle line (red) snakes between Σ+ and Σ- therefore lives in $\overline{M}$ after surgery.

Braided (Unitary-Modular) Tensor Category

(rigid=unit and inverse)

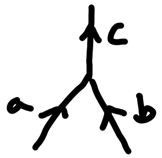
Quantum numbers $a, b, c, \ldots$



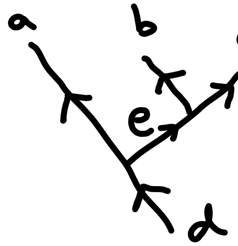


$= d_a$

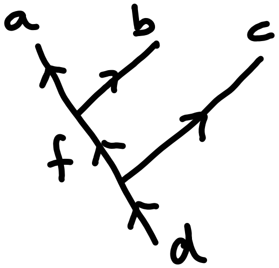
Fusions $a \times b = \sum_c N_{ab}^c \; c$



F-matrices (6-j)

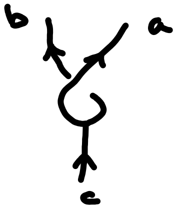


$= \sum_f F_{def}^{abc}$

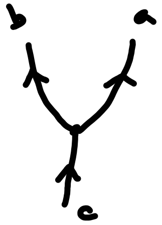


Levin-Wen (and Turaev-Viro) is defined with any tensor category – but if we add

R-matrix



$= R_{ab}^c$



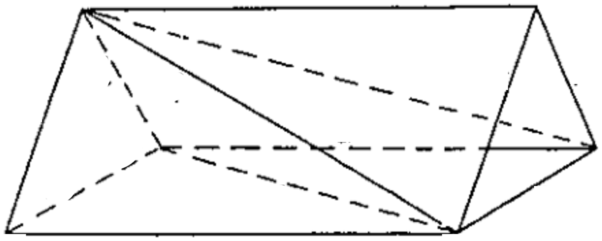
For safety want modular, and no pseudo-real fields

.. we have a model of anyons "C".

(Example: Chern-Simons theory)

Until we specify R, we do not know ex, the chirality

The Turaev Viro State Sum



Take a triangulated 3 manifold M

Color edges with the quantum numbers from the anyon model.

Construct the sum

$$TV[M] = \mathcal{D}^{\#\ \text{vertices}} \sum_{\text{colorings}} \prod_{edges=a} d_a \prod_{faces=abc} N_{abc} \prod_{tetrahedra=abcdef} F^{def}_{abc}$$

- Independent of triangulation (topological invariant)

$$TV[M] = Z_{WRT}[M] Z_{WRT}[\overline{M}]$$

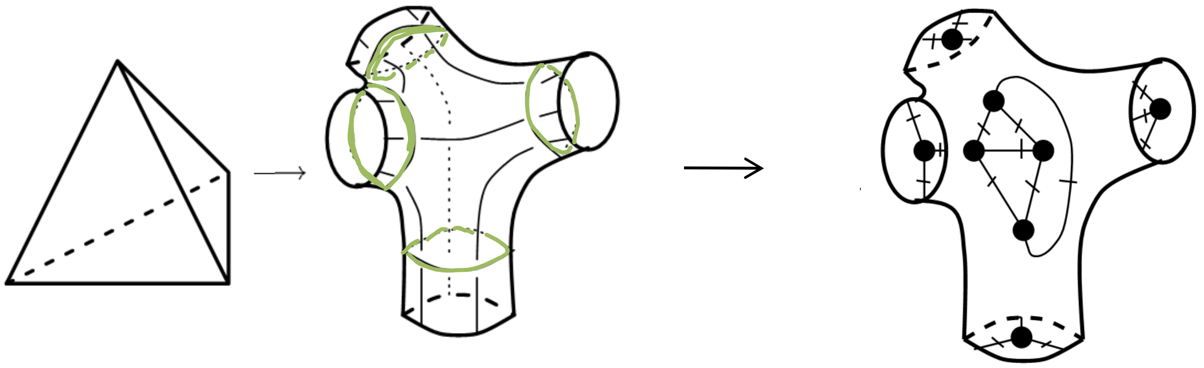
(Walker, Turaev)

“ $Z_{WRT} = \int_M \mathcal{D}A\ e^{iS_{CS}(A)}$ “

No concept of a “quasiparticle” in TV yet

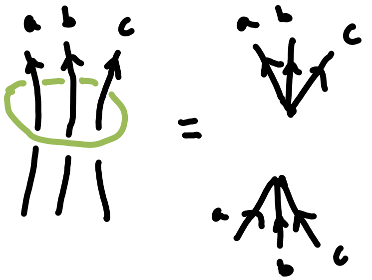
Why is Chain-Mail = TV[M]

$$TV[M] = \mathcal{D}^{\#\text{ vertices}} \sum_{\text{colorings}} \prod_{edges=a} d_a \prod_{faces=abc} N_{abc} \prod_{tetrahedra=abcdef} F^{def}_{abc}$$



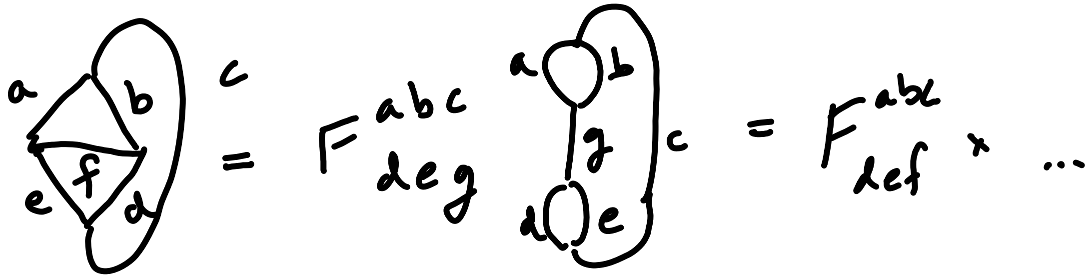
Uses

(1)

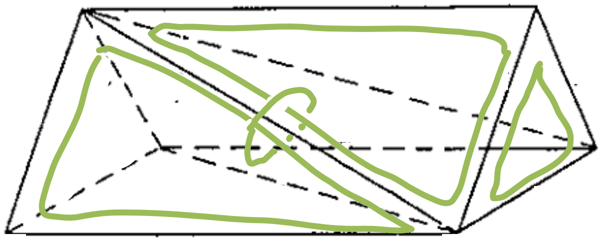


If abc can fuse to zero.
Otherwise = 0

(2)



Chain Mail (J. Roberts)

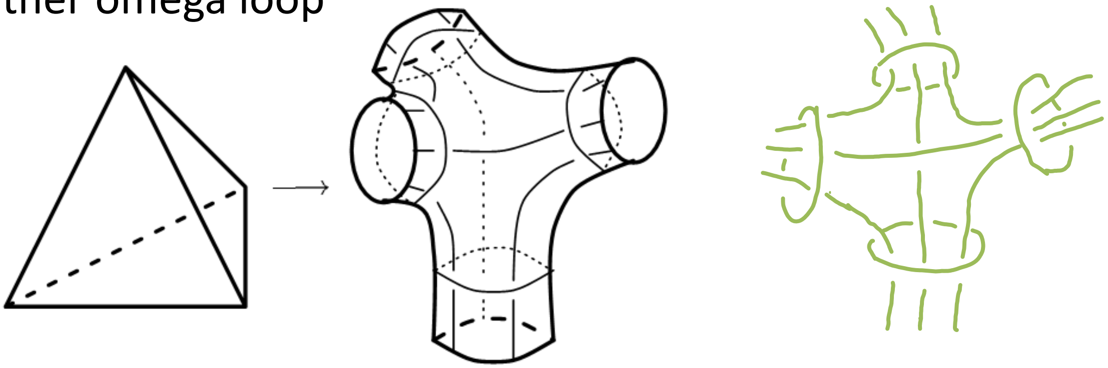


Take a triangulated 3 manifold M

Add omega loops around all plaquettes

Each edge has several omegas running along it
Bind them together with another omega loop

This is the Chain Mail Link.



Evaluate this knot

Roberts: The value of this link is just TV[M]

Comment: You have to choose an R matrix to “define the knot invariant”
But the end result is independent of the R matrix you choose!

About 3+1 D (in progress)

There is a generalization of Turaev-Viro to 4D known as Crane-Yetter

$$TV[M^3] = \mathcal{D}^{\#\text{ vertices}} \sum_{\text{colorings}} \prod_{edges=a} d_a \prod_{faces=abc} N_{abc} \prod_{3-cells} 6j$$

$$CY[M^4] = \mathcal{D}^{\#\text{ vertices}} \sum_{\text{colorings}} \prod_{edges} d_a \prod_{faces} N_{abc} \prod_{3-cells} 6j \prod_{4-cells} 15j$$

And it can be reduced to a similar chain mail.

Can we construct a nontrivial topological theory in 3+1 this way?

Unfortunately, CY[M^4] is almost trivial - sensitive to only the signature of M^4

However, if M^4 has a boundary

$$CY[M^4] \sim Z_{WRT}[\partial M^4]$$

This is a very nontrivial topological insulator

Two chiralities are separated on the opposite surfaces

Example of a Topological Quantum Field Theory: Chern-Simons Theory

$$\left.\begin{array}{l}\text{Gauge Group G}\\ \text{integer “level” k}\end{array}\right\} \mathrm{G}_{\mathrm{k}}$$

spacetime manifold M

A is a one-form taking values in the Lie algebra of G

CS Action

$$\boxed{S_{CS}[A,M] = \frac{k}{4\pi}\int_M tr[A \wedge dA + \frac{2}{3} A \wedge A \wedge A]}$$

Invariant for “small” gauge transformations

Under “large” gauge transform ations $S_{CS} \to S_{CS} + 2\pi n k$

CS Vacuum
Partition Function

$$\boxed{Z_{CS}[M] \text{ “}=\text{” } \int \mathcal{D}[A] e^{iS_{CS}[A,M]}}$$

= topological invariant of the manifold

Example of a Topological Quantum Field Theory: Chern-Simons Theory

Chern-Simons Action

$$S_{CS}[k, A, M]$$

Integer “level” k
[Frequently not written]

Gauge Field taking values in the Lie Algebra of Gauge Group G

Spacetime Manifold M

$e^{iS_{CS}}$ is gauge invariant and independent of the spacetime metric

Chern-Simons Vacuum Partition Function

$$\boxed{Z_{CS}[M] \text{ “} = \text{”} \int \mathcal{D}[A] e^{iS_{CS}[A,M]}}$$

Called G_k

= topological invariant of the manifold

Example of a Topological Quantum Field Theory: Chern-Simons Theory

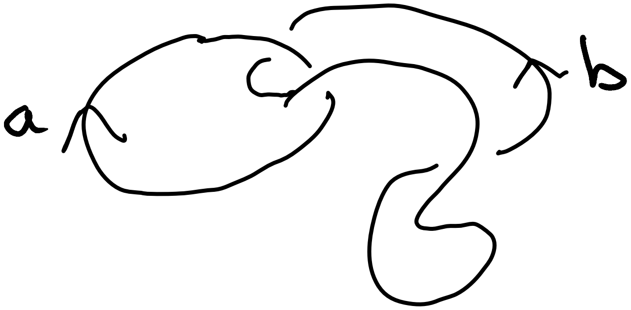
Wilson Loop Operators

$$\boxed{W_a[C] = tr_a\left[\mathcal{P}\exp\{i\int_C A\}\right]}$$

C is a directed spacetime path

a is a representation of the gauge group or a "particle type"

NOTE: Only a finite set of particle types are allowed: Depending on the gauge group and level



$$\boxed{Z_{CS}[M, Link] = \int \mathcal{D}[A]\, W_a(C_1) W_b(C_2)\, e^{iS_{CS}[A,M]}}$$

Topological link invariant of "colored" link in manifold M

(Witten-Jones)

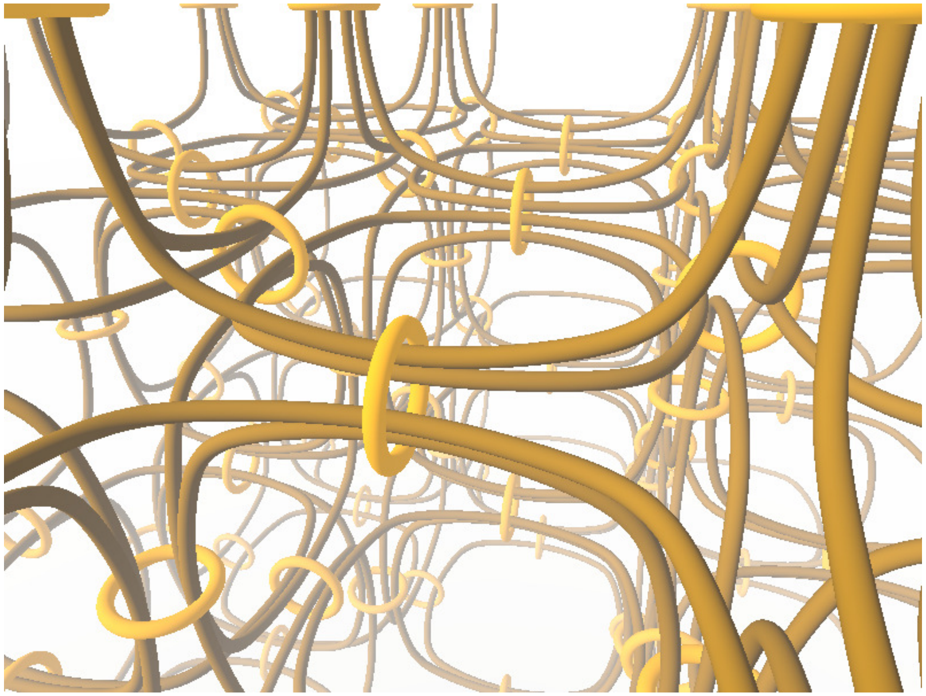
“Chiral” Quasiparticles

$$\boxed{Z_{CS}[M; ChainMail + L_{chiral}] = Z_{CS}[M; L_{chiral}]Z_{CS}[\overline{M}]}$$

Chiral particle world lines in lattice model.
Runs along edges, violates vertices

Same world lines
No underlying chainmail

If the chiral particles run along the edge, they live in M not $\overline{\text{M}}$.



Chainmail

Roberts '95 for "SU(2)$_k$" models
=Turaev-Viro State Sum Invariant

Independent of lattice and decomposition of manifold

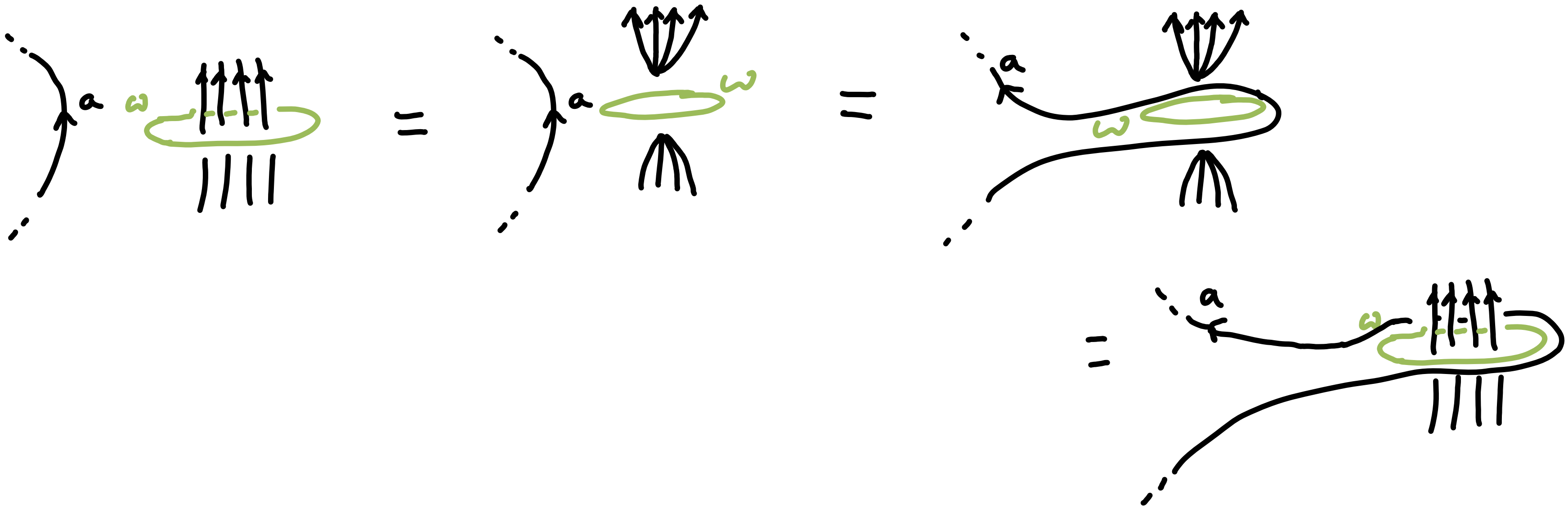
$$\sum_{\Psi_{i_1},\Psi_{i_2},\ldots} \ldots |\Psi_{i_n}\rangle\langle\Psi_{i_n}|P|\Psi_{i_{n-1}}\rangle\langle\Psi_{i_{n-1}}|V|\Psi_{i_{n-2}}\rangle\langle\Psi_{i_{n-2}}|P|\Psi_{i_{n-3}}\rangle\langle\Psi_{i_{n-3}}|V|\ldots$$

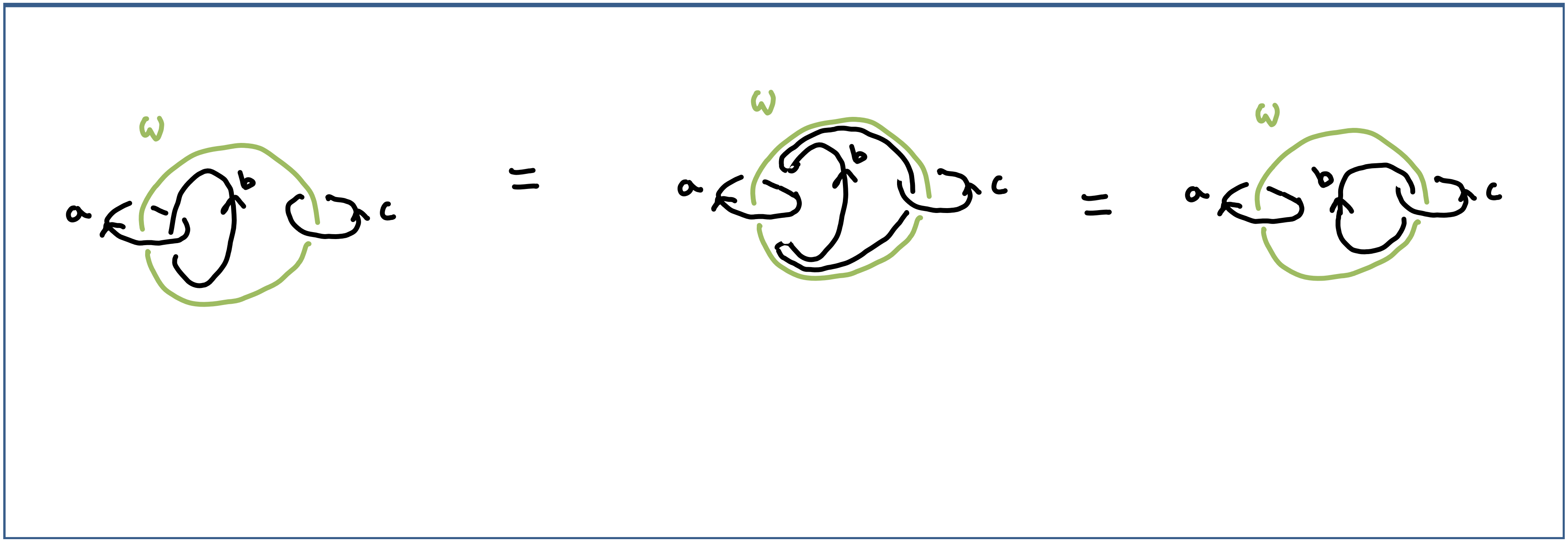
This is *not quite* the action associated with Levin-Wen's

$$H = -\sum_{vertices=i} V_i \quad - \sum_{plaquettes=j} P_j$$

Partition function of
the ground state sector.

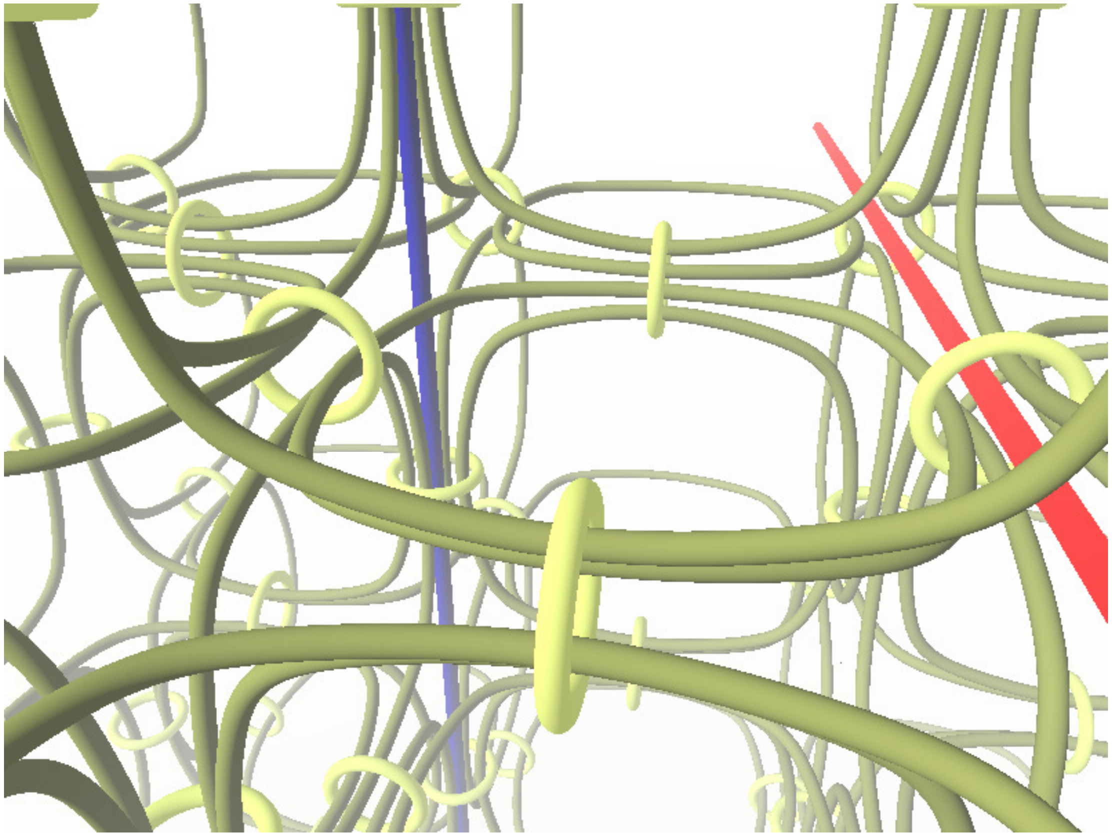
Handleslide From Killing





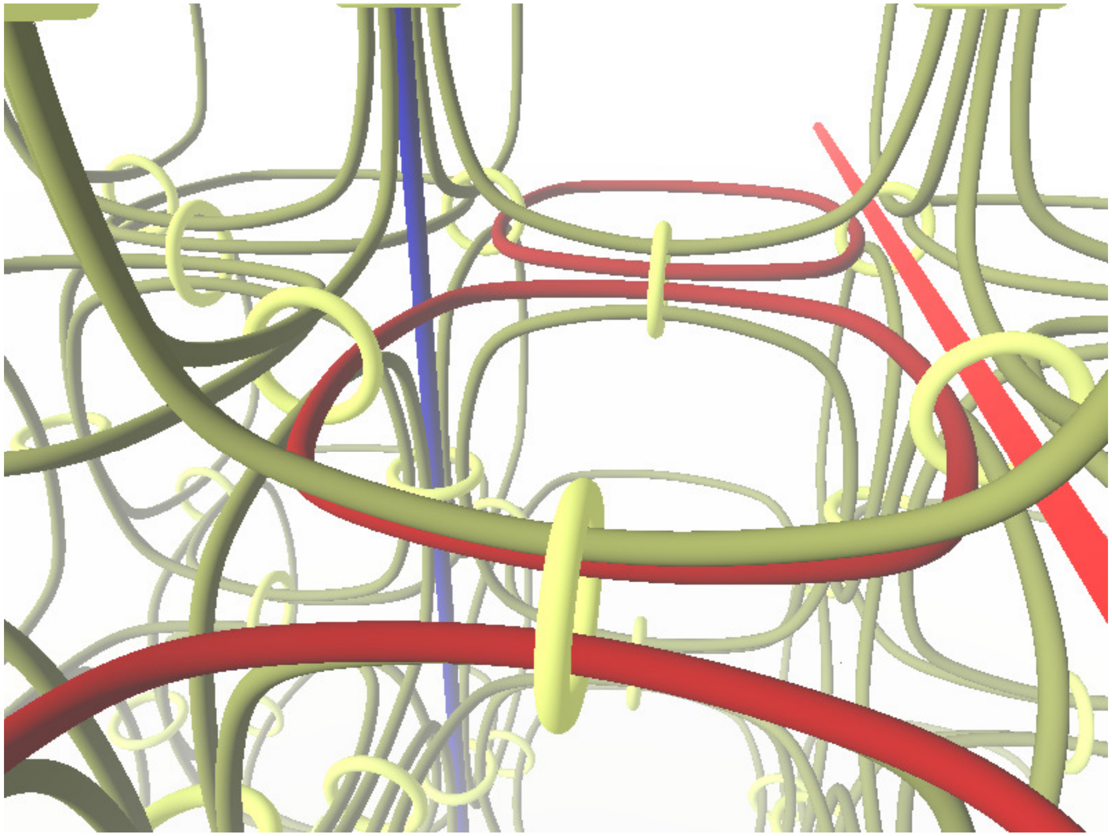
START HERE FOR SHOWING VERTEX PARTICLE BEHAVIOR

“Chiral” Quasiparticles



Chiral particles cannot handle slide through each other

“Chiral” Quasiparticles



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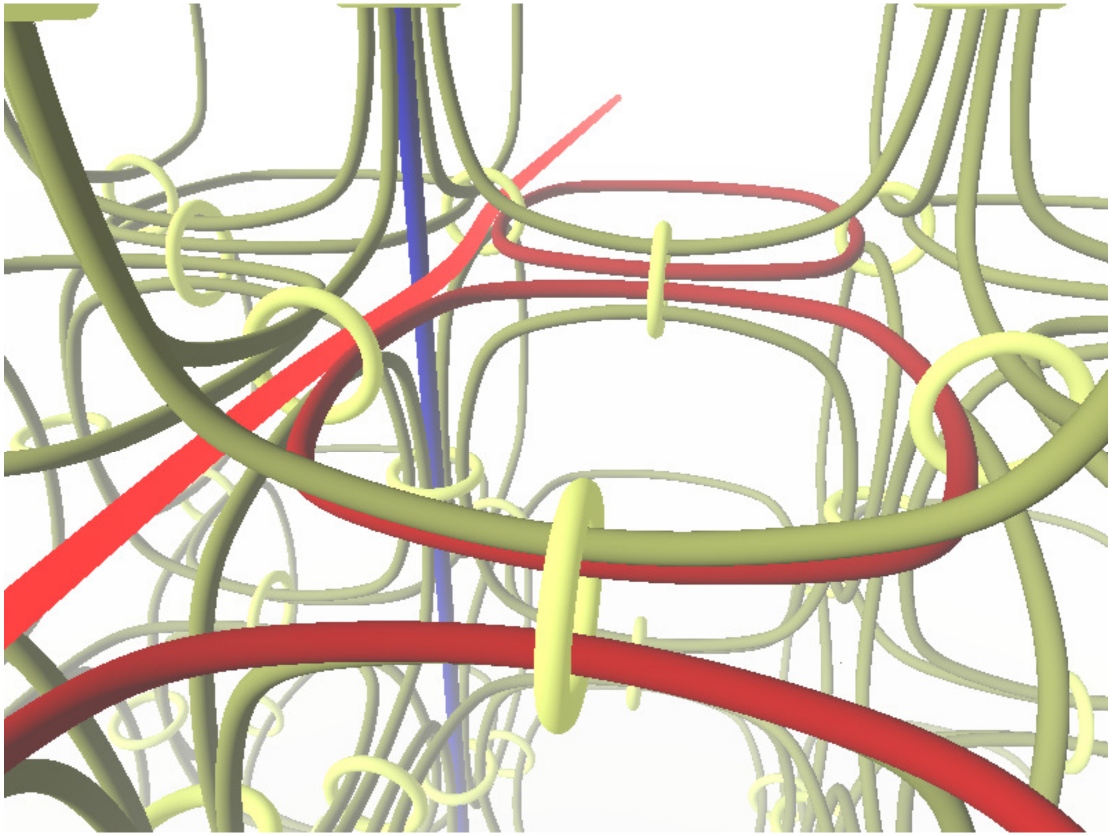

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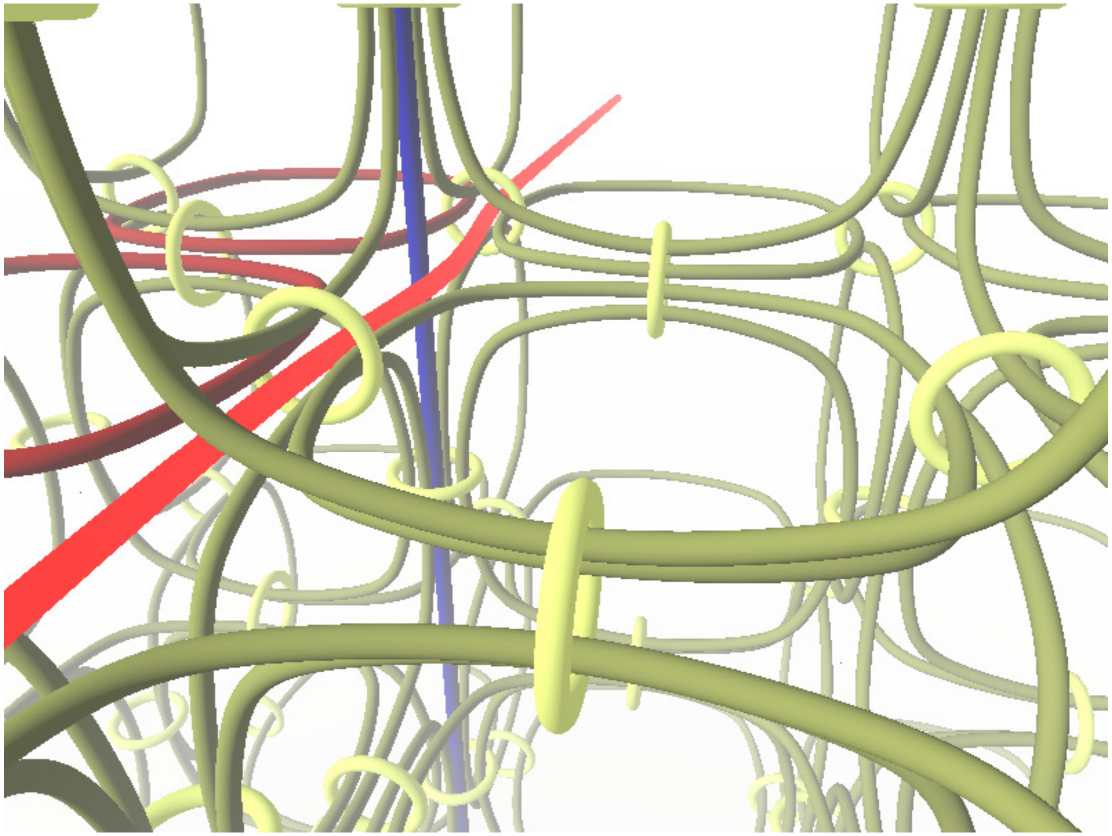

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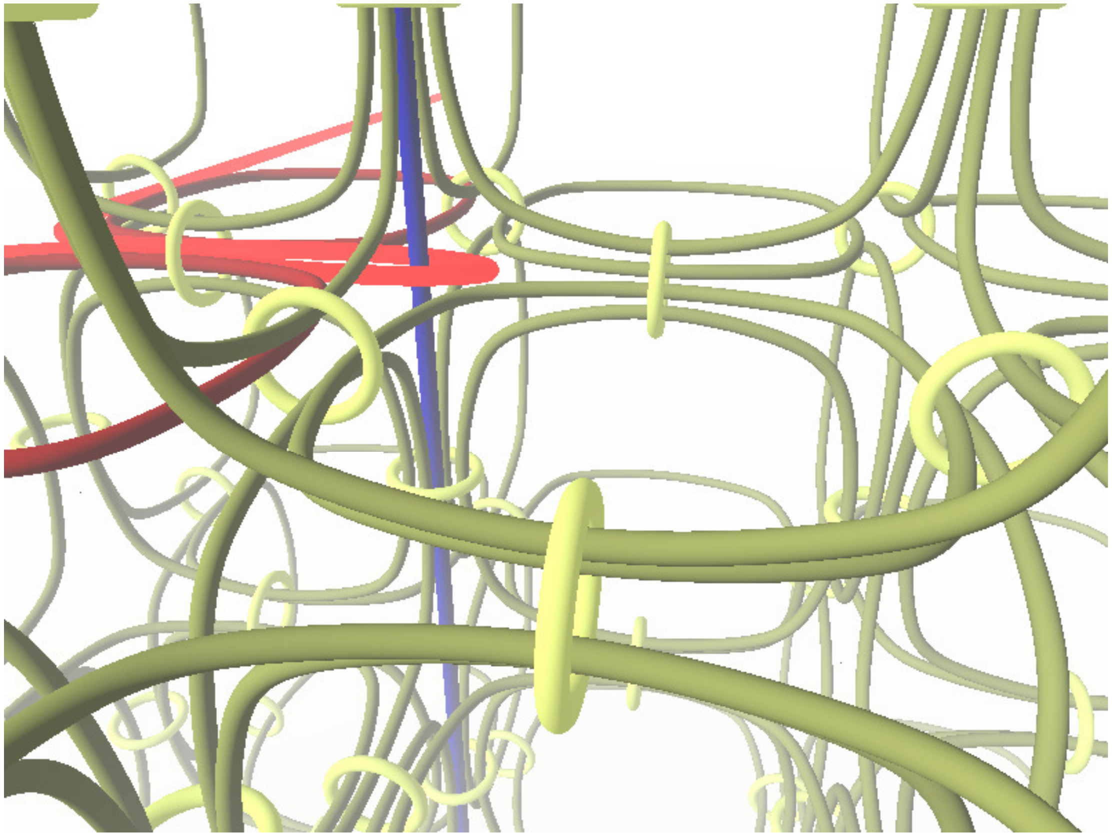
“Chiral” Quasiparticles

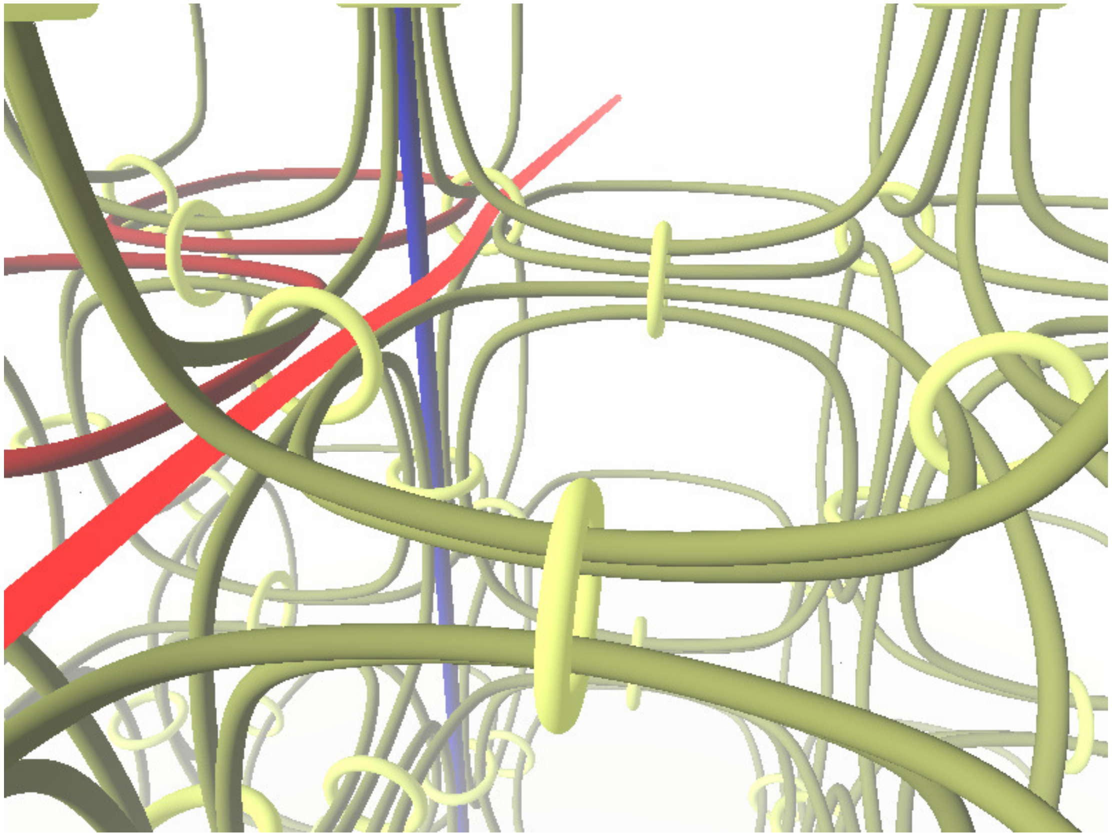


Chiral particles cannot handle slide through each other

Chiral particles have *nontrivial braid statistics*

“Chiral” Quasiparticles

Can handleslide everything to a single plane – but must keep track of over and undercrossings

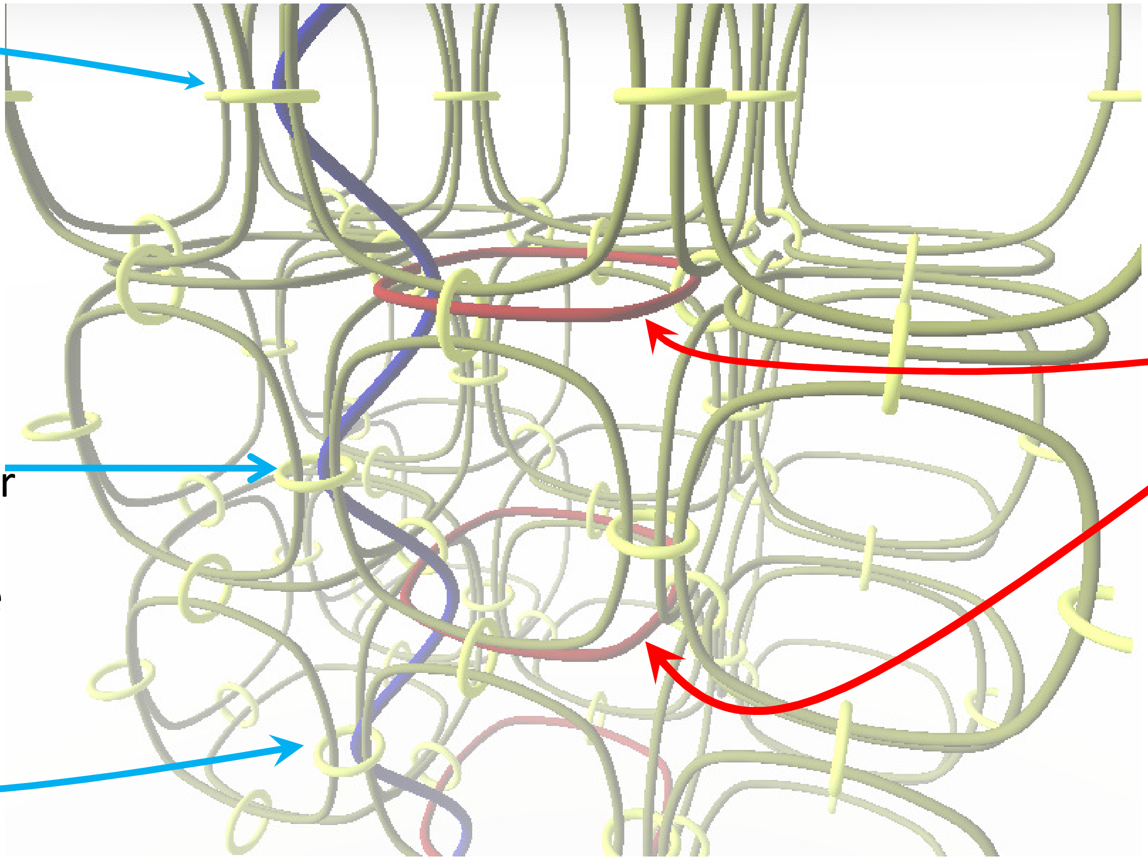


Chiral particles cannot handle slide through each other

Chiral particles have *nontrivial braid statistics*

START HERE FOR SHOWING MIRROR PARTICLE BEHAVIOR

Mirror Quasiparticles



Blue string marked with quantum number $\overline{a}$

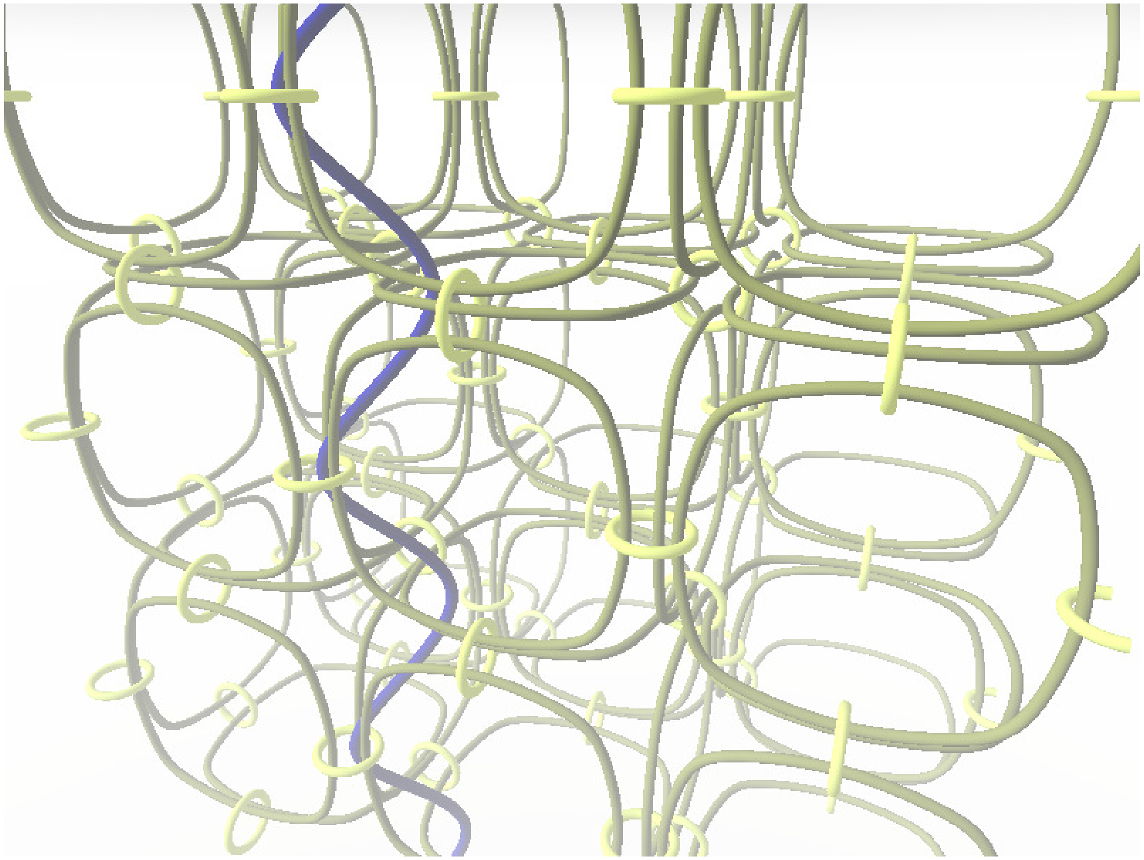
Forces the bonds entering this vertex to fuse to quantum number "a" at three consecutive time steps.

Also forces Flux "a" Through These plaquettes

Mirror quasiparticles must go *through* plaquettes when they cross between cells.

Mirror Quasiparticles

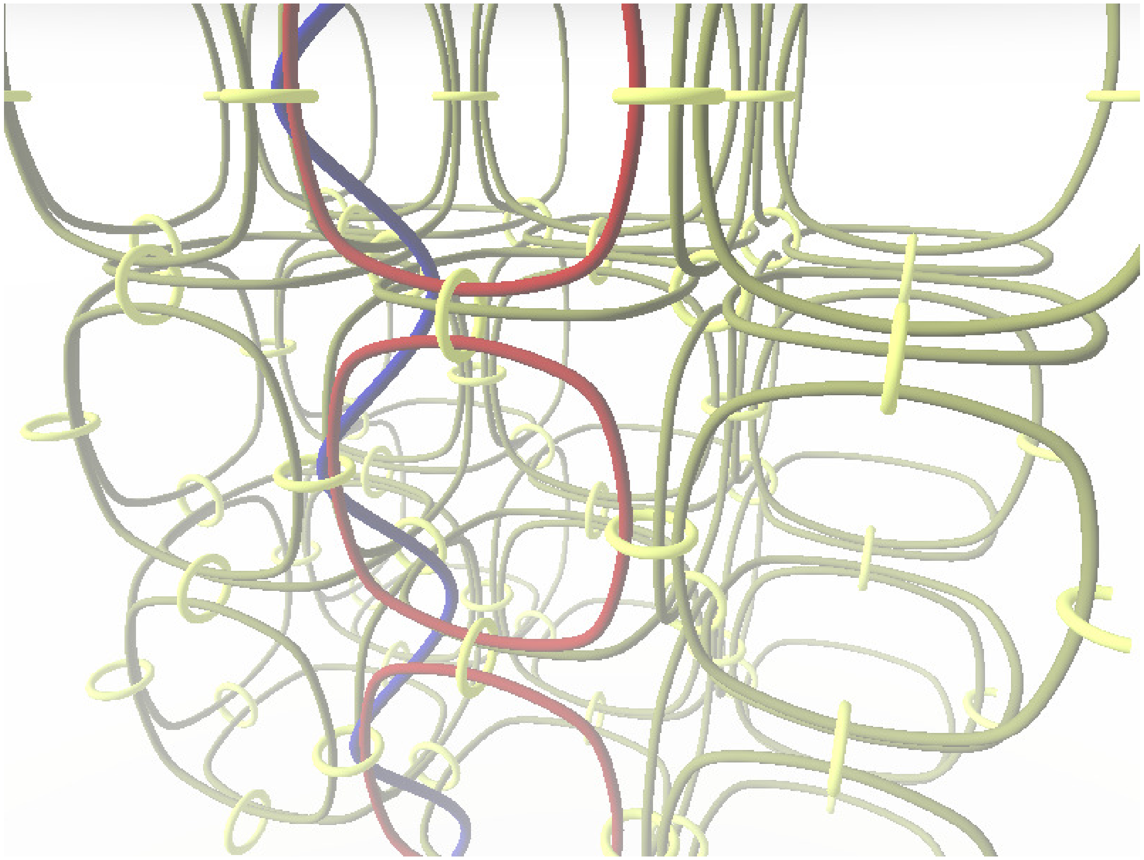
Mirror Handleslide



Mirror quasiparticles must go *through* plaquettes when they cross between cells.

Mirror Quasiparticles

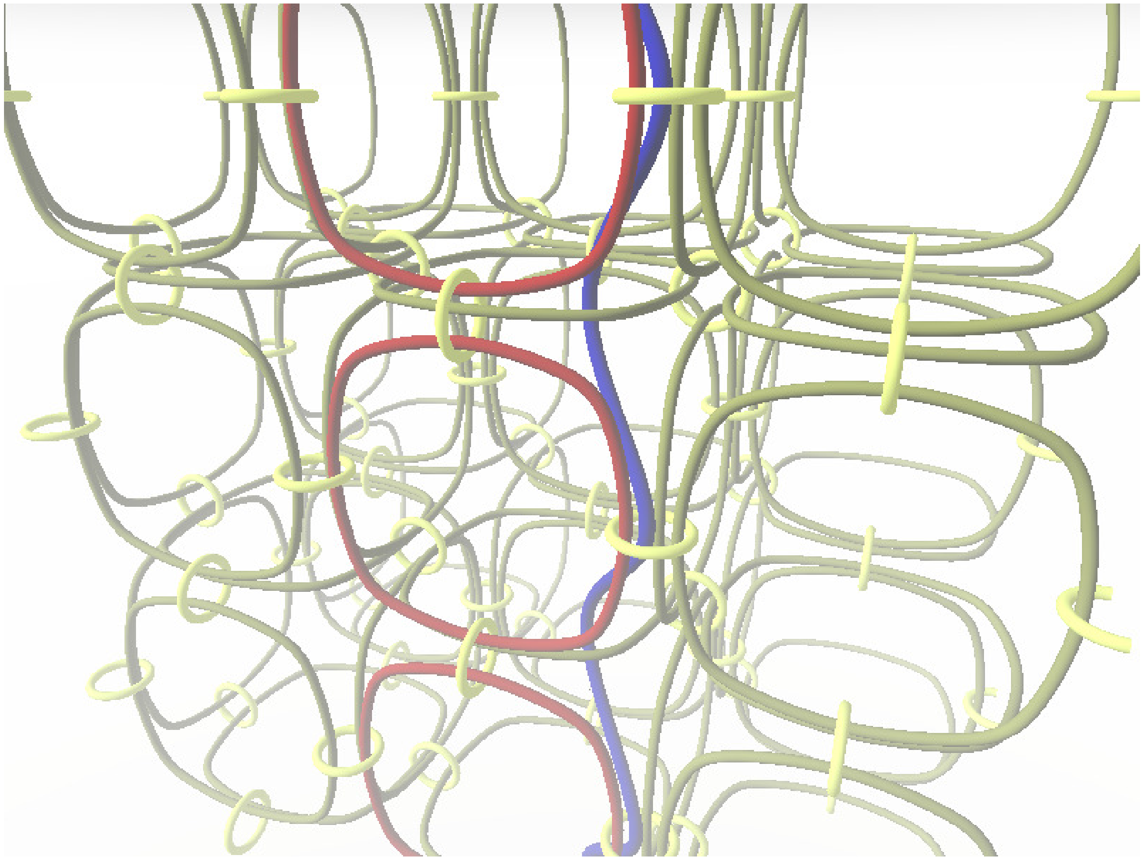
Mirror Handleslide



Handleslide over plaquette

Mirror Quasiparticles

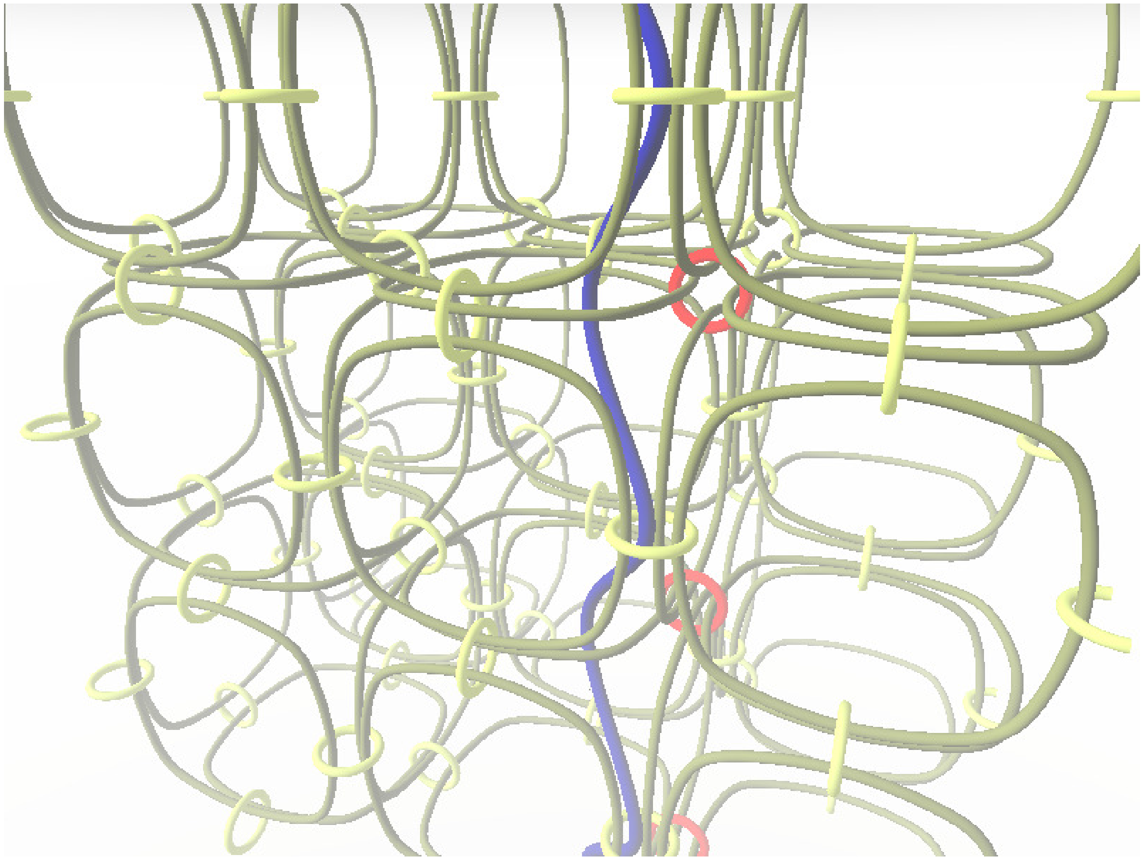
Mirror Handleslide



Handleslide over plaquette

Mirror Quasiparticles

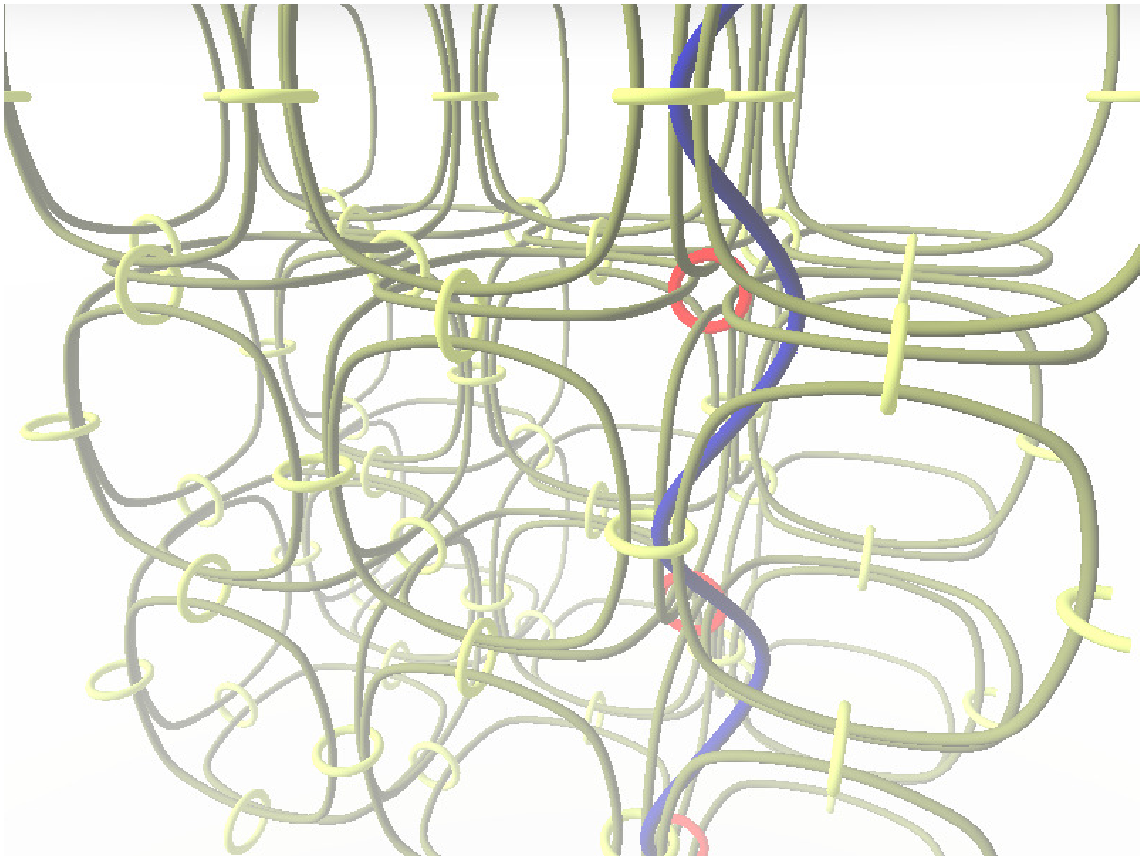
Mirror Handleslide



Handleslide over plaquette… followed by slide over chainmail link

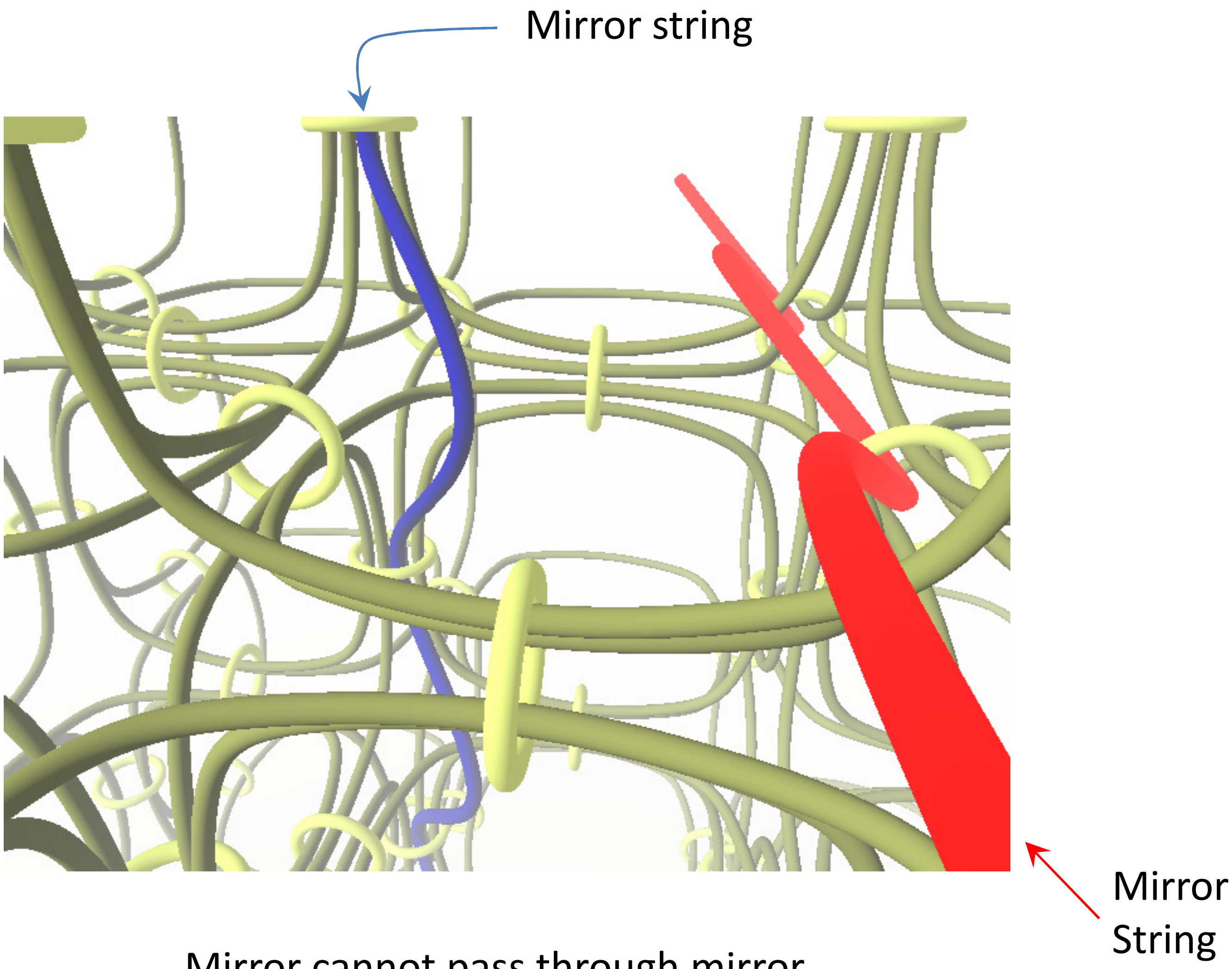
Mirror Quasiparticles

Mirror Handleslide



Handleslide over plaquette… followed by slide over chainmail link

Mirror Quasiparticles

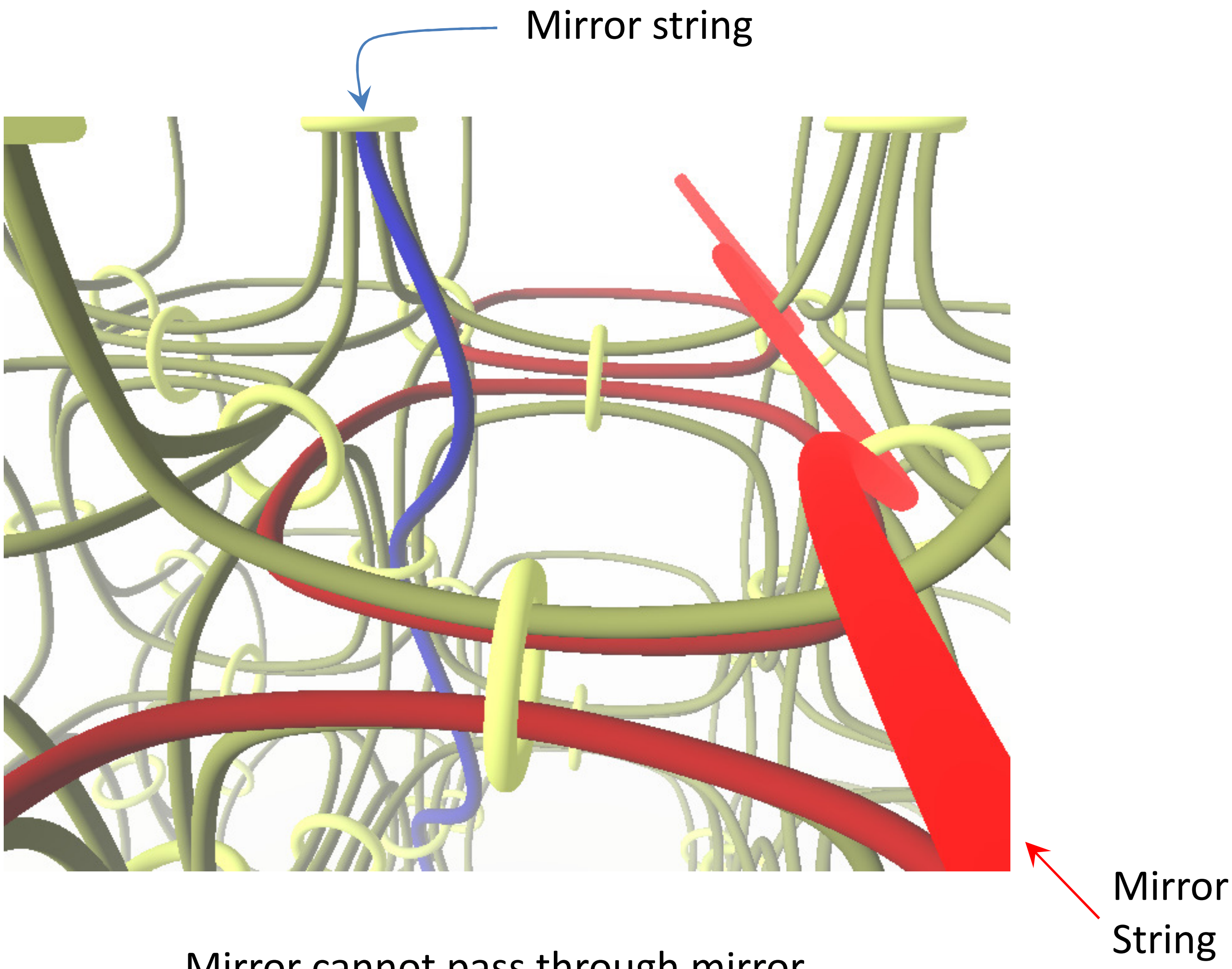


Mirror string

Mirror String

Mirror cannot pass through mirror

Mirror Quasiparticles



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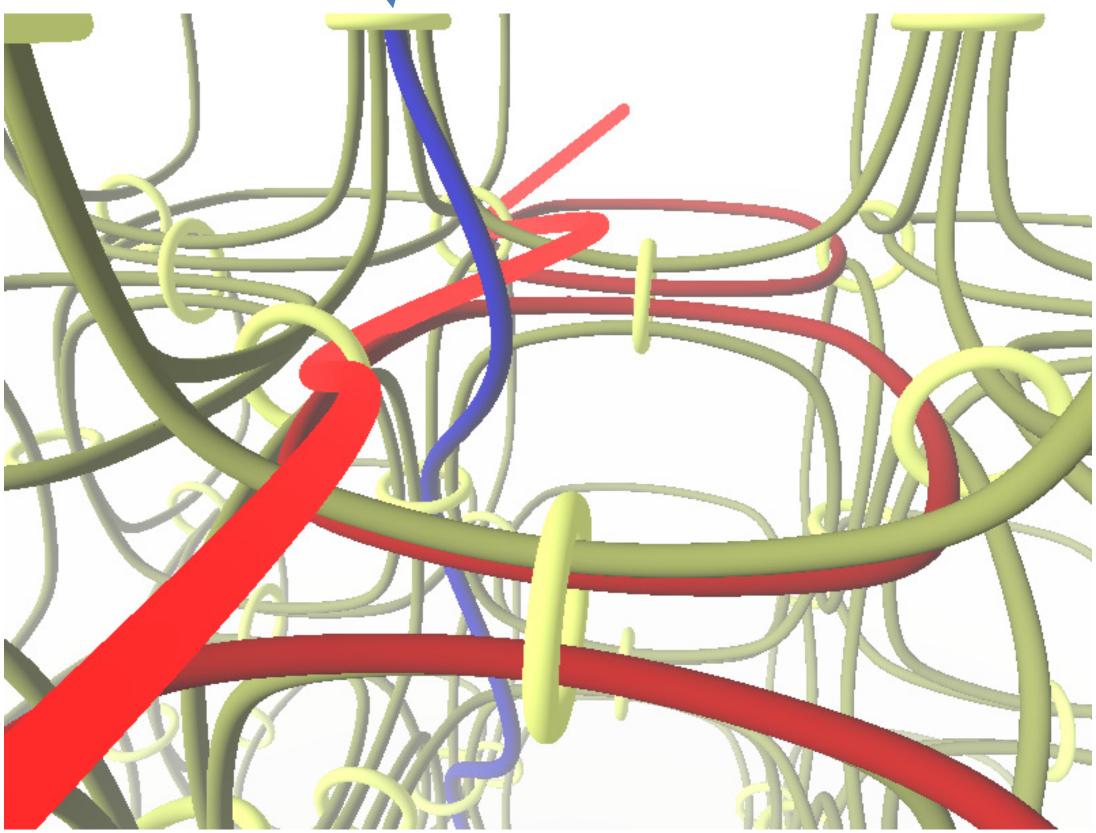
Mirror Quasiparticles



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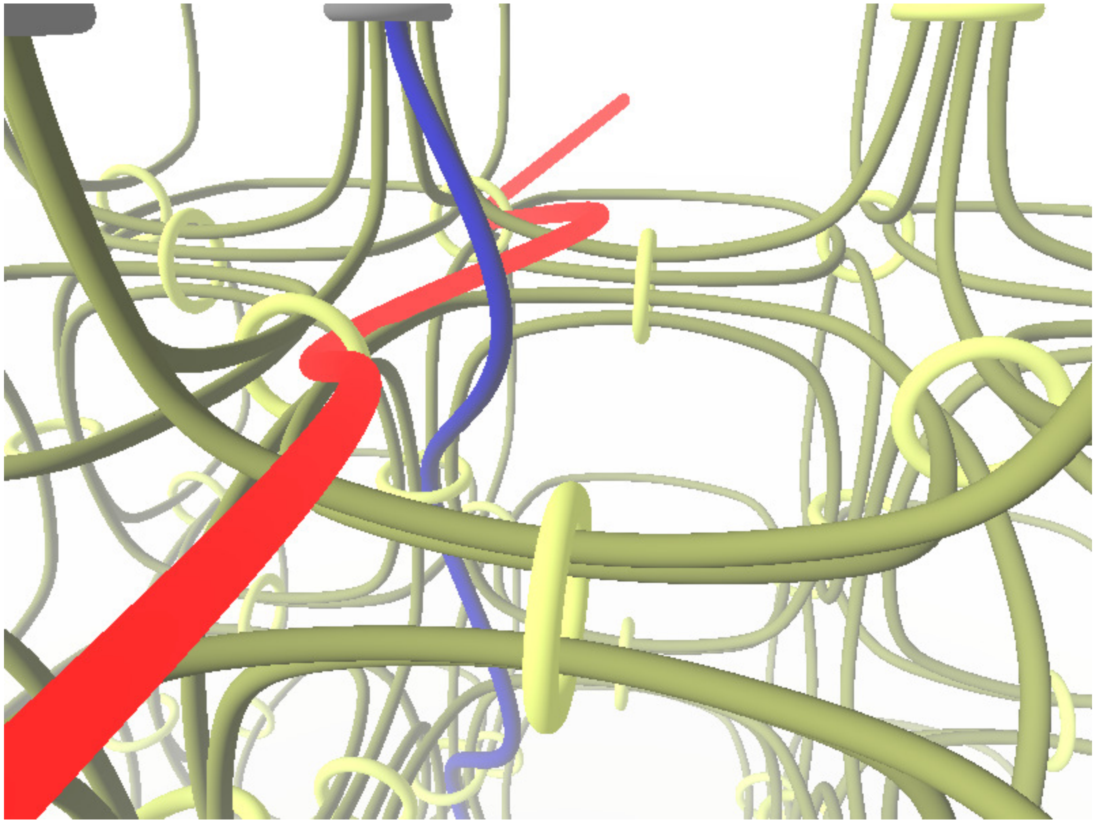

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String

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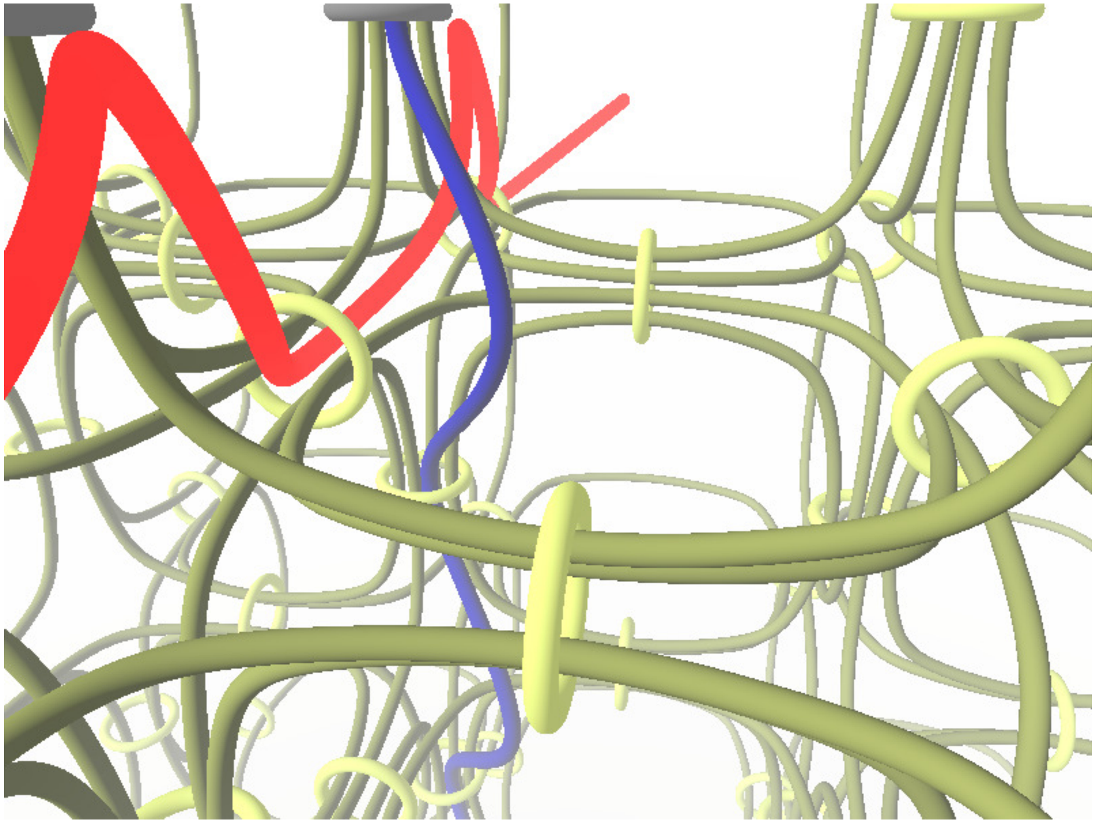
Mirror Quasiparticles



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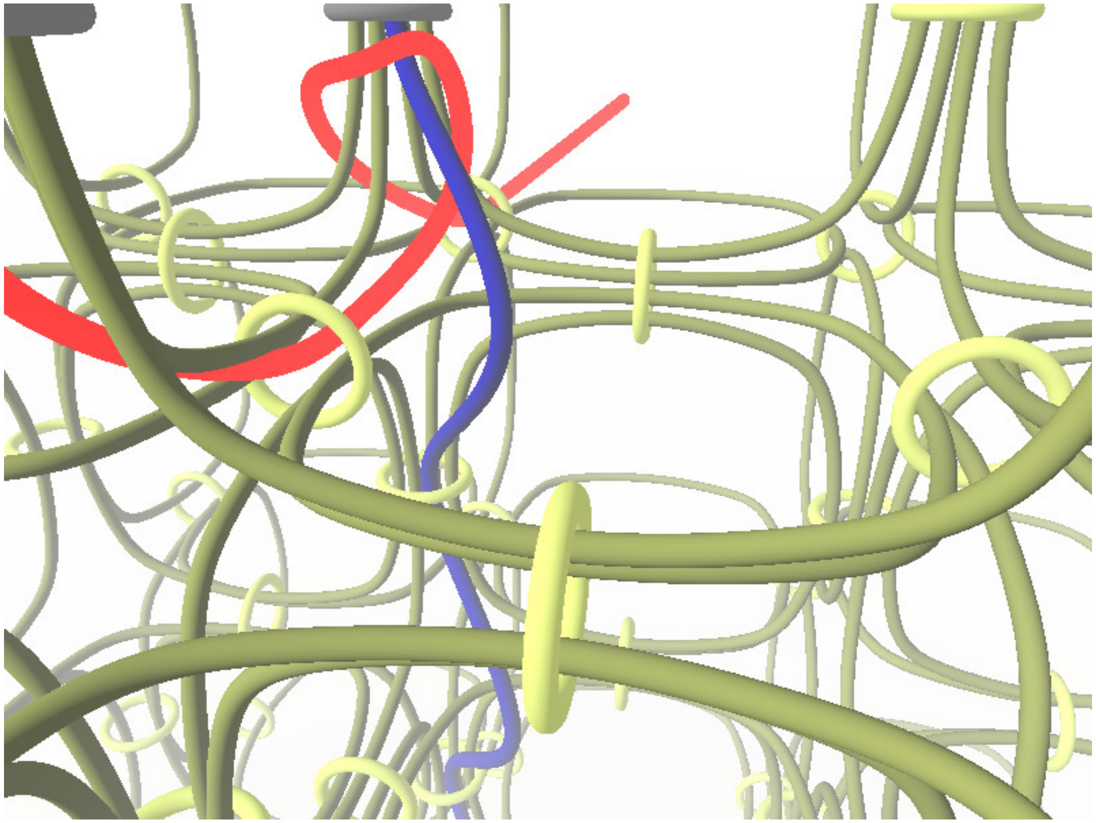

Mirror string

Mirror String

Mirror cannot pass through mirror

START HERE FOR SHOWING CHIRALITY REVERSAL