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Non-Abelian Statistics as a Berry Phase in the Honeycomb Lattice Model

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Ville Lahtinen and Jiannis K. Pachos
DAQIST workshop, 18 September 2009, Maynooth

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Anyons in an exactly solvable model

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Kitaev's honeycomb lattice model:
A.Y. Kitaev, *Annals of Physics*, 321:2, 2006

- An exactly solvable 2D spin model on a honeycomb lattice
- Known to support non-Abelian Ising anyons based on Chern number and CFT arguments

Anyons in an exactly solvable model

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- An exactly solvable 2D spin model on a honeycomb lattice
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<u>We can do more:</u>

- Demonstrate the fusion rules from the spectrum
- Calculate the non-Abelian statistics from the eigenstates
- Understand the non-Abelian behavior microscopically
- Provide methods and predictions for future experiments

Anyons in an exactly solvable model

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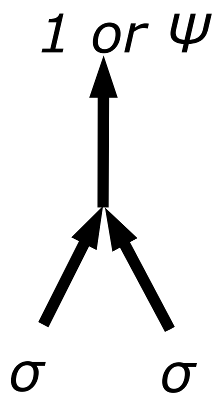
To demonstrate the Ising anyons, one needs to demonstrate:

(1) <u>Ising fusion rules</u>



$$\boxed{\sigma \times \sigma = 1 + \Psi}$$

$$\Psi \times \sigma = \sigma$$

$$\Psi \times \Psi = 1$$

DONE!

VL et al., Ann. Phys. 323, 9 (2008)

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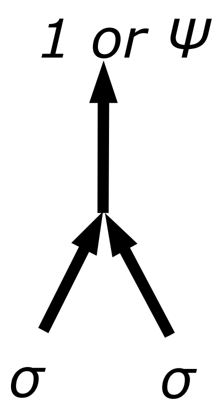

$$\sigma \times \sigma = 1 + \psi$$
$$\psi \times \sigma = \sigma$$
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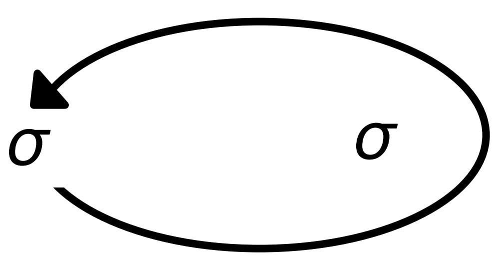
VL et al., Ann. Phys. 323, 9 (2008)

(2) <u>Statistics</u>



$$R^2 = e^{-i\pi/4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

?

The honeycomb lattice model

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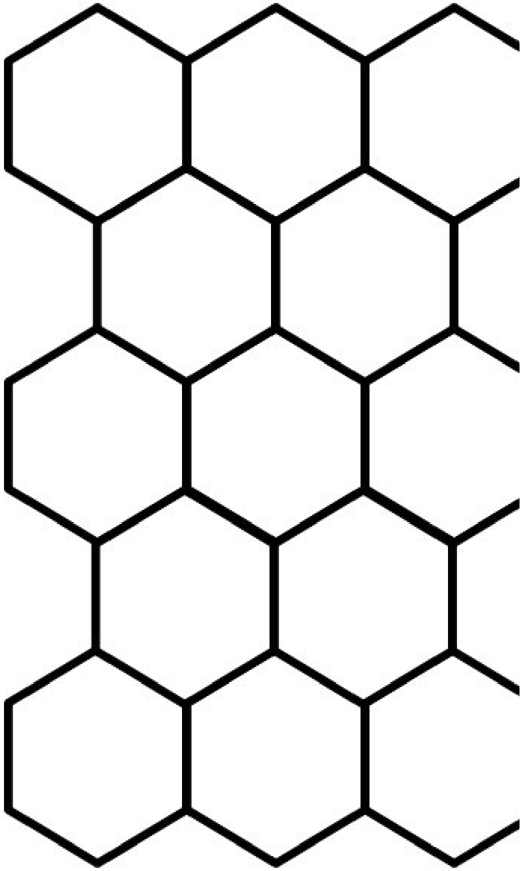
- The Hamiltonian

$$H = -\sum_{\nu \in \{x,y,z\}} \sum_{(i,j) \in \nu\text{-links}} J_{ij}^{\nu} \sigma_i^{\nu} \sigma_j^{\nu} - \sum_{(i,j,k)} K_{ijk} \sigma_i^x \sigma_j^y \sigma_k^z$$



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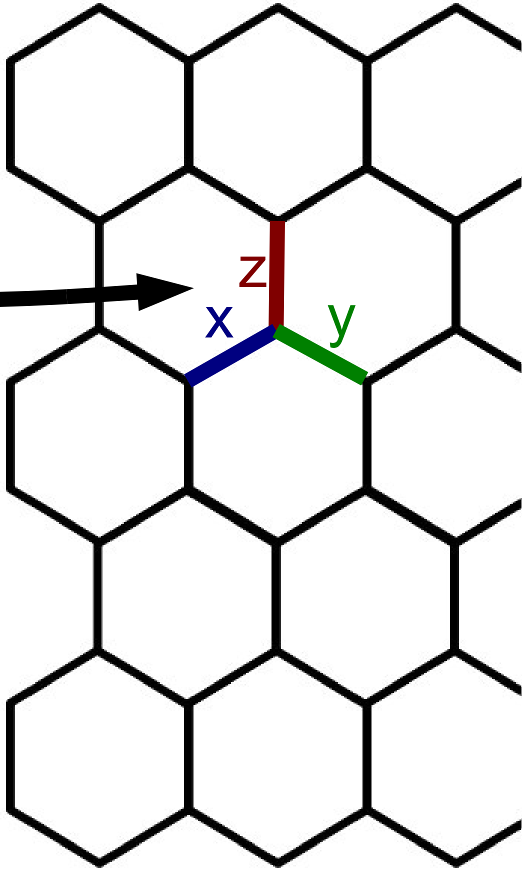
Anisotropic nearest neighbour couplings



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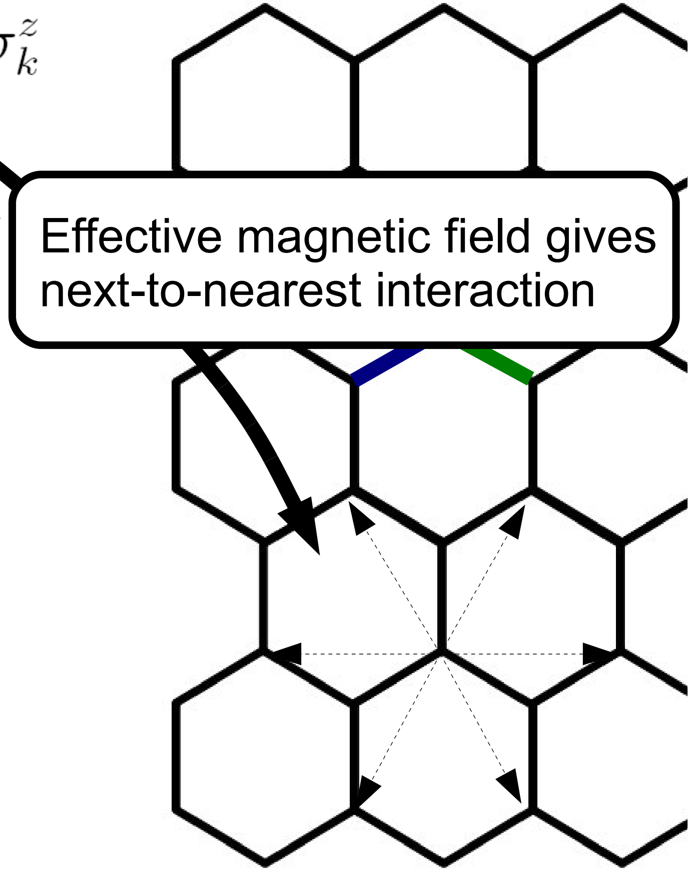
$$\sum_{(i,j,k) \in p} K_{ijk} \sigma_i^x \sigma_j^y \sigma_k^z = K_{123} \sigma_1^z \sigma_2^y \sigma_3^x + K_{234} \sigma_2^x \sigma_3^z \sigma_4^y + K_{345} \sigma_3^y \sigma_4^x \sigma_5^z + K_{456} \sigma_4^z \sigma_5^y \sigma_6^x + K_{561} \sigma_5^x \sigma_6^z \sigma_1^y + K_{612} \sigma_6^y \sigma_1^x \sigma_2^z .$$



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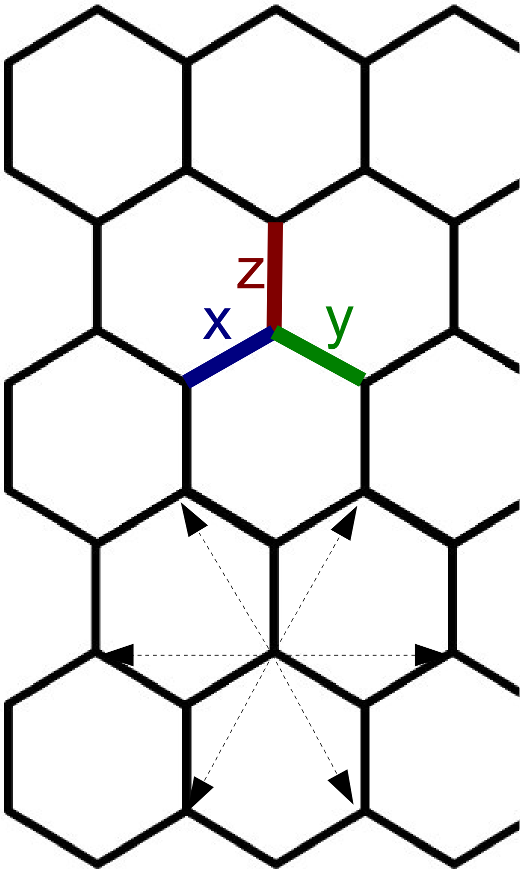
Represent Pauli operators by Majorana fermions

$$H = \frac{i}{4}\sum_{i,j} \hat{A}_{ij}c_i c_j \qquad \hat{A}_{ij} = 2J_{ij}\hat{u}_{ij} + 2\sum_k K_{ijk}\hat{u}_{ik}\hat{u}_{jk}$$



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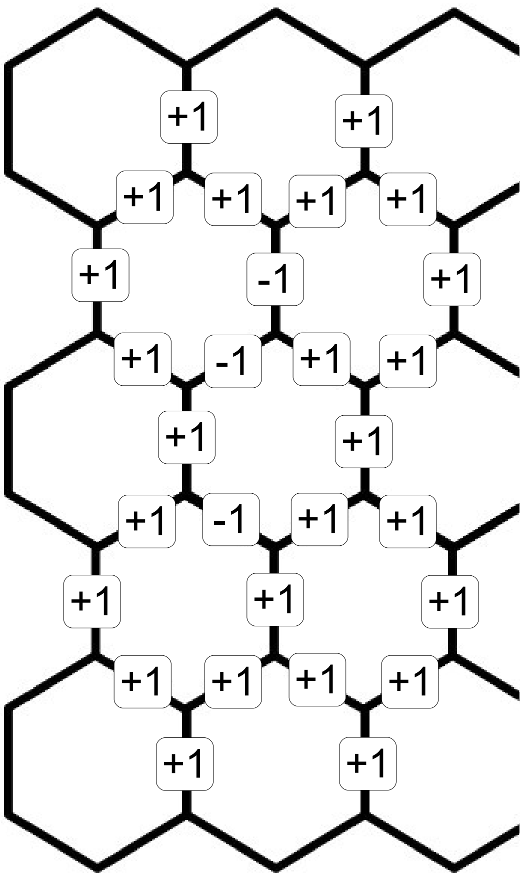
- Static background Z2 gauge field living on every link
- $[H, \hat{u}_{ij}] = 0$
- Fixing *u* fixes the gauge and the physical sector



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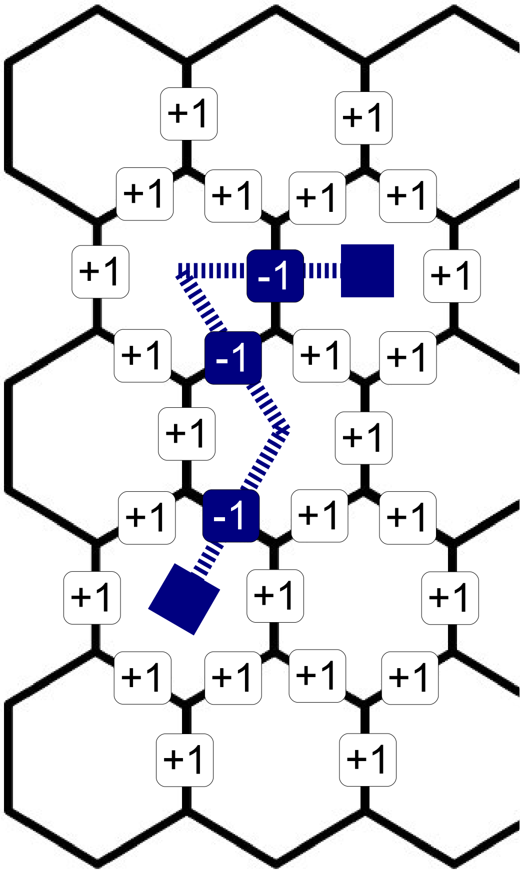
- Eigenvalue w_p = -1: a vortex at plaquette *p*



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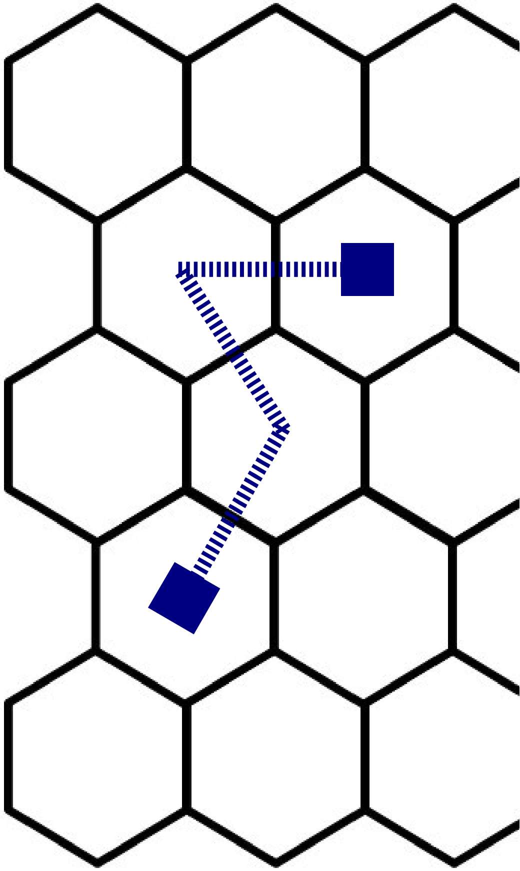

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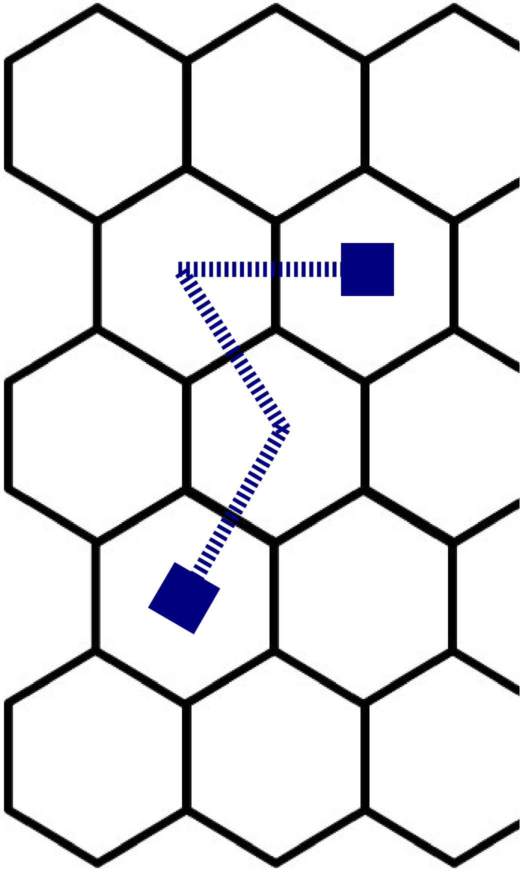
To solve the model:

- Choose the system size and fix the boundary conditions (we work on torus).



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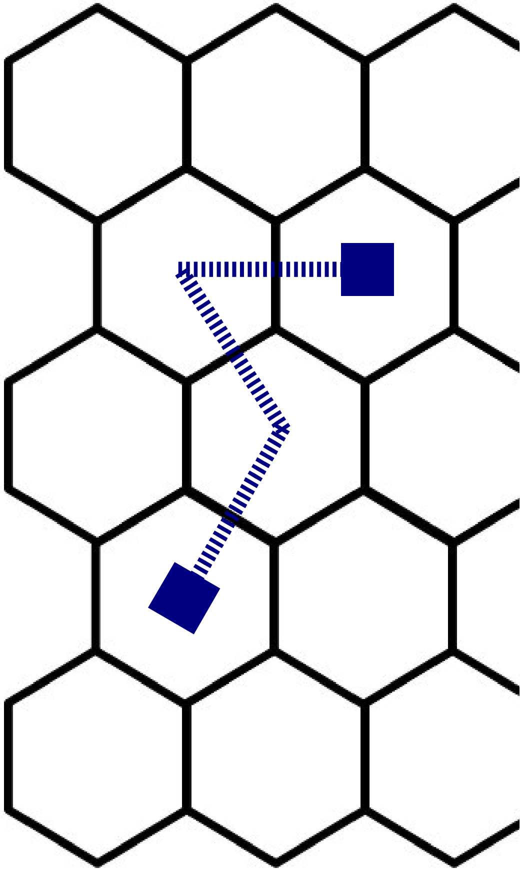

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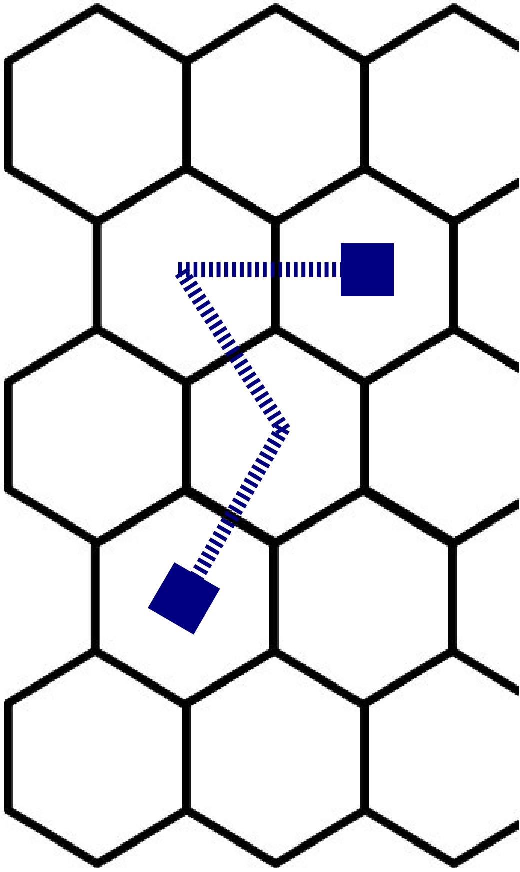

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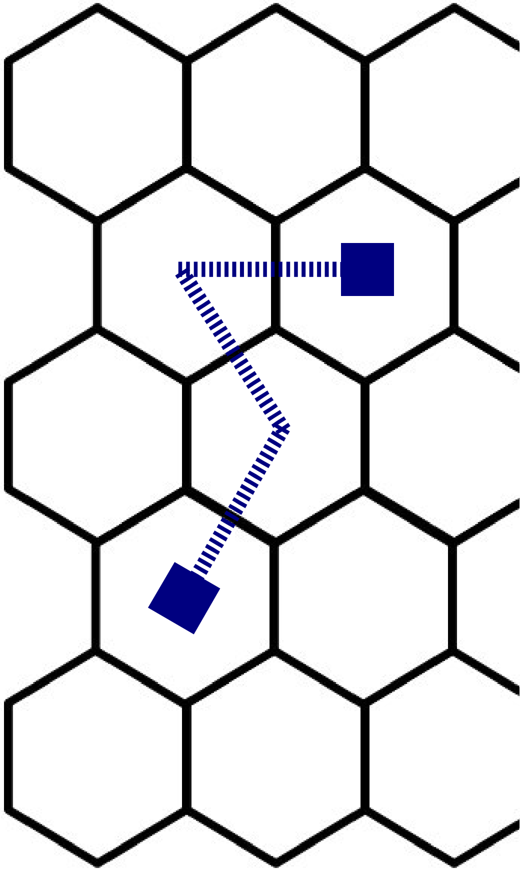

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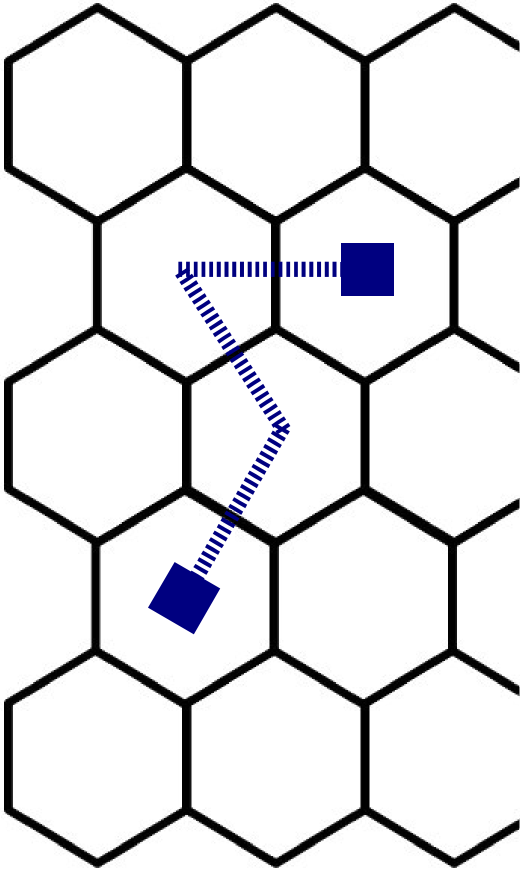
- Wait.



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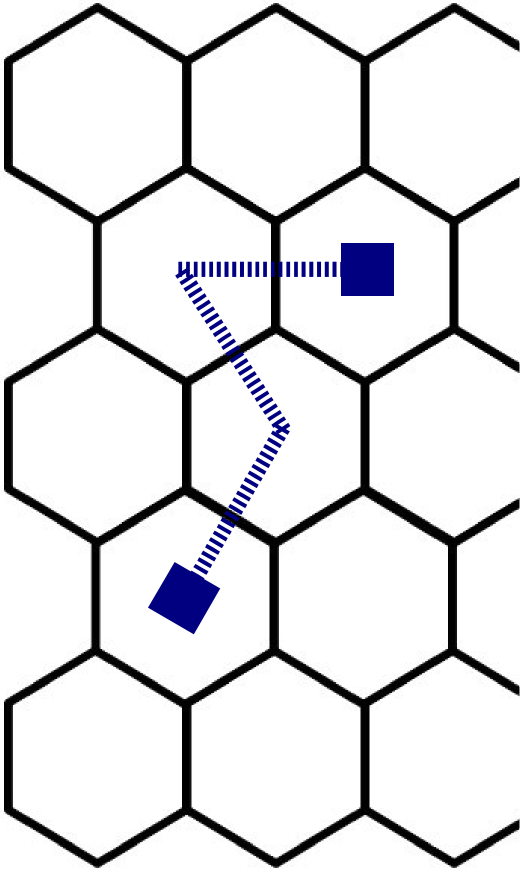
- ... while waiting figure out what to do with the spectrum and eigenvectors ..



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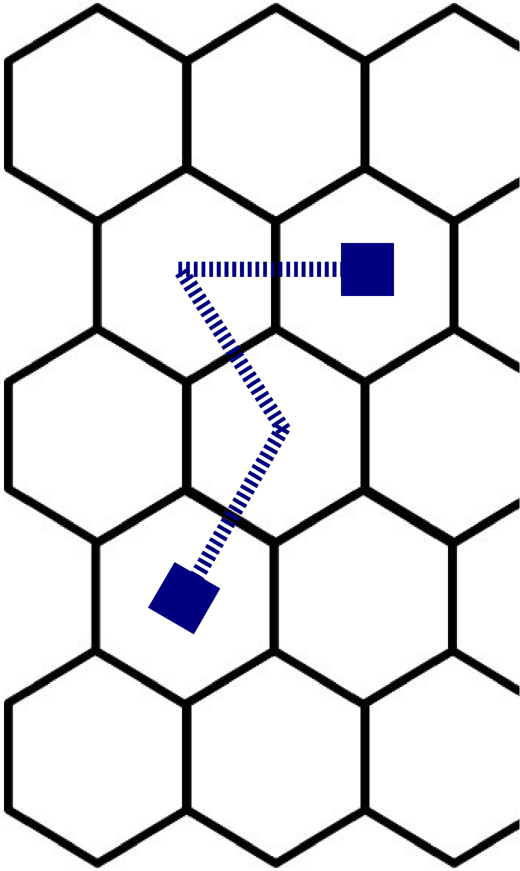

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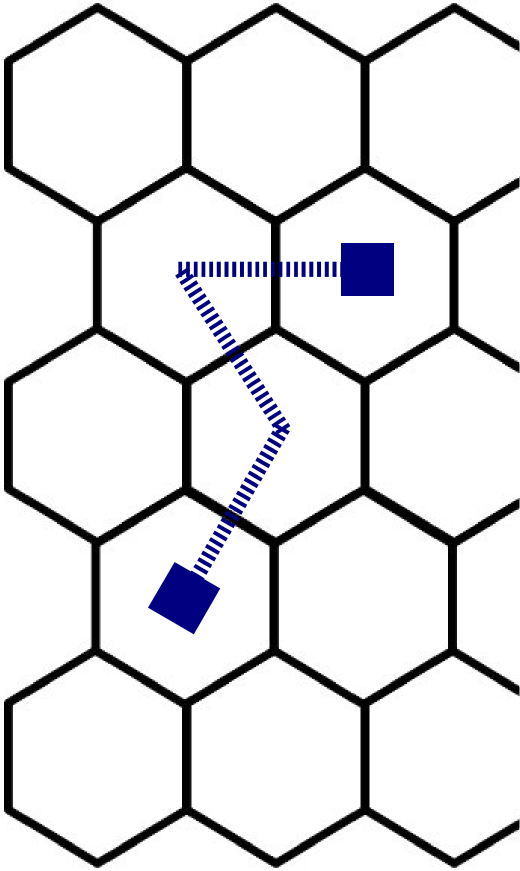
- Get the results, change the parameters and repeat.



Vortex transport

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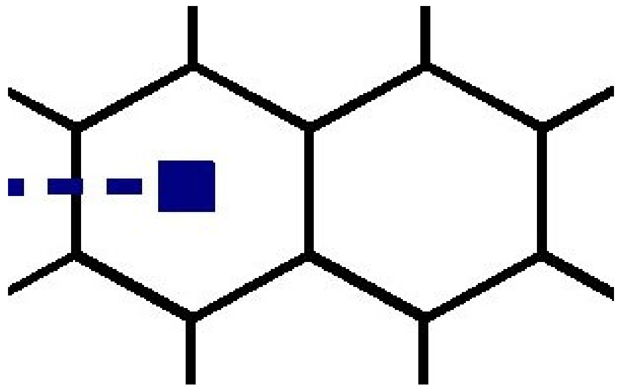
How to simulate the transport of vortices?

The parameters J_{ij} and K_{ikj} appear in the Hamiltonian always paired with the local gauge fields u_{ij}.

$$H = \frac{i}{4}\sum_{i,j} \hat{A}_{ij} c_i c_j \qquad \hat{A}_{ij} = 2J_{ij}\hat{u}_{ij} + 2\sum_k K_{ijk}\hat{u}_{ik}\hat{u}_{jk}$$



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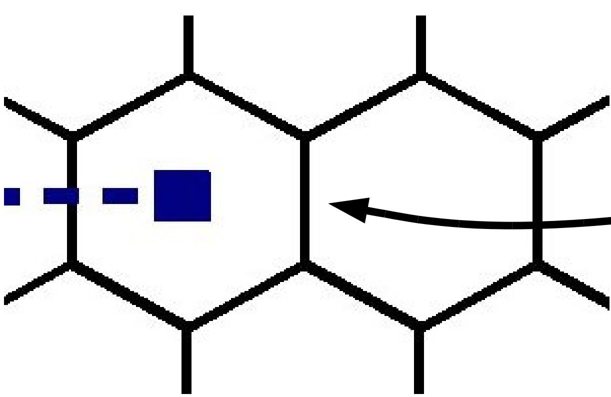

- Assume local control on link (*ij*)

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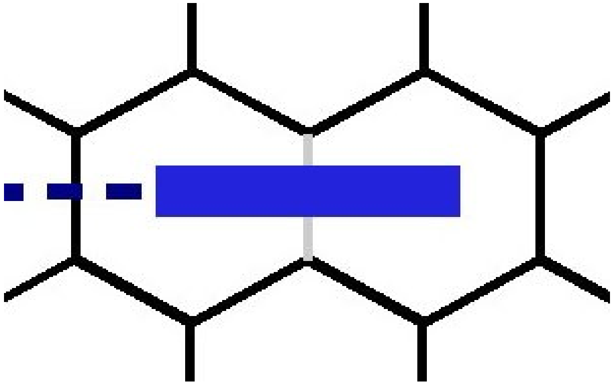

- Assume local control on link (*ij*)

  $J_{ij}, K_{ikj} \rightarrow 0$

- The vortex occupies both plaquettes

Vortex transport

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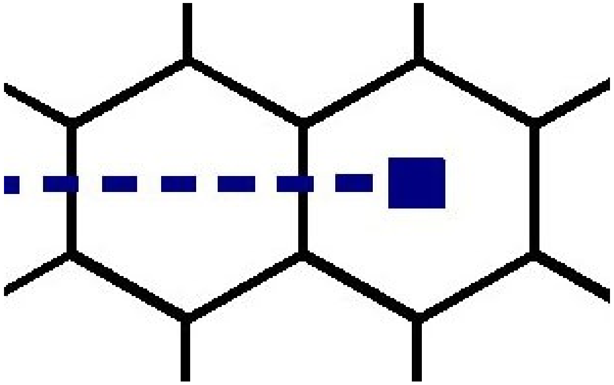
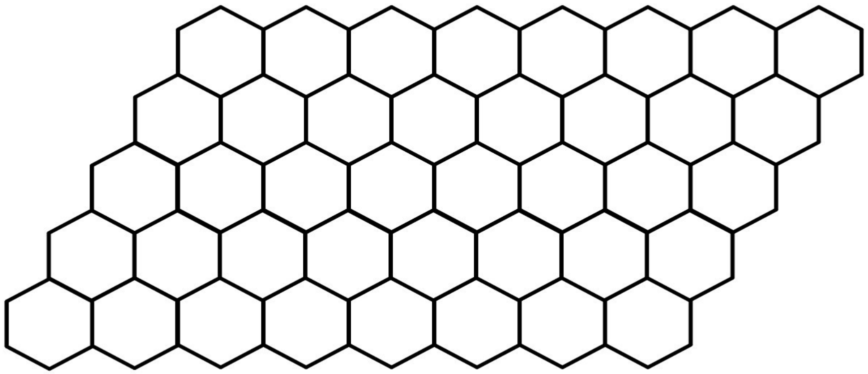


- Assume local control on link (*ij*)

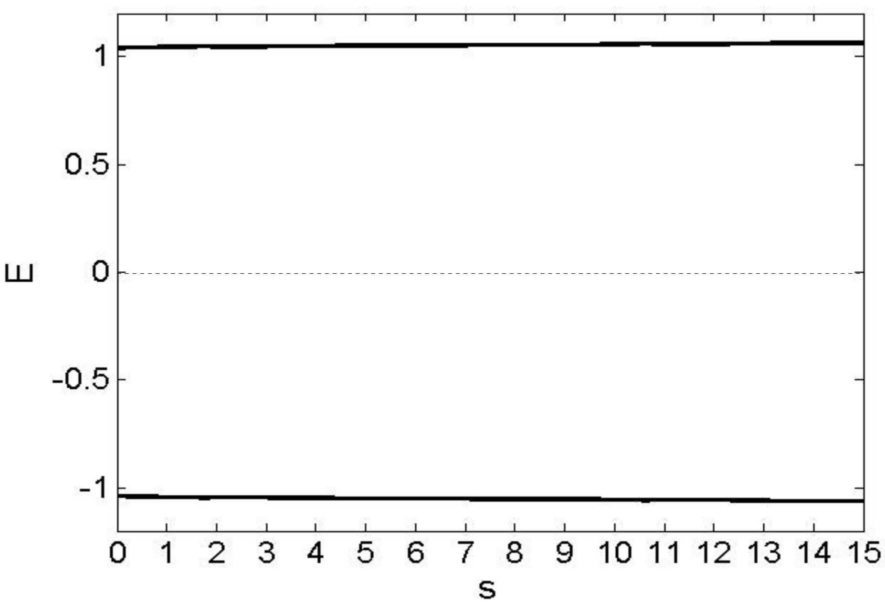
$$J_{ij}, K_{ikj} \rightarrow -J_{ij}, -K_{ikj}$$

- Equivalent to changing u_{ij} -> $-u_{ij}$.

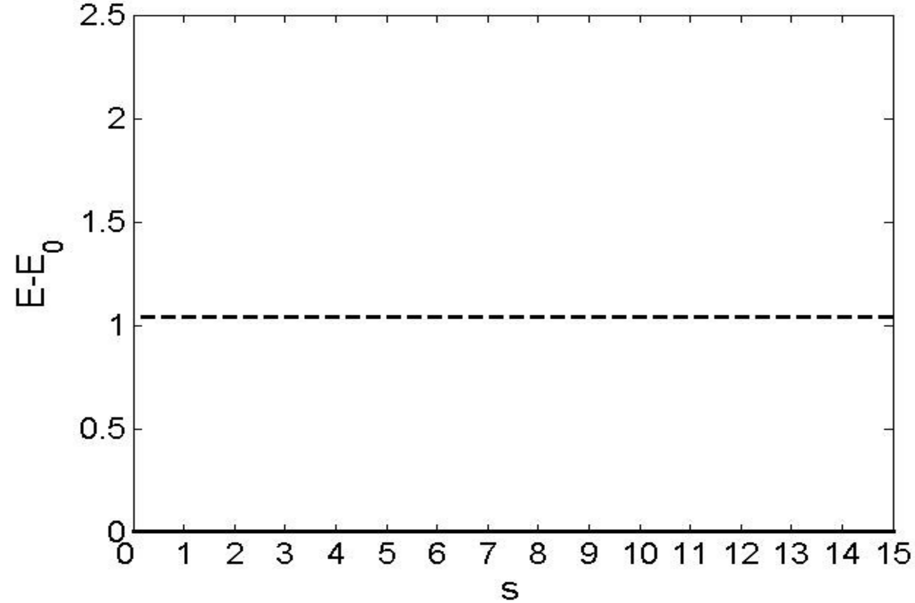
Zero modes and fusion rules



Mode spectrum $A\psi_k^{\pm} = \pm\epsilon_k\psi_k^{\pm}$



Full low-energy spectrum



Zero modes and fusion rules

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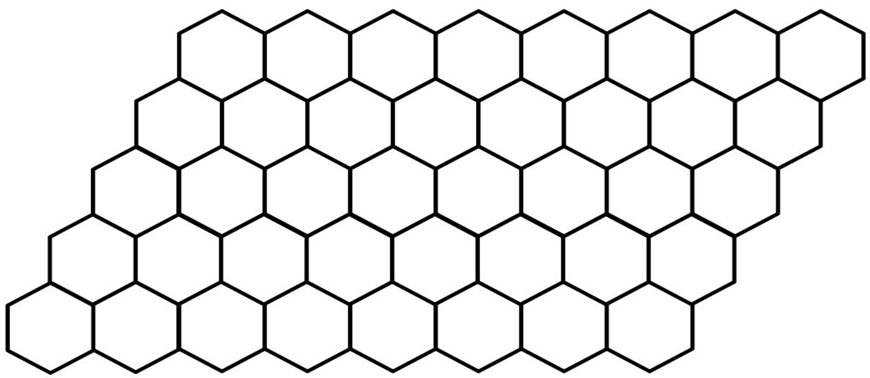

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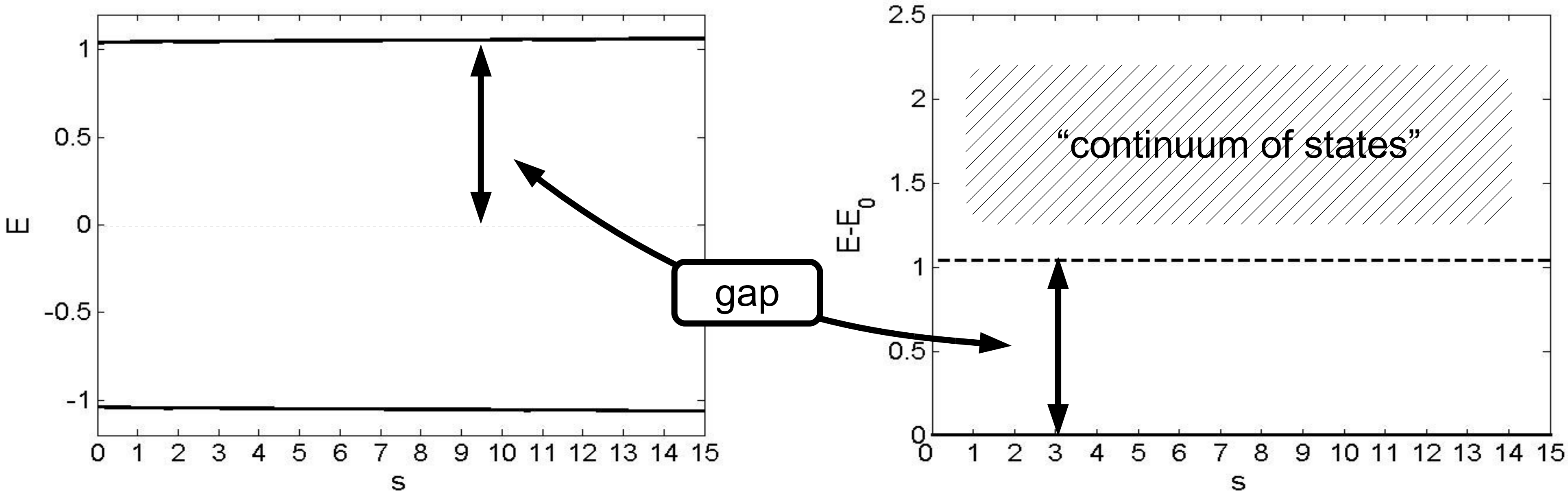

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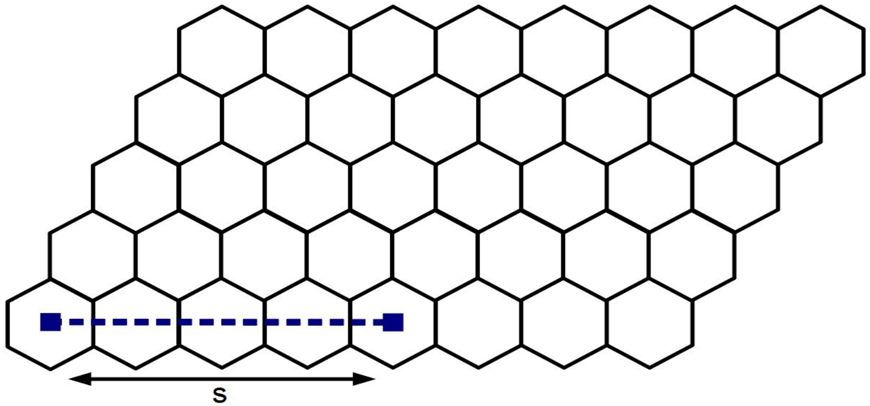

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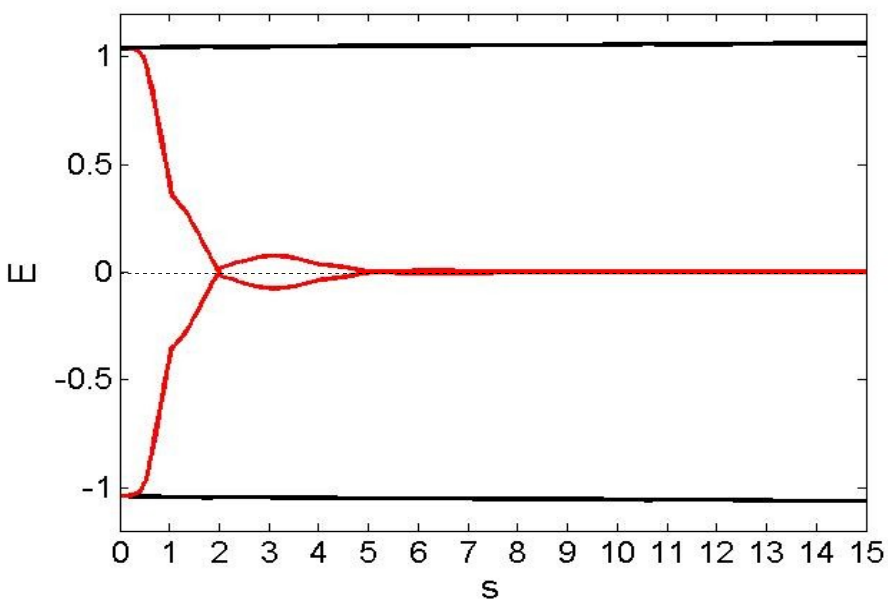

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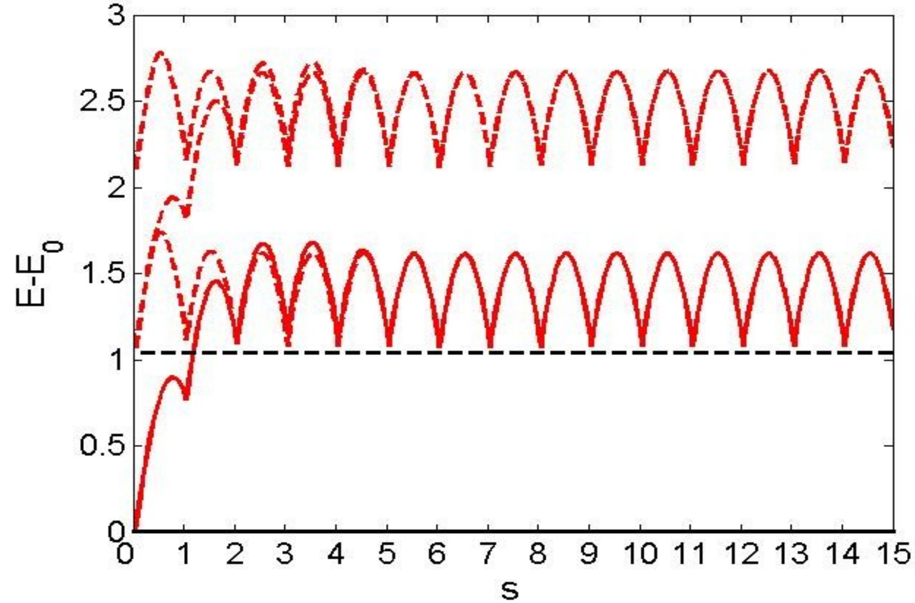

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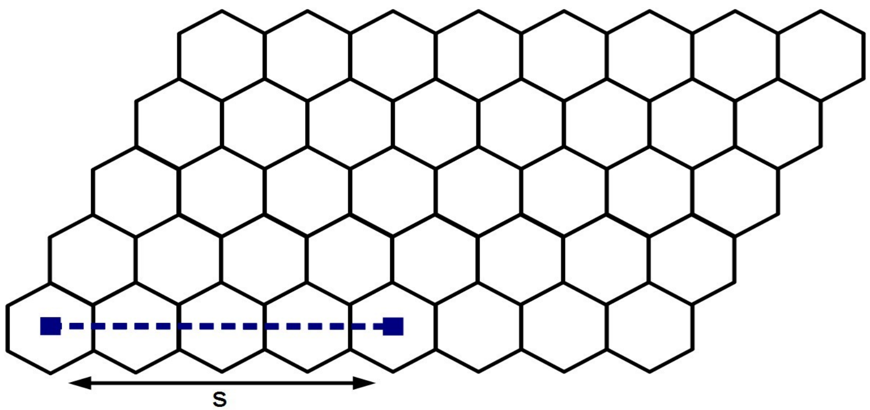

- 2^n-fold degeneracy for 2n vortices

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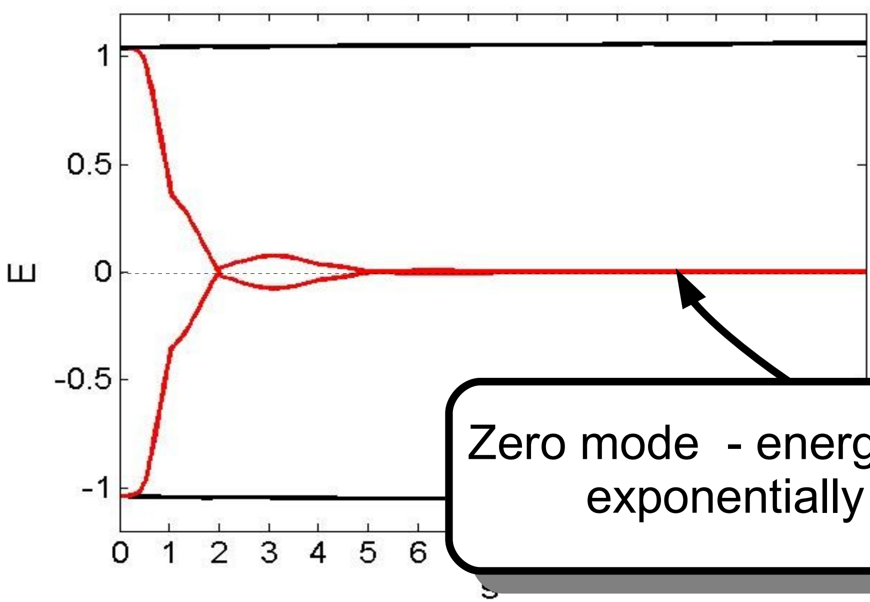
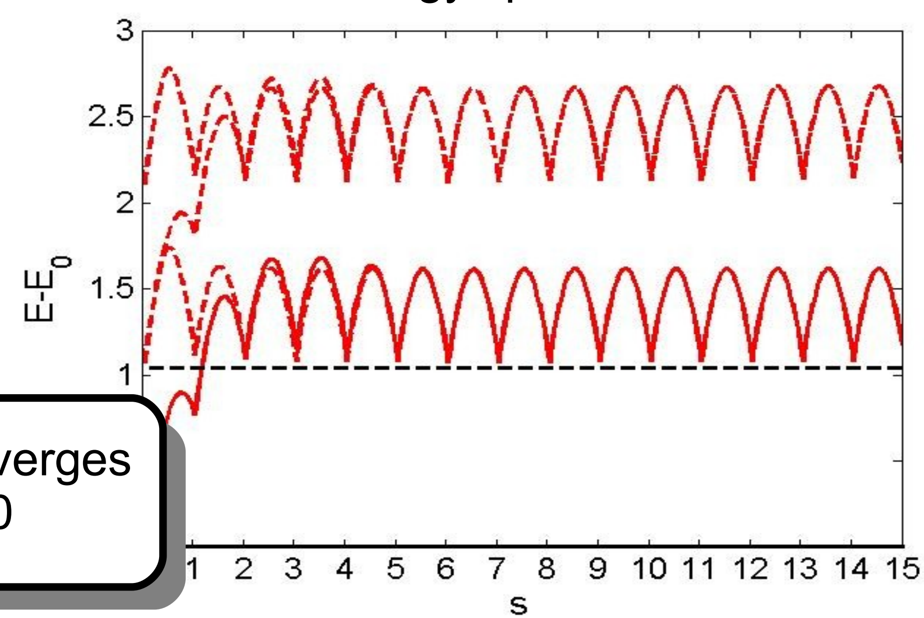


Zero mode - energy converges exponentially to E=0

Full low-energy spectrum



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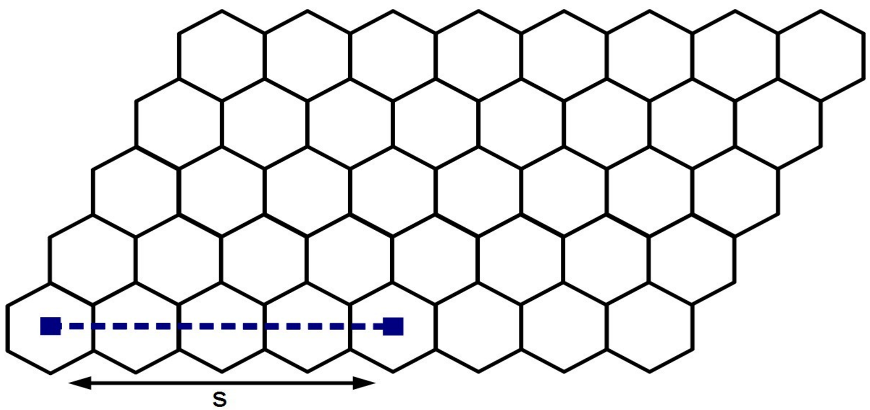

- 2^n-fold degeneracy for 2n vortices
- Interactions lift degeneracy when vortices nearby each other

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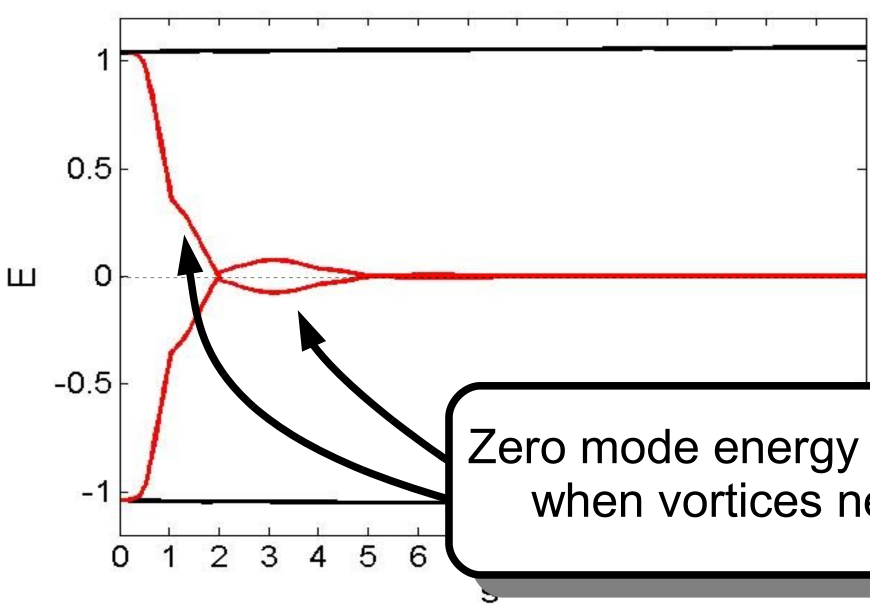
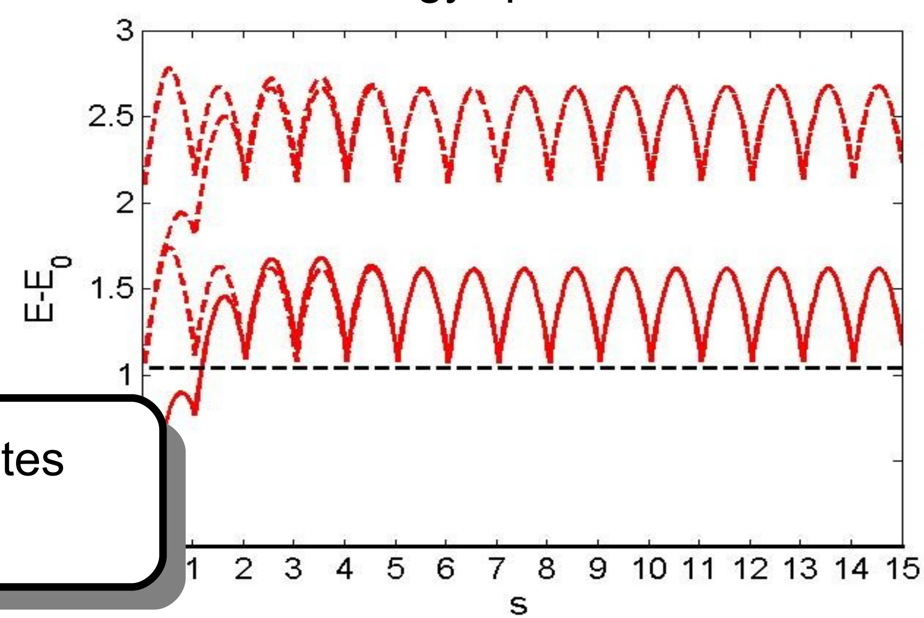


Zero mode energy oscillates when vortices nearby.

Full low-energy spectrum



Zero modes and fusion rules

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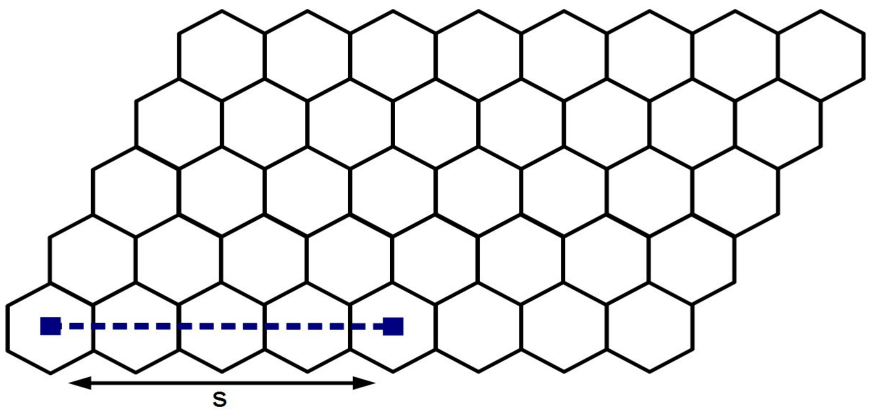

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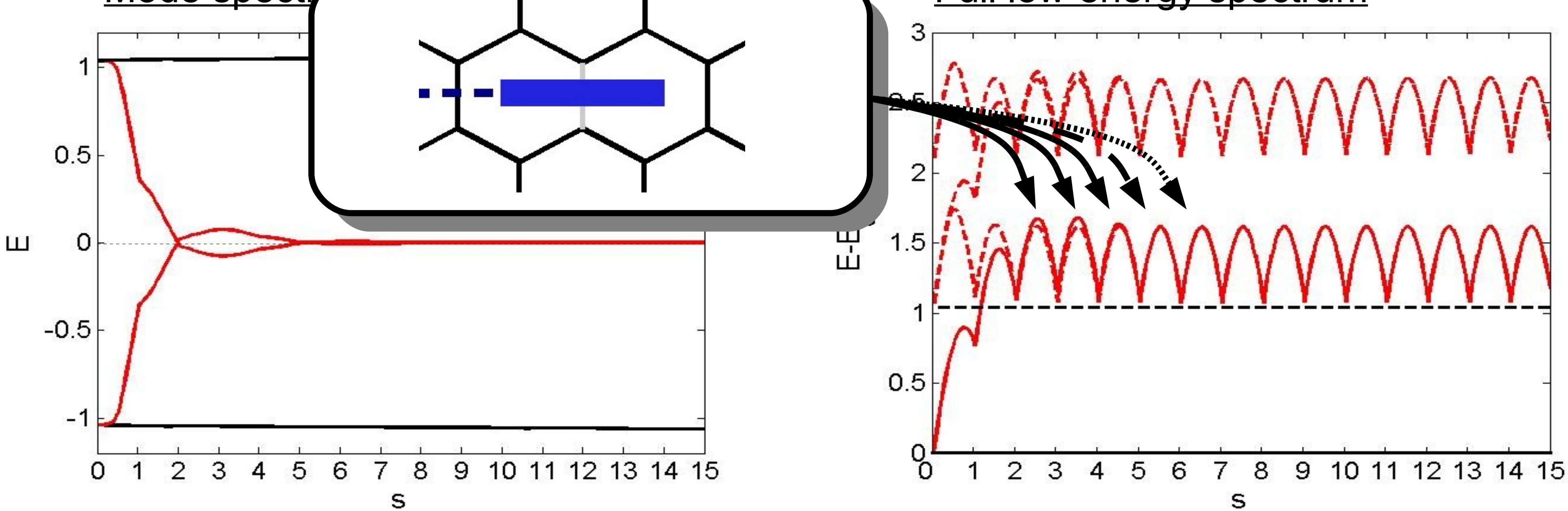

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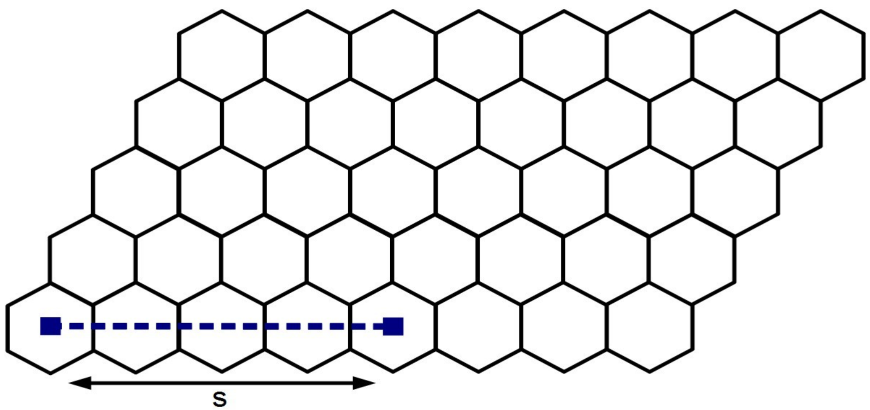
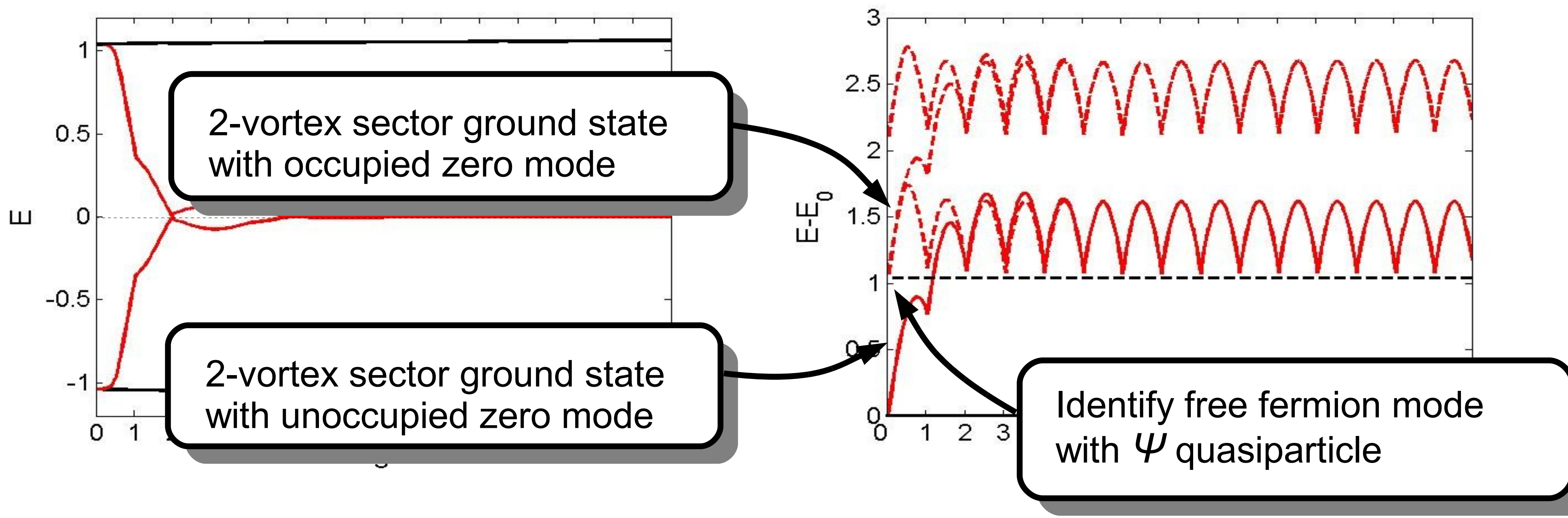


- 2^n-fold degeneracy for 2n vortices
- Interactions lift degeneracy when vortices nearby each other
- Occupation of a zero mode corresponds to the fusion channel

$$\sigma \times \sigma = 1 + \Psi$$

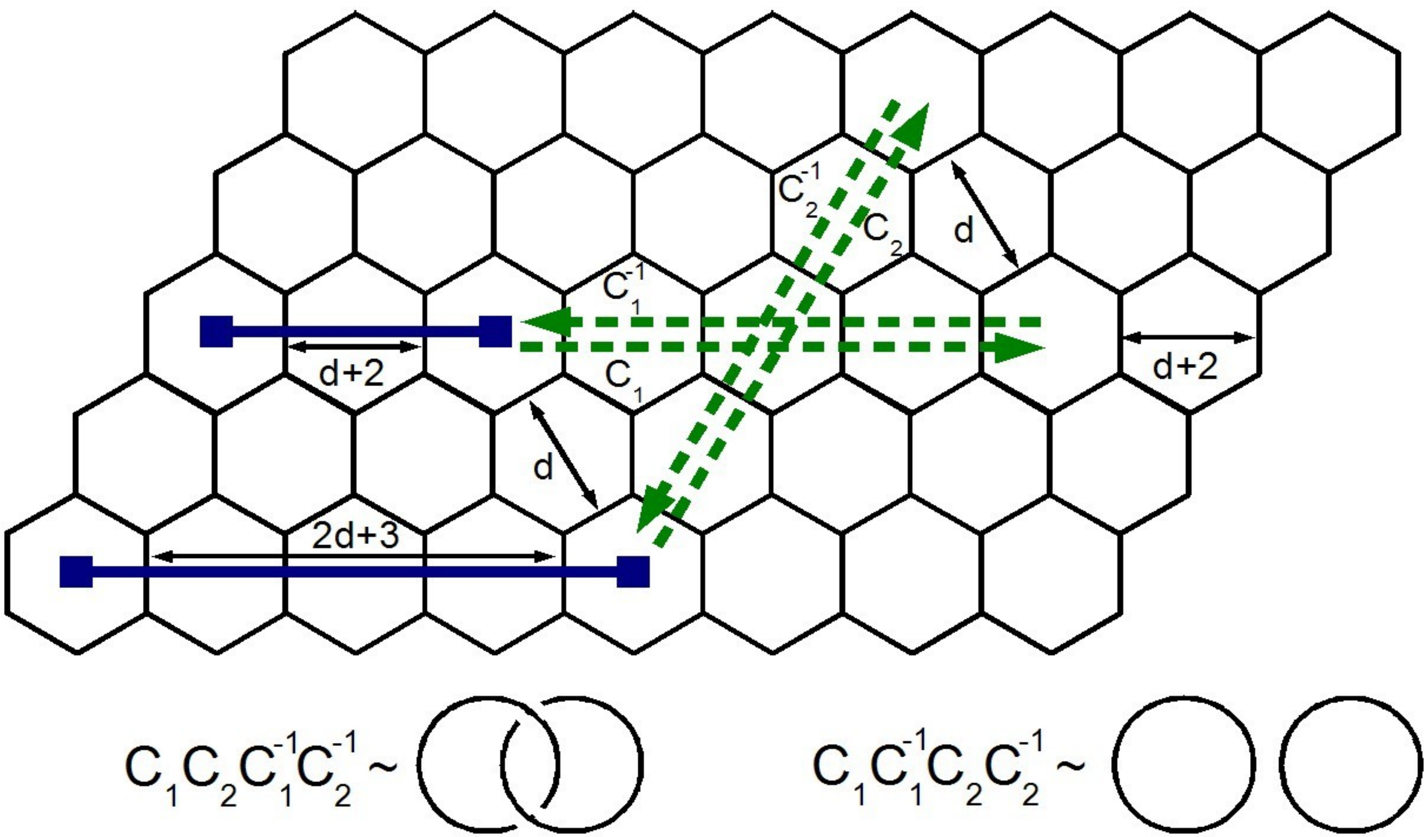
Mode spectrum $A\psi_k^{\pm} = \pm\epsilon_k\psi_k^{\pm}$

Full low-energy spectrum



Non-Abelian statistics as a Berry phase

Can we evaluate the corresponding evolution of the system?



Non-Abelian statistics as a Berry phase

Under adiabatic evolution degenerate states evolve according to the non-Abelian Berry phase:

$$\Gamma_C = P \exp \oint_C A^\mu(\lambda) d\lambda_\mu = P \prod_{t=1}^{T} \left(\sum_{\alpha=1}^{n} |\Psi_\alpha(\lambda(t))\rangle\langle\Psi_\alpha(\lambda(t))| \right)$$

C ~ a loop in a parameter space (space of 4-vortex configurations)

T ~ total number of discrete steps on *C*

t ~ particular step on *C*

P ~ “time ordering” in *t*

n ~ ground state degeneracy (twofold for four vortices)

Non-Abelian statistics as a Berry phase

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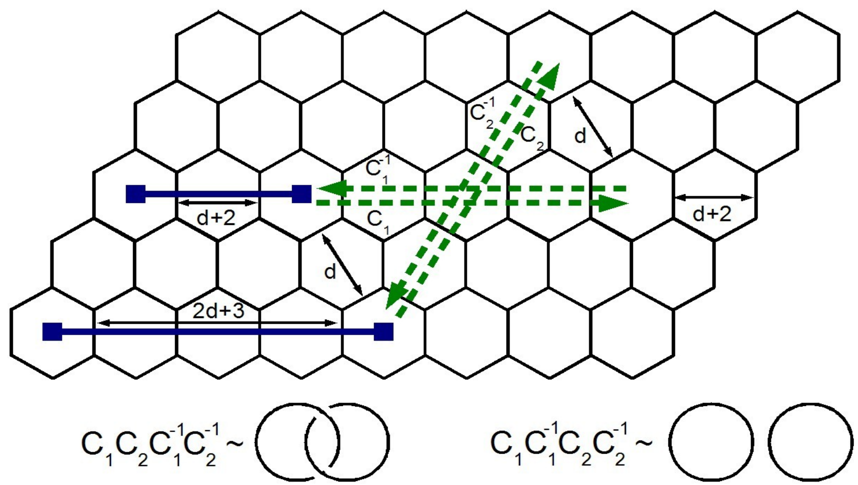


Strategy:

1) Diagonalize the Hamiltonian for every *t*

2) Construct the projector to the ground state space

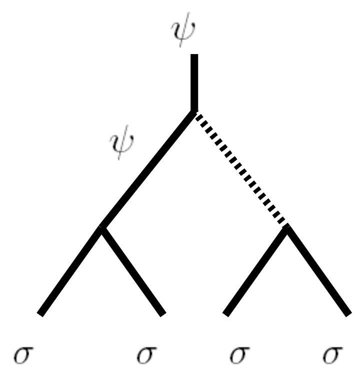
3) Multiply them together to evaluate Γ_C

Non-Abelian statistics as a Berry phase

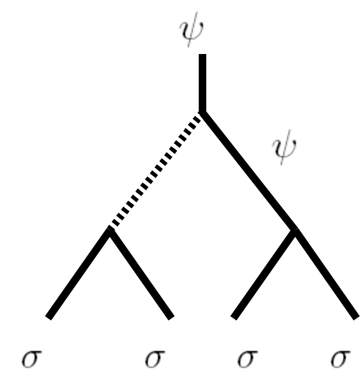
<u>How to construct the degenerate ground states?</u>

- Restrict to overall Ψ-fusion channel of four vortices

$|\Psi_{10}> =$



$|\Psi_{01}> =$



- The spectrum has two zero modes

$$|\,\Psi_{\alpha_1\alpha_2}\rangle \;=\; (b_1^\dagger)^{\alpha_1}(b_2^\dagger)^{\alpha_2}\,|\,\mathrm{gs}\rangle \qquad |\,\mathrm{gs}\rangle = \prod_{k=1}^{MN} b_k\,|\,\phi\rangle \qquad b_k^\dagger\,|\,\phi\rangle = 0$$

- The ground states can be represented by:

$$|\,\Psi_\alpha\rangle = \sum_{\substack{\{k,\dots,l=1|\\ k,\dots,l\neq\alpha\}}}^{MN-1} \frac{\varepsilon_{k,\dots,l}}{\sqrt{(MN-1)!}}\psi_k^- \otimes\cdots\otimes\psi_l^- \qquad A\psi_k^\pm = \pm\epsilon_k\psi_k^\pm$$

Non-Abelian statistics as a Berry phase

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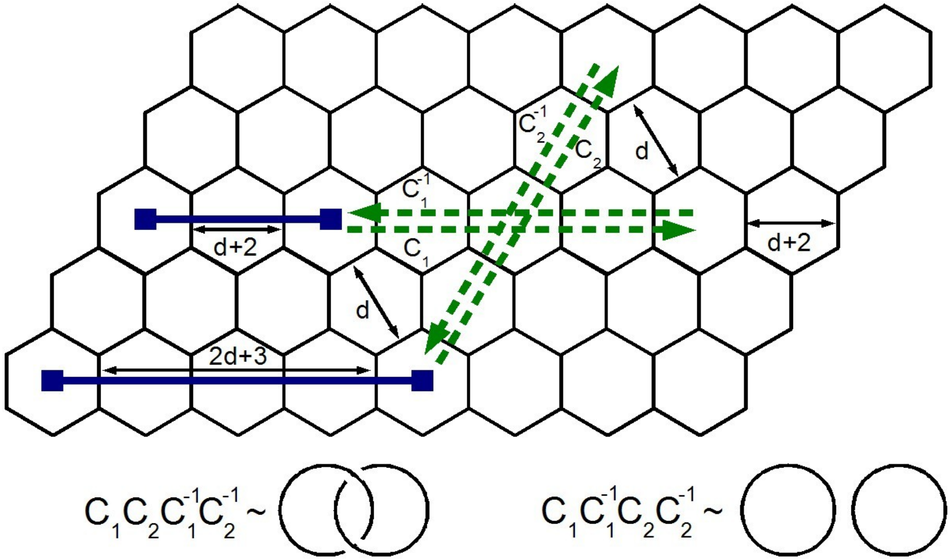
- A finite system of *2MN* spins on a torus.
- Consider the range 0 < *K* < 0.15 to study magnetic field dependance
- Evaluate the Berry phase for three parametrizations to study scaling with system size

	d	S	T	$2MN$
(i)	1	$2{\cdot}10^3$	$32{\cdot}10^3$	120
(ii)	2	$2{\cdot}10^3$	$48{\cdot}10^3$	224
(iii)	3	$4{\cdot}10^3$	$128{\cdot}10^3$	360

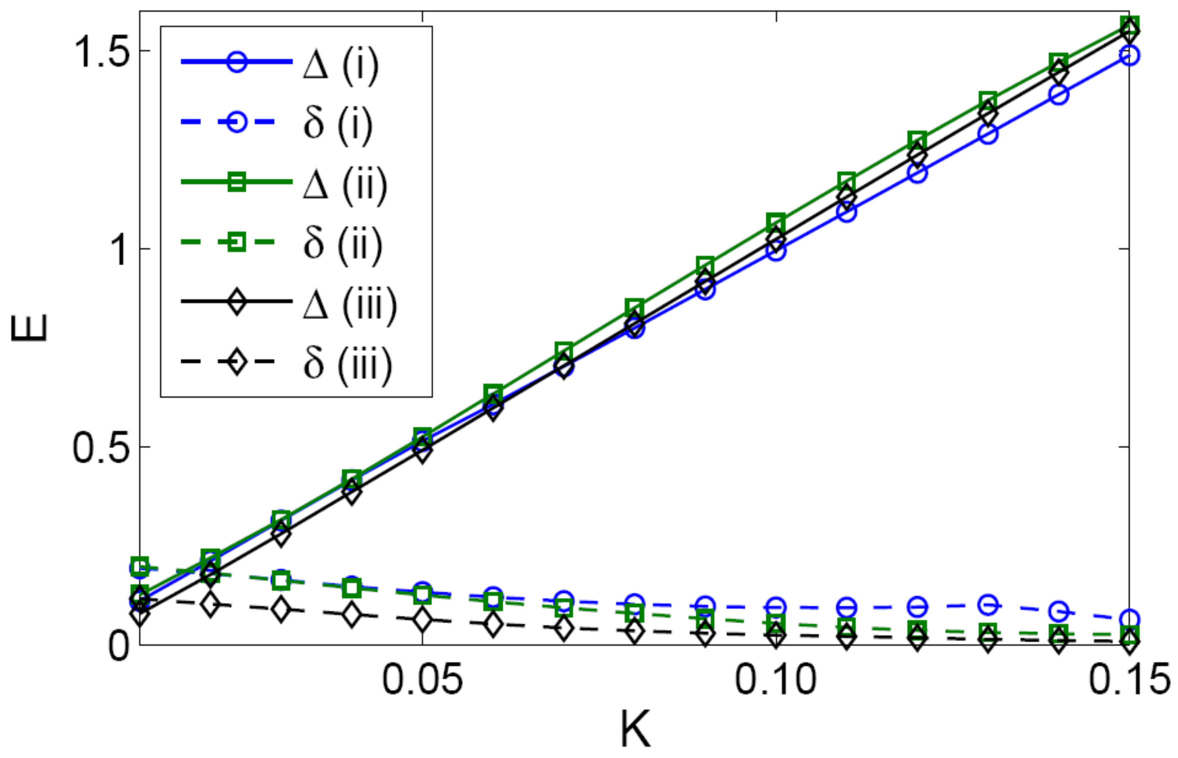


Non-Abelian statistics as a Berry phase

- Under adiabatic approximation the Berry phase corresponds to the exact time evolution when *Δ* >> *δ*

Energy gap: Degeneracy:

$$\Delta = \min_t(\epsilon_3^t - \epsilon_2^t), \quad \delta = \max_t(\epsilon_2^t - \epsilon_1^t)$$



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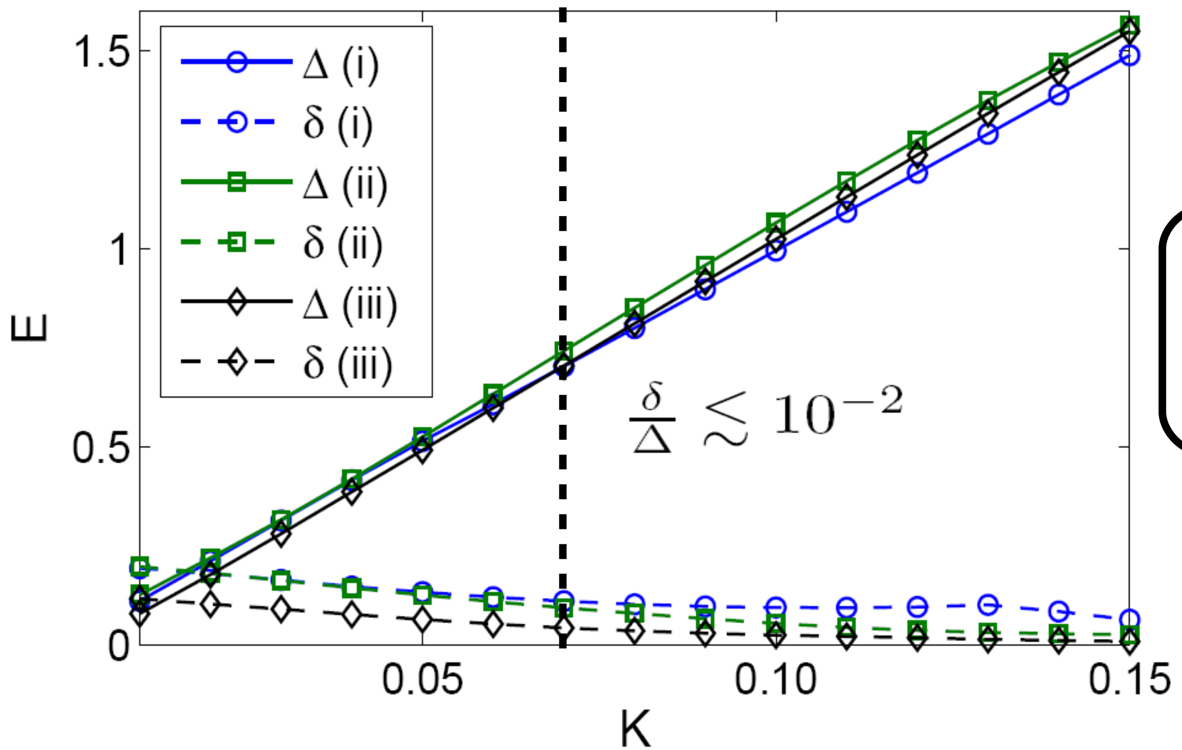

K = 0.07 lower bound for a stable topological phase.

Non-Abelian statistics as a Berry phase

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- Introduce fidelity measures for the holonomy

$$s(U,V) \;=\; \frac{1}{4}\mathrm{tr}\,(UV^\dagger + VU^\dagger) \qquad\qquad s(U,V)\;\leq\;1$$

- When the off-diagonal elements of Γ_C are $re^{i\theta}$ and $R^2 = e^{-\frac{\pi}{4}i}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$s(1\!\!1, \Gamma_{C_l}\Gamma^\dagger_{C_l})$$ (measure of unitarity)

$$s(|R^2|, |\Gamma_{C_l}|) \;=\; r$$ (measure of off-diagonality)

$$\bar{s}(R^2, \Gamma_{C_l}) \;=\; \tfrac{1}{2}[r\cos(\tfrac{\pi}{4}+\theta)+1]$$ (measure of total fidelity)

Non-Abelian statistics as a Berry phase

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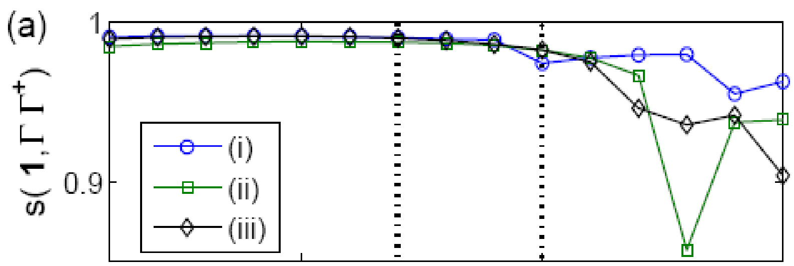

Unitarity

- >0.98 when $K < 0.11$
- Upper bound for stable topological phase

Non-Abelian statistics as a Berry phase

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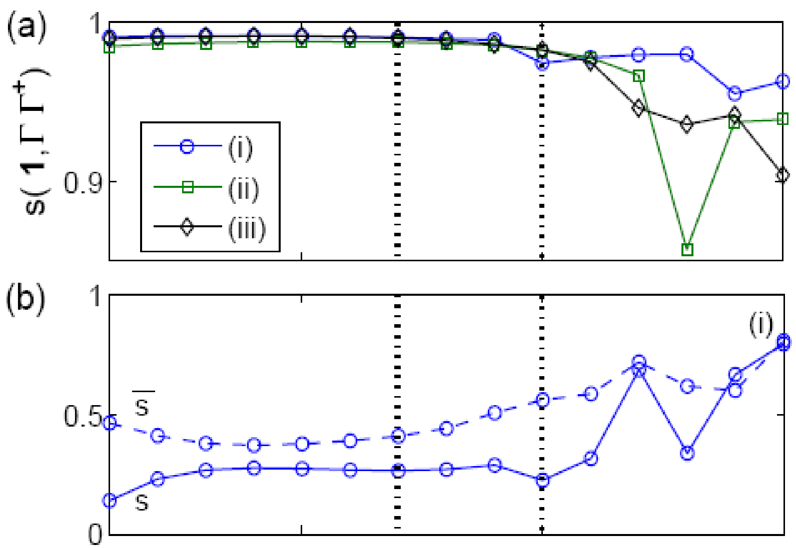

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(i)

- No off-diagonality

Non-Abelian statistics as a Berry phase

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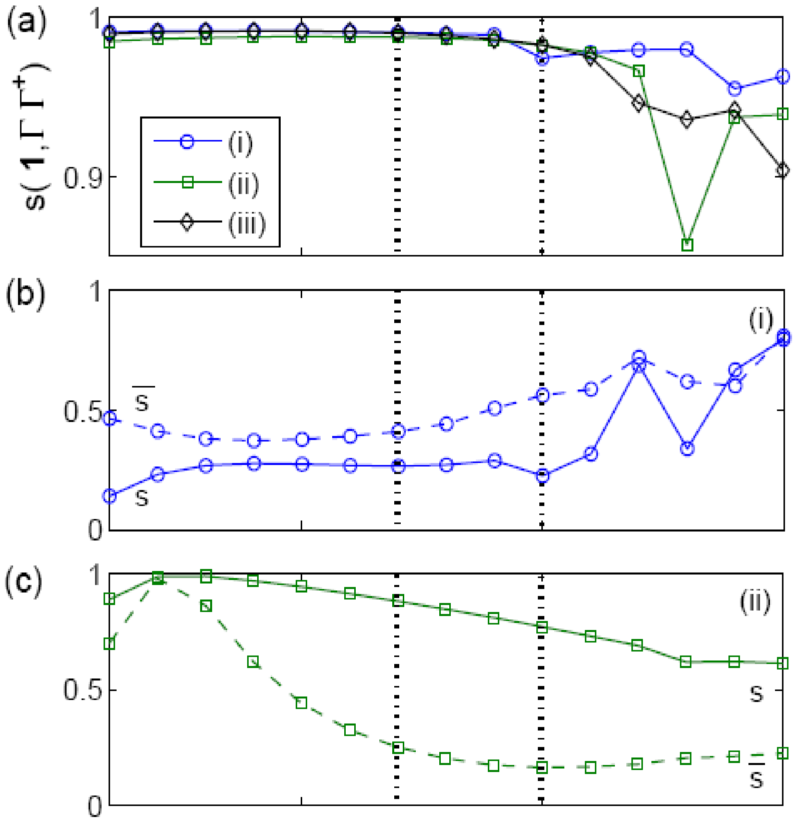

<u>Unitarity</u>

- >0.98 when $K < 0.11$
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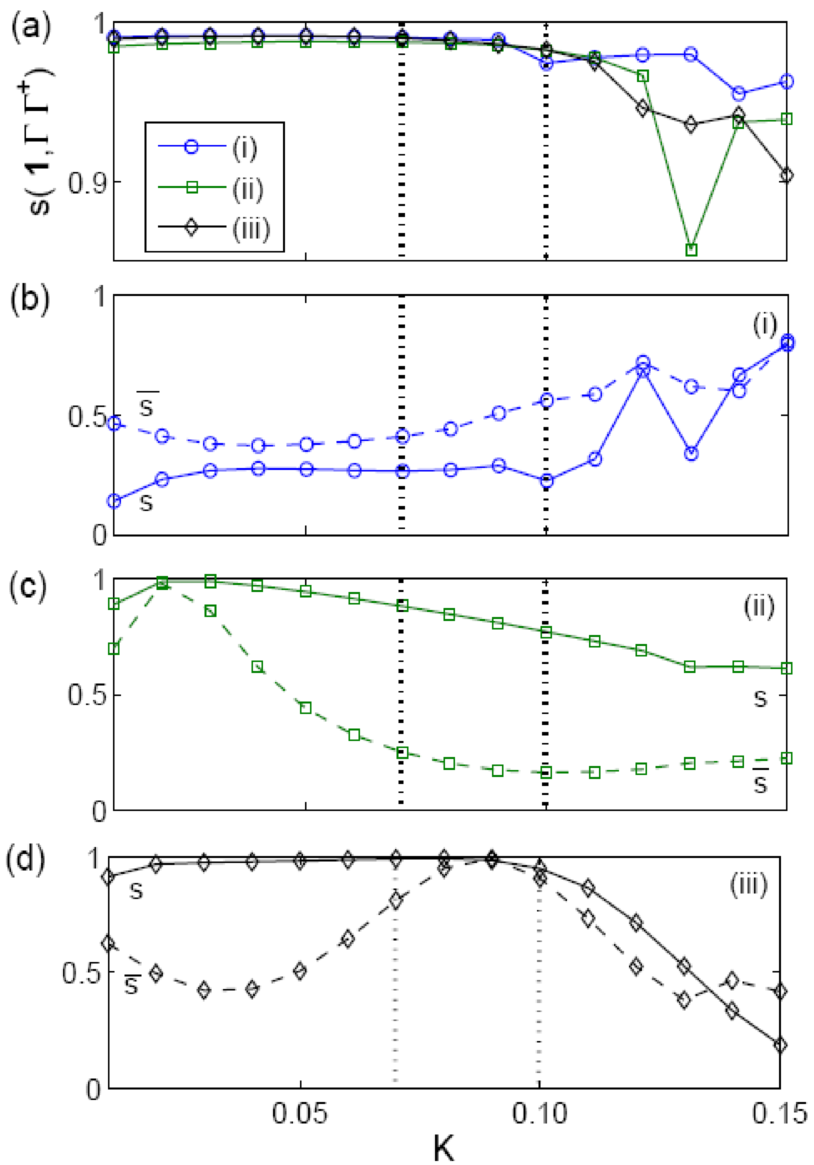
<u>(i)</u>

- No off-diagonality

<u>(ii)</u>

- Decaying off-diagonality
- The phase does not match

Non-Abelian statistics as a Berry phase



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(i)

- No off-diagonality

(ii)

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- The phase does not match

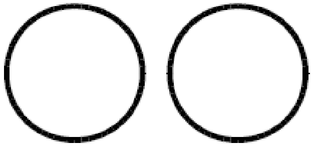
(iii)

- Stable off-diagonality
- At K=0.09 total fidelity > 0.99!

Non-Abelian statistics as a Berry phase

Further checks of the topological nature of the Berry phase:

Topology of the path

$\Gamma_{C_o} \approx \mathbb{1}$ With total fidelity > 0.98 when $C_1 C_1^{-1} C_2 C_2^{-1} \sim$ 

$\Gamma_{C_l} = \Gamma_{C_l'}$ Insensitive to perturbations of the path

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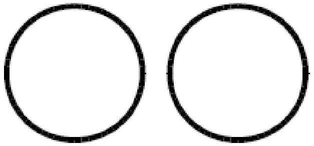
$\Gamma_{C_o} \approx \mathbb{1}$ With total fidelity > 0.98 when $C_1 C_1^{-1} C_2 C_2^{-1} \sim$ 

$\Gamma_{C_l} = \Gamma_{C_{l'}}$ Insensitive to perturbations of the path

Orientation of the path

$\Gamma_{C_l^{-1}} = \Gamma_{C_l}^{\dagger}$ Exact when direction reversed

Non-Abelian statistics as a Berry phase

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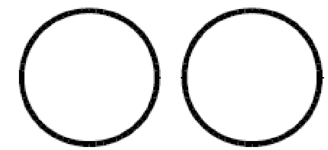
$\Gamma_{C_o} \approx \mathbb{1}$ With total fidelity > 0.98 when $C_1 C_1^{-1} C_2 C_2^{-1} \sim$ 

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Statistics only in the non-Abelian phase

Γ_{C_l} Vanishes for $K=0$

Conclusions

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Conclusions

- Explicit demonstration of non-Abelian statistics (better numerics desirable)
- The calculation discriminates between Ising and $SU(2)_2$
- The transport protocol experimentally realistic given sufficient site addressability
- Could be applied to other models
- Interesting to study robustness of the holonomy under perturbations