Quantum Information Group
School of Physics and Astronomy
UNIVERSITY OF LEEDS

Ising anyons in Kitaev's honeycomb lattice model

Ann. Phys. 323:9 (2008)
arXiv:0712.1164

Ville Lahtinen
Jiannis K. Pachos

Dublin 09/08

Motivation

Kitaev's honeycomb lattice model:
A.Y. Kitaev, *Annals of Physics*, 321:2, 2006

- An exactly solvable 2D spin model on a honeycomb lattice.
- Conjectured to support non-abelian Ising anyons
  (if you know FQHE and believe in Chern number and CFT arguments...)

Motivation

Kitaev's honeycomb lattice model:
A.Y. Kitaev, *Annals of Physics*, 321:2, 2006

- An exactly solvable 2D spin model on a honeycomb lattice.
- Conjectured to support non-abelian Ising anyons
  (if you know FQHE and believe in Chern number and CFT arguments...)

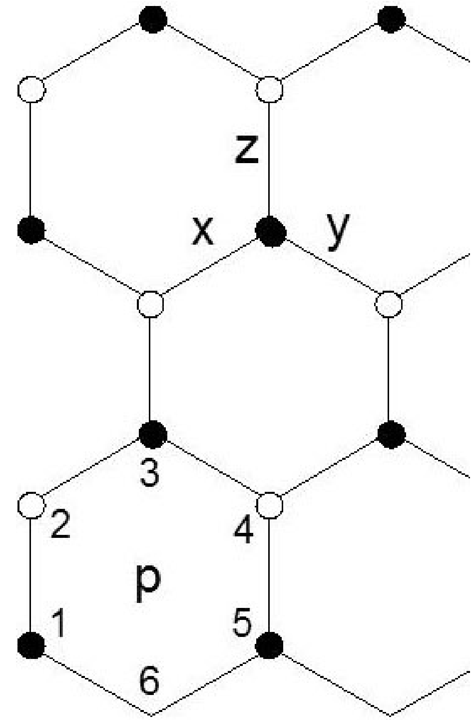
We solve the model and ...

- Demonstrate the fusion rules from the spectral behavior.
  Lahtinen et.al., *Ann. Phys*. 323:9 (2008)
- Calculate the braid statistics as a holonomy
  (work in progress)

The honeycomb lattice model

UNIVERSITY OF LEEDS

$$H = -\sum_{\alpha \in \{x,y,z\}} J_\alpha \sum_{\alpha-\text{links}} \sigma_i^\alpha \sigma_j^\alpha - K \sum_p (\sigma^x \sigma^y \sigma^z)_p$$



Represent spins by Majorana fermions:

$$H = \frac{i}{4} \sum_{i,j \in \Lambda} \hat{A}_{ij} c_i c_j, \qquad \hat{A}_{ij} = 2J_{ij}\hat{u}_{ij} + 2K \sum_k \hat{u}_{ik}\hat{u}_{kj}.$$

$$[H, \hat{u}_{ij}] = 0 \qquad \hat{w}_p = \prod_{i,j \in \partial p} \hat{u}_{ij}$$

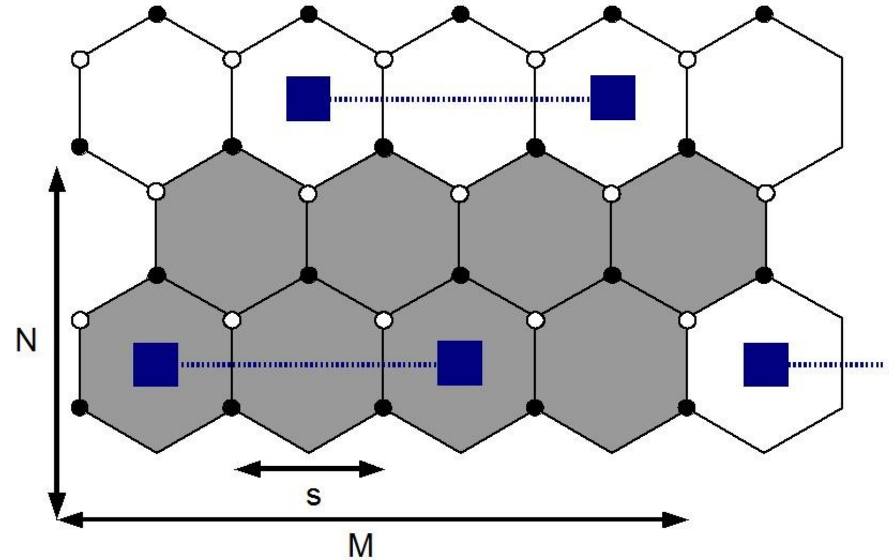
- Fix the eigenvalues u_{ij} on all links *(ij)*
  - fix the eigenvalues w_p

    = fix the underlying vortex configuration

Solving the model

UNIVERSITY OF LEEDS

- Choose a (*M,N*)-unit cell containing *MN* plaquettes
- Create a *n*-vortex configuration with vortex separation *s*
- Fourier transform with respect to the unit cell
- Diagonalize the Hamiltonian



$$H = MN \int_{-\pi/M}^{\pi/M} \frac{dp_x}{2\pi} \int_{-\pi/N}^{\pi/N} \frac{dp_y}{2\pi} \left[\sum_{i=n+1}^{MN} |\epsilon_i(\mathbf{p})| b_i^\dagger b_i + \sum_{i=1}^{n} |\alpha_i^s(\mathbf{p})| z_i^\dagger z_i - \left(\sum_{i=n+1}^{MN} \frac{|\epsilon_i(\mathbf{p})|}{2} + \sum_{i=1}^{n} \frac{|\alpha_i^s(\mathbf{p})|}{2} \right) \right]$$

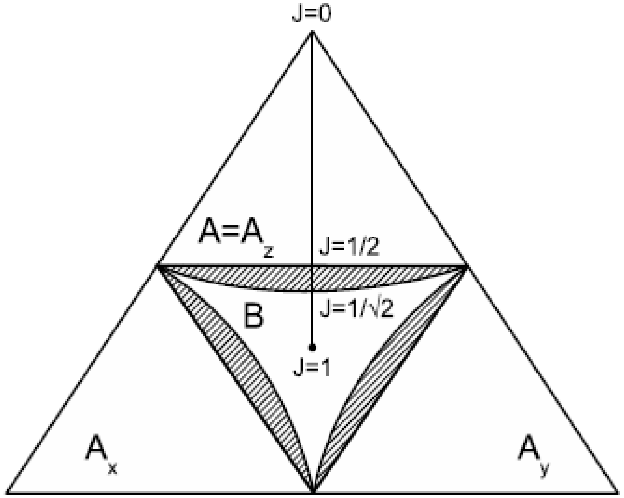
- We study the spectrum as a function of J_i, *K*, *n* and *s* using a (20,20)-unit cell

Phase space geometry

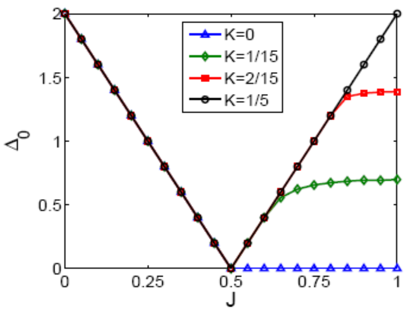
The fermion gap: $\Delta = \min_{\mathbf{p}} |\epsilon_1(\mathbf{p})|$

- A phase always gapped (toric code)
- B phase gapped when *K*>0 (Ising)
- Phase boundaries at Δ → 0 depend on underlying vortex configuration
- Boundaries:
  - Vortex-free: *J=1/2*
  - Full-vortex: *J=1/√2*
  - Sparse: *1/2 ≤ J ≤ 1/√2*

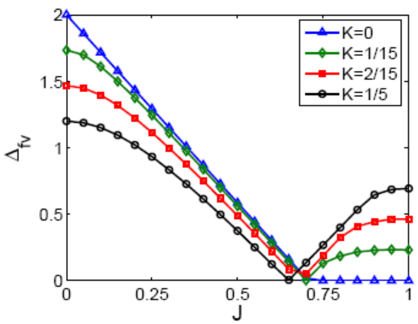
Temperature - vortex density: Tunable parameter that induces phase transition



(J_z = 1 and J = J_x = J_y)



vortex-free



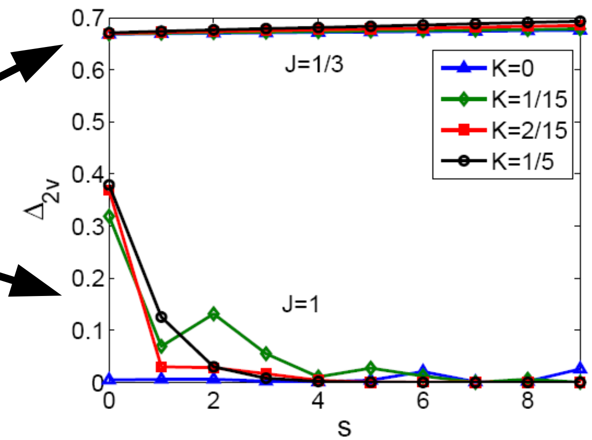
full-vortex

Zero modes

UNIVERSITY OF LEEDS

The fermion gap Δ_{2v} above a 2-vortex configuration

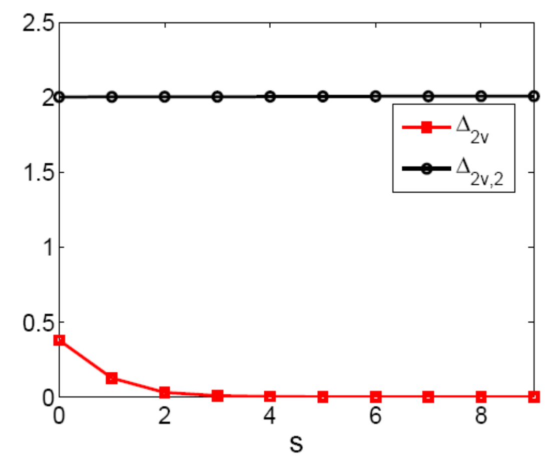
- Insensitive to *s* in the abelian phase
- Vanishes with *s* in the non-abelian phase



$$|1\rangle = b_1^\dagger(\mathbf{p}_0)|gs\rangle, \qquad \Delta_{2v} = \min_{p_0} |\epsilon_1(\mathbf{p})|$$
$$|2\rangle = b_2^\dagger(\mathbf{p}_0)|gs\rangle, \qquad \Delta_{2v,2} = \min_{p_0} |\epsilon_2(\mathbf{p})|$$

- Twofold degenerate ground state at large *s*
- Degeneracy lifted at small *s*
- $\Delta_{2v,2}$ insensitive to *s*

→ ***One zero mode per two vortices***



Zero modes

Degeneracy at large *s*:

- 4-vortex: fourfold degeneracy
- 6-vortex: eightfold degeneracy

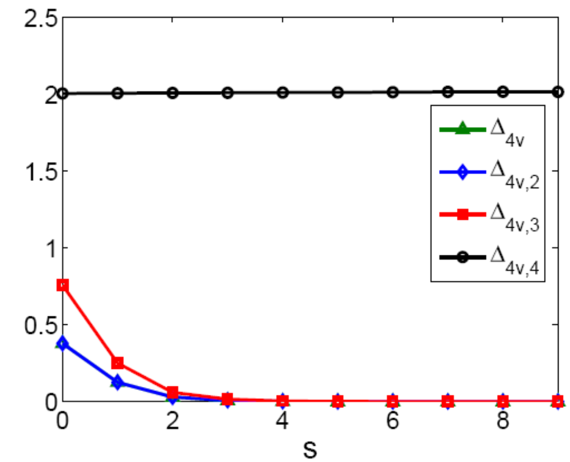
Degeneracy at small *s:*

- 4-vortex: 1st excited state twofold degenerate
- 6-vortex: 1st and 2nd excited states threefold degenerate

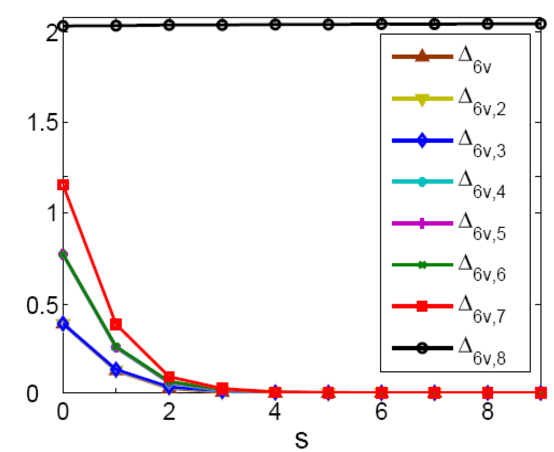
***p*-wave superconductors:**

2^n-fold degeneracy in the presence of 2n well separated Ising vortices

Lifting of degeneracy at short ranges due to vortex interactions



4-vortex



6-vortex

Zero modes and fusion rules

Ising fusion rules: $\psi \times \psi = 1, \quad \psi \times \sigma = \sigma, \quad \sigma \times \sigma = 1 + \psi$

Identify: ψ ~ fermion mode, σ ~ vortex

Interpret: Unoccupied zero mode ~ $\sigma x \sigma \rightarrow 1$
Occupied zero mode ~ $\sigma x \sigma \rightarrow \psi$

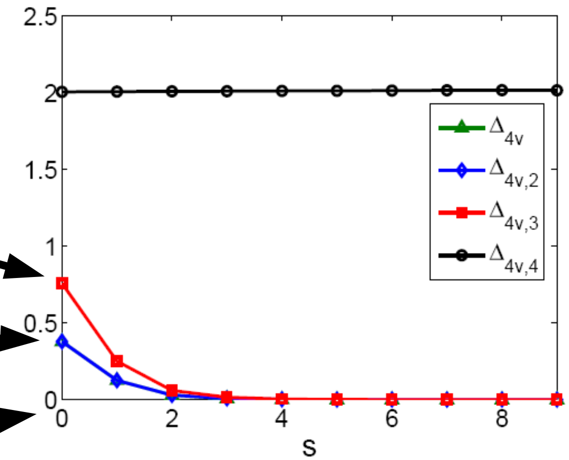
E.g. $\sigma \times \sigma \times \sigma \times \sigma = 1 + 1 + \psi + \psi$

Four fusion channels:

$$(\sigma \times \sigma) \times (\sigma \times \sigma) \rightarrow \psi \times \psi = 1$$

$$(\sigma \times \sigma) \times (\sigma \times \sigma) \rightarrow 1 \times \psi = \psi$$
$$(\sigma \times \sigma) \times (\sigma \times \sigma) \rightarrow \psi \times 1 = \psi$$

$$(\sigma \times \sigma) \times (\sigma \times \sigma) \rightarrow 1 \times 1 = 1$$



4-vortex

The low-energy spectrum

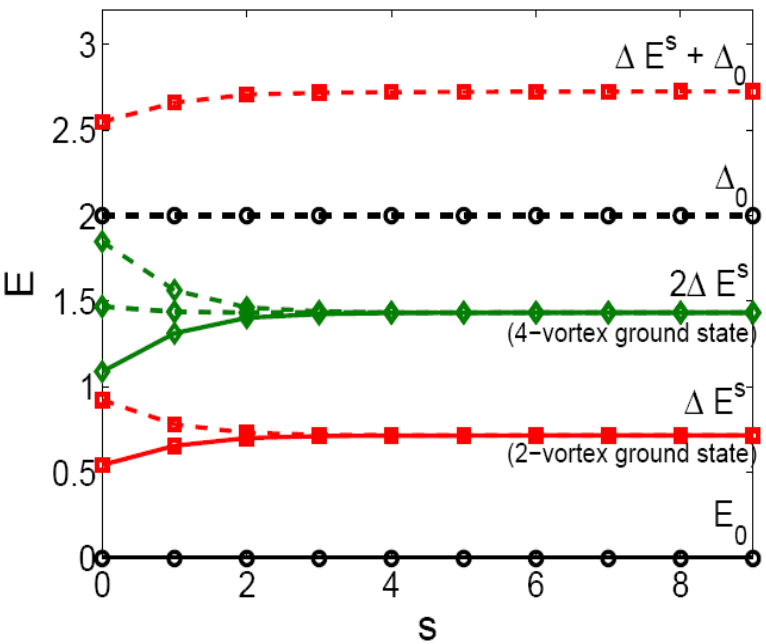
UNIVERSITY OF LEEDS

For large vortex separations (*s>2*)

- Consist of only vortices
- Applies for arbitrary number of well separated vortices
- Agrees with the Ising prediction

For small vortex separations (*s<2*)

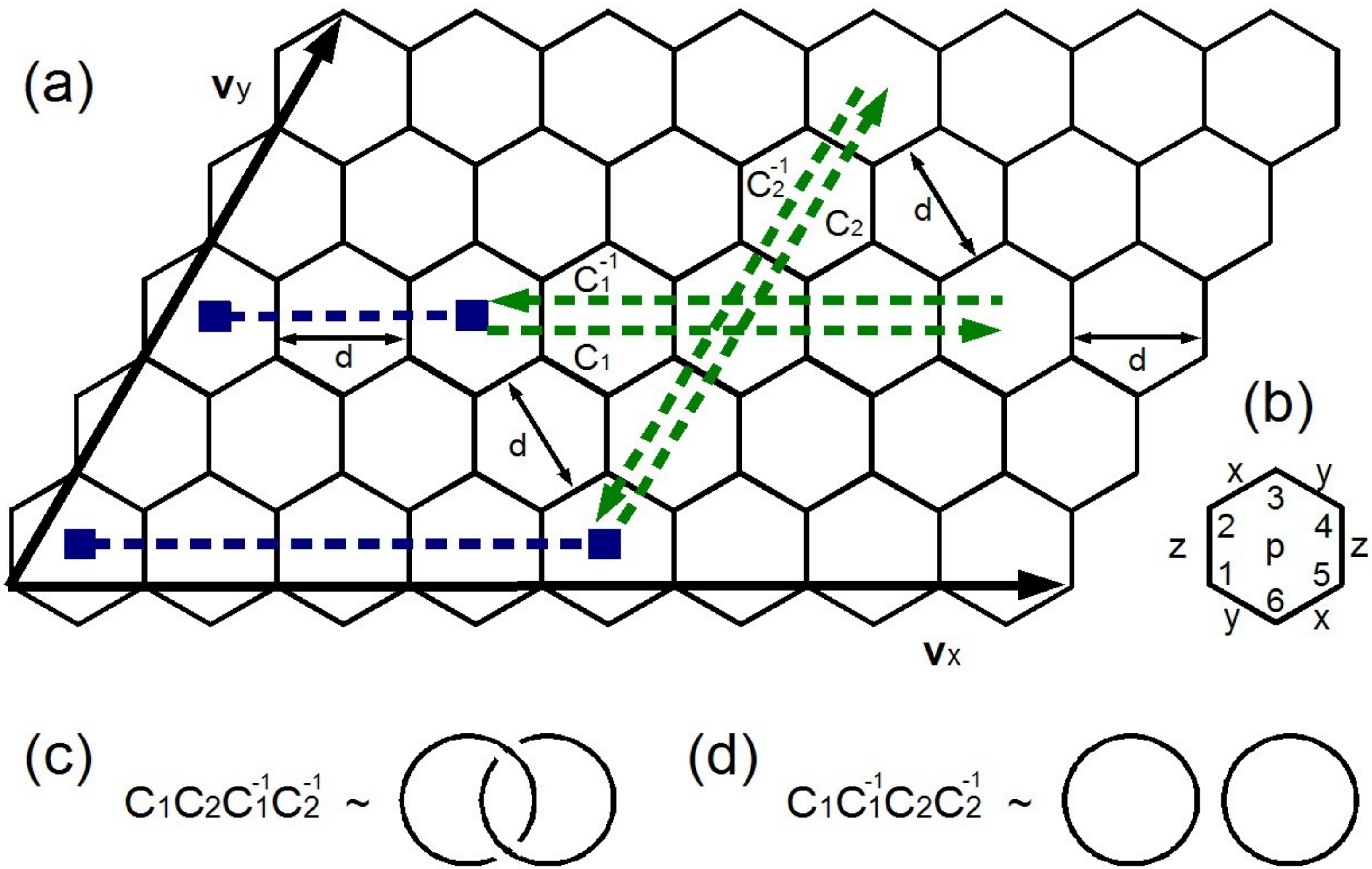
- In the absence of vortices spectrum purely fermionic
- Vacuum fusion channels tend towards ground state
- Fermion fusion channels tend towards higher energies



(J=1 and K=1/5)

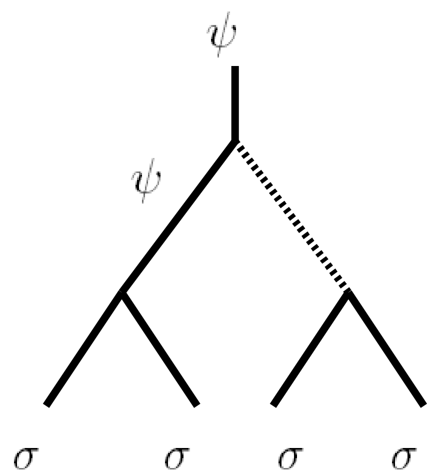
Braid statistics as a holonomy

UNIVERSITY OF LEEDS

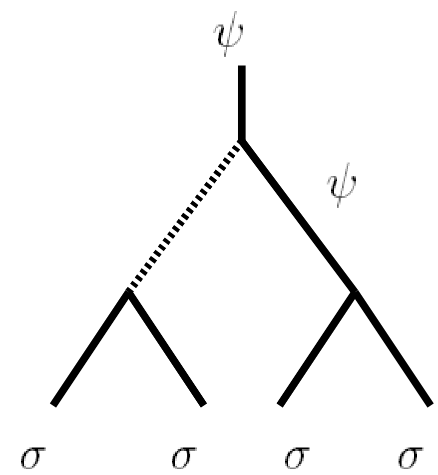


Braid statistics as a holonomy

Degenerate ground states and fusion channels



$|\Psi_{10}>$



$|\Psi_{01}>$

Braiding acts on these states as:

$$R^2 = e^{-\frac{\pi}{4}i} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Braid statistics as a holonomy

Discrete form of non-abelian Berry phase:

$$\Gamma_C = P \exp \oint_C A^\mu(\lambda) d\lambda_\mu \quad = P \prod_{t=1}^{T} \left(\sum_{i=1}^{n} |\Psi_i(t)\rangle\langle\Psi_i(t)| \right)$$

C ~ a loop in a parameter space (space of 4-vortex configurations)

T ~ total number of discrete steps on *C*

t ~ particular step on *C*

P ~ “time ordering” in *t*

n ~ ground state degeneracy (twofold for four vortices)

Strategy:

1) Diagonalize Hamiltonian for every *t*

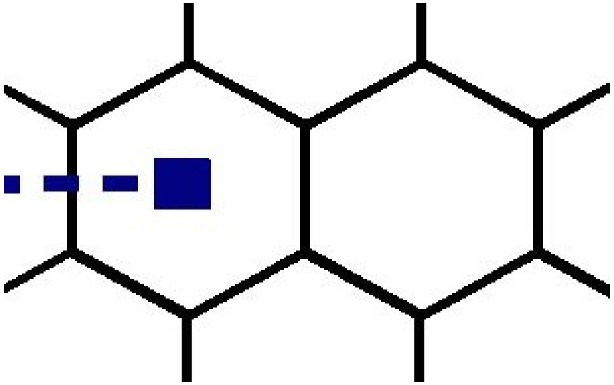
2) Construct the projector to the ground state space

3) Multiply them together to evaluate Γ_C

Braid statistics as a holonomy

$$\hat{A}_{ij} = 2J_{ij}\hat{u}_{ij} + 2K\sum_k \hat{u}_{ik}\hat{u}_{jk}$$

- Assume that J_{ij} and K can be tuned independently at every link

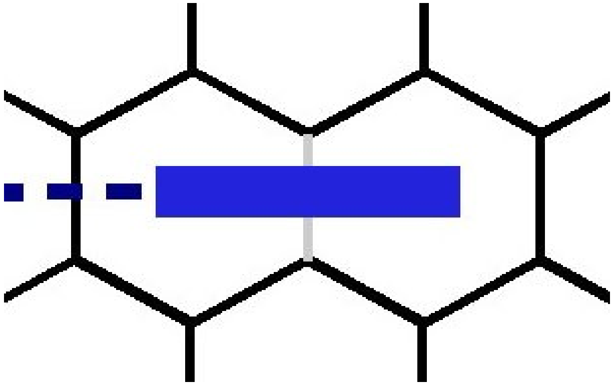


J_{ij} , K_{ij}

Braid statistics as a holonomy

$$\hat{A}_{ij} = 2J_{ij}\hat{u}_{ij} + 2K \sum_k \hat{u}_{ik}\hat{u}_{jk}$$

- Assume that J_{ij} and K can be tuned independently at every link

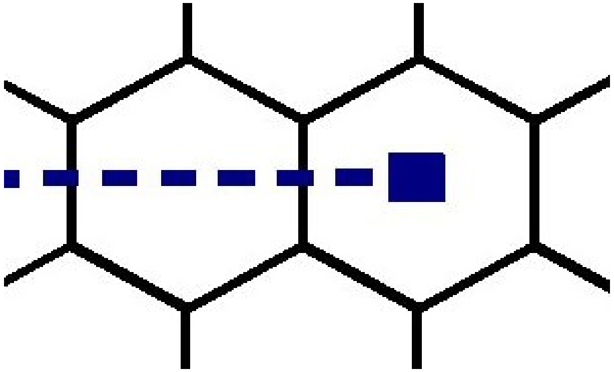


$$J_{ij} \rightarrow 0, \quad K_{ij} \rightarrow 0$$

Braid statistics as a holonomy

$$\hat{A}_{ij} = 2J_{ij}\hat{u}_{ij} + 2K\sum_k \hat{u}_{ik}\hat{u}_{jk}$$

- Assume that J_{ij} and K can be tuned independently at every link



$$J_{ij} \rightarrow -J_{ij}, \quad K_{ij} \rightarrow -K_{ij}$$

- Can be done “continuosly” in S steps!

Braid statistics as a holonomy

So, when C spans Q plaquettes and moving a vortex involves S steps, the holonomy can be evaluated with $T=QS$ diagonalizations...

Braid statistics as a holonomy

UNIVERSITY OF LEEDS

So, when *C* spans *Q* plaquettes and moving a vortex involves *S* steps, the holonomy can be evaluated with *T*=*QS* diagonalizations...

Numerical diagonalization gives:

$$H = \int \mathrm{d}^2\mathbf{p} \sum_{i=1}^{MN} \frac{\epsilon_i(\mathbf{p})}{2} \left[b_i^\dagger(\mathbf{p}) b_i(\mathbf{p}) - d_i^\dagger(\mathbf{p}) d_i(\mathbf{p}) \right] \qquad d_i^\dagger(-\mathbf{p}) = b_i(\mathbf{p})$$

The degenerate ground states can be represented by:

$$|\,\Psi_{i_1 i_2}(\mathbf{p})\rangle \;=\; \sum_{k,\ldots,l=1}^{MN+1} \frac{\epsilon_{k,\ldots,l}}{\sqrt{(MN+1)!}} \;\; a_k^\dagger(\mathbf{p})\,|\,0\rangle \otimes \cdots \otimes a_l^\dagger(\mathbf{p})\,|\,0\rangle$$

$$a_k^\dagger(\mathbf{p}) \in \{d_1^\dagger(\mathbf{p}), \ldots, d_{MN}^\dagger(\mathbf{p}), b_n^\dagger(\mathbf{p})\}$$

$i_1 i_2$ = (01),(10). $i_n = 1$

Braid statistics as a holonomy

UNIVERSITY OF LEEDS

Inner product of two such vectors is given by...

$$\langle \Psi_{i_1 i_2}(\mathbf{p}, t) \,|\, \Psi_{j_1 j_n}(\mathbf{p}, t') \rangle = \det(A^{tt'}_{i_1 j_1}(\mathbf{p})) \qquad [A^{tt'}_{i_1 j_1}(\mathbf{p})]_{kl} = \langle 0 \,|\, a_k(\mathbf{p}, t) a^\dagger_l((\mathbf{p}, t') \,|\, 0 \rangle$$

... and hence the holonomy unitary can be written as:

$$\Gamma_C(\mathbf{p}) = P \prod_{t=1}^{T} \begin{pmatrix} \det\left(A^{t,t+1}_{00}(\mathbf{p})\right) & \det\left(A^{t,t+1}_{01}(\mathbf{p})\right) \\ \det\left(A^{t,t+1}_{10}(\mathbf{p})\right) & \det\left(A^{t,t+1}_{11}(\mathbf{p})\right) \end{pmatrix}$$

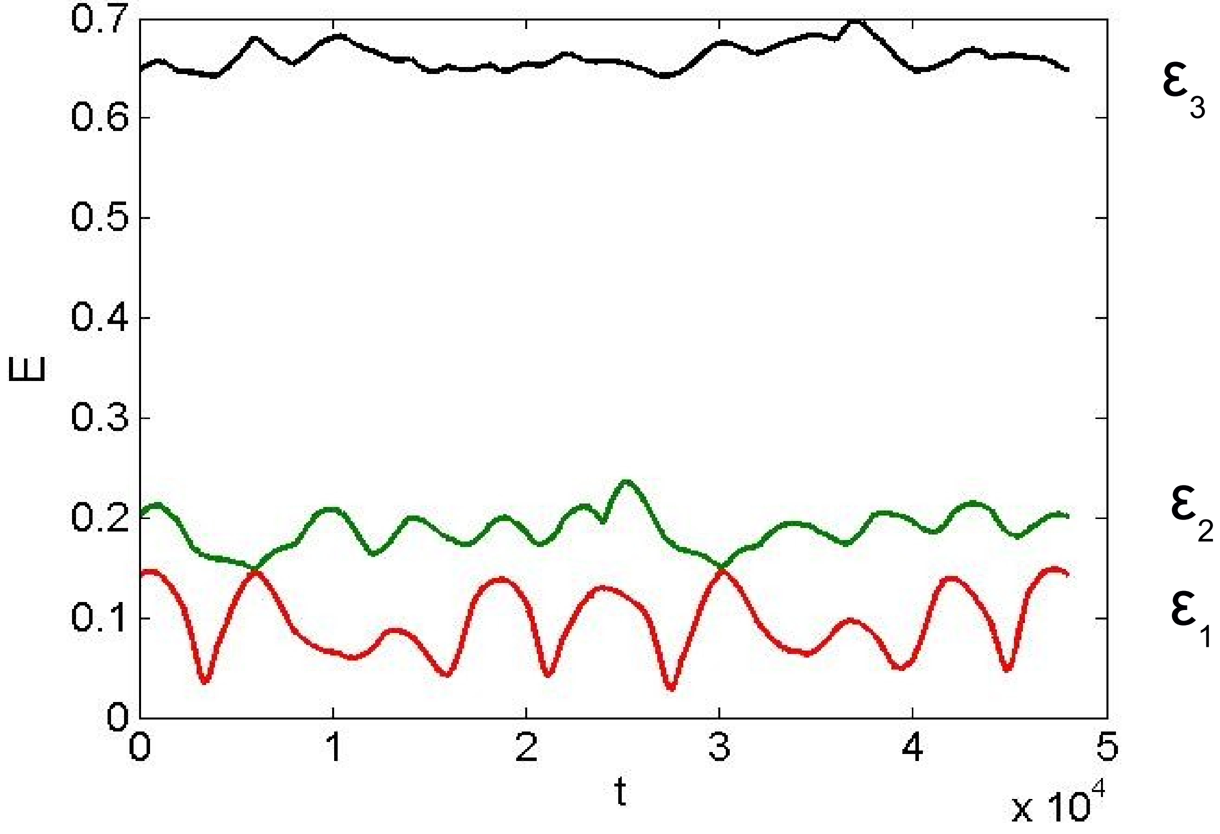
However, this applies only to for a single value of momentum...

- Describes a finite periodic system with twisted boundary conditions

Braid statistics as a holonomy

UNIVERSITY OF LEEDS

Fermion gap and the zero modes

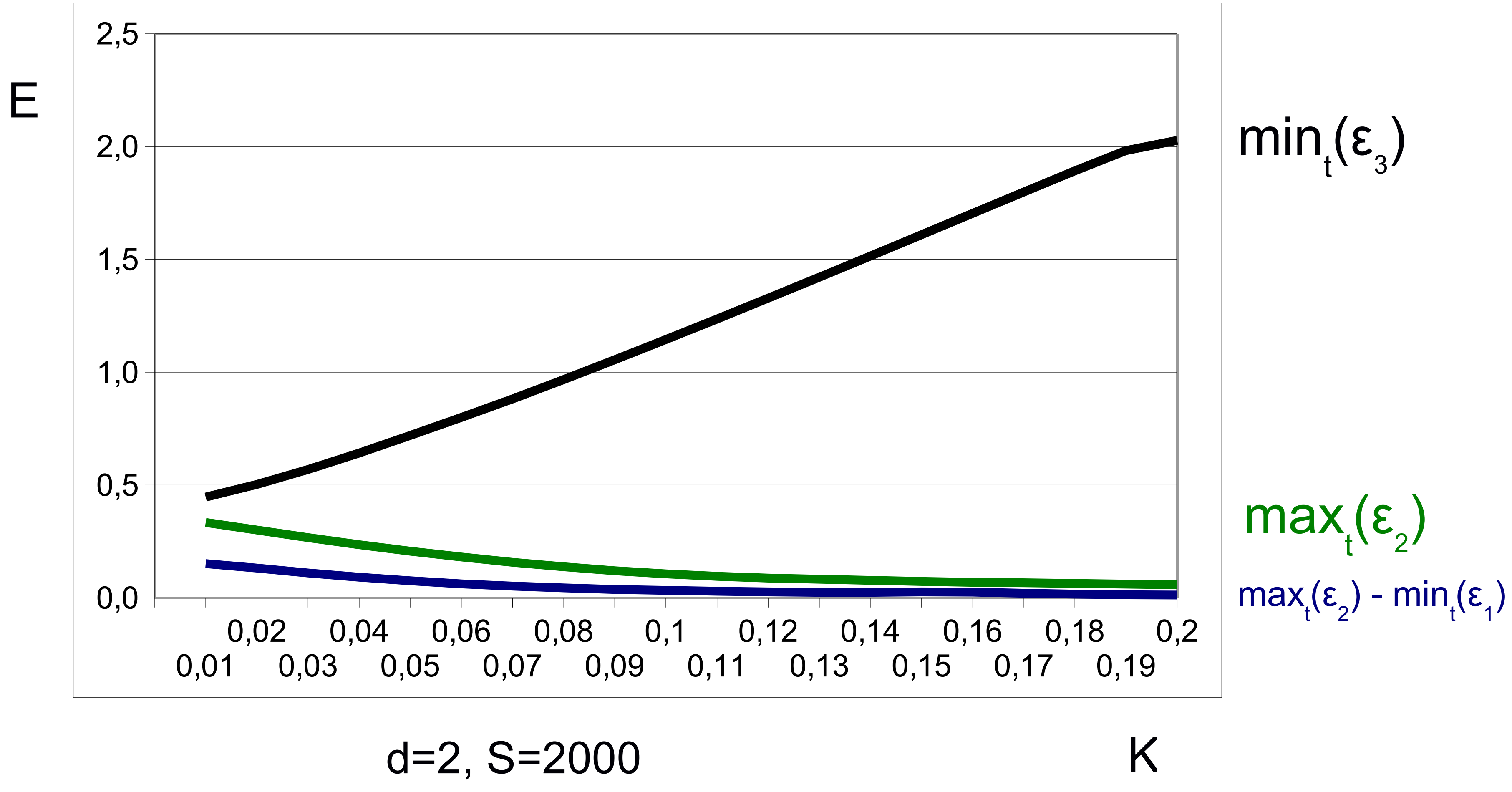


K=0.04, d=2, S=2000

Braid statistics as a holonomy

UNIVERSITY OF LEEDS

Fermion gap and the zero modes



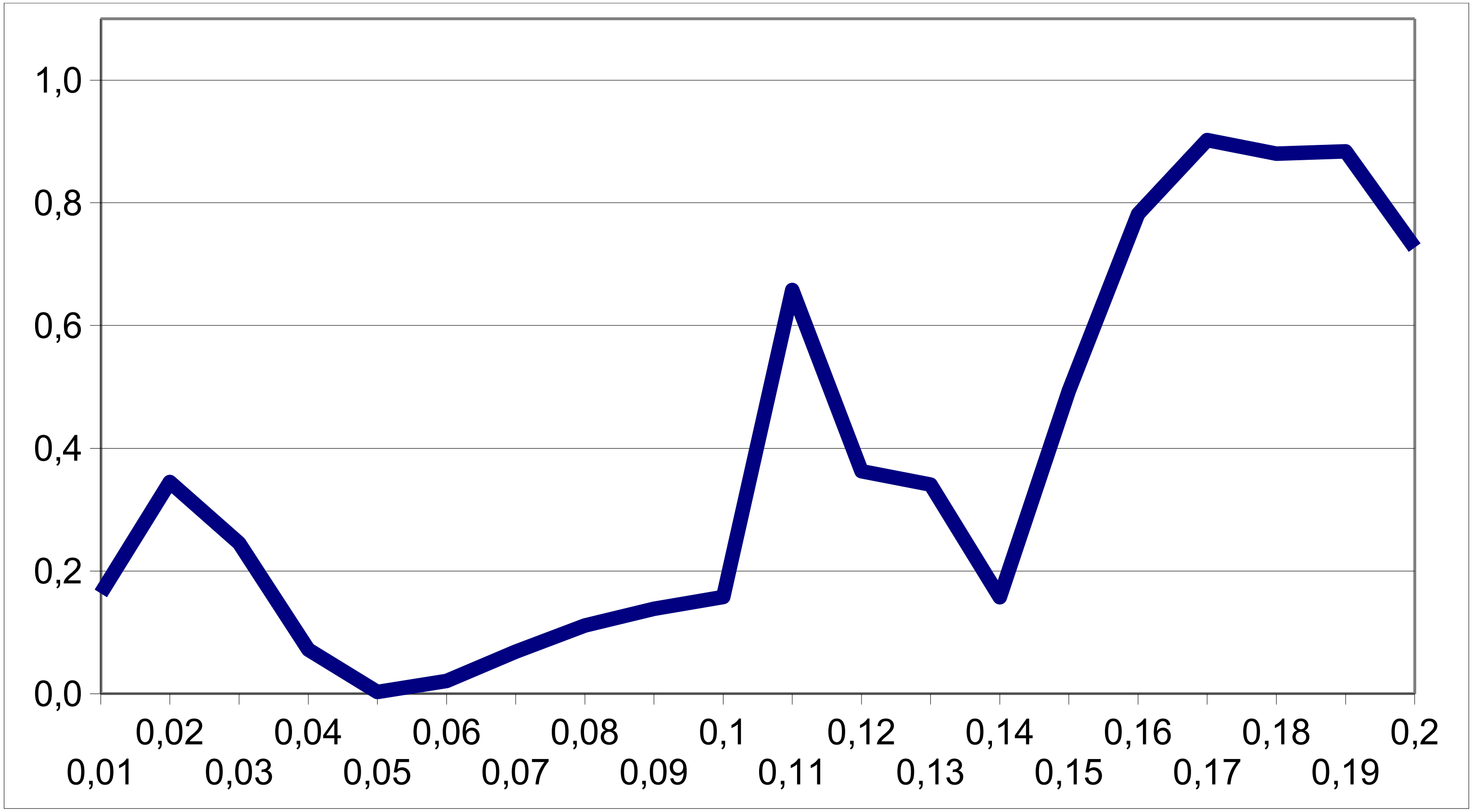
d=2, S=2000

Braid statistics as a holonomy

UNIVERSITY OF LEEDS

Fidelity of the holonomy unitary

$|(\Gamma_C)_{12}|^2$



d=2, S=2000

K