Efficient descriptions of many-body systems using tensor network states

Maynooth University, October 14th, 2014
MANY-BODY QUANTUM SYSTEM

PHYSICAL SYSTEM

- Dynamics
- Thermal equilibrium (T)

Computation time/memory scales exponentially with the number of constituents
MANY-BODY QUANTUM SYSTEM

MODEL

HILBERT SPACE

\[ c_1 |00..0\rangle + c_2 |00..1\rangle + ... + c_{2^N} |11..1\rangle \]
**However:**

- Interactions in nature are very special: few-body Hamiltonians, local
- Physical states occupy a small corner of Hilbert space
- Tensor networks: efficient description of that corner
PROJECTED ENTANGLED-PAIR STATES

- PEPS provide efficient descriptions of states with local interactions and in thermal equilibrium

\[ D = \text{poly}(N) \]

A. Molnar, N. Schuch, F. Verstraete, IC, arXiv:1406.2973

- PEPS provide simple descriptions of complex many-body states
**Bulk-boundary correspondence in lattice systems at T=0:**
T. Walh, H.H. Tu, S. Yang (MPQ),
N. Schuch (Aachen), D. Perez-Garcia (Madrid), D. Poilblanc (Tolouse)+ F. Verstraete (Vienna)

**Chiral topological models:**
T. Wahl, S. Yang, H.H. Tu (MPQ), N. Schuch (Aachen)
PROJECTED ENTANGLED PAIR STATES (PEPS)
Spins on a lattice in 2D at zero temperature:

- Many-body state: $|\psi\rangle$
- Parent Hamiltonian (local)
  
  $$H |\psi\rangle = E_0 |\psi\rangle$$
Entanglement swapping

\[ \ket{\phi} = \sum_{n=1}^{D} \ket{n,n} \]

\( a \)

\[ \ket{\Psi_1} \quad \ket{\Psi_1} \quad \rightarrow \quad \ket{\Phi} \]

\( b \)

\[ \ket{\Psi_1} \quad \rightarrow \quad \ket{\Phi} \]
PROJECTED ENTANGLED-PAIR STATES

\[ |\Psi_1\rangle \]

\[ |\Psi_2\rangle \]

\[ |\Psi_N\rangle \]

\[ |\Psi_{\nu,2}\rangle \]

\[ |\Psi_{Nv,Nh}\rangle \]
PROJECTED ENTANGLED-PAIR STATES

PLANE

TORUS

CYLINDER
PROJECTED-ENTANGLED PAIR STATES

$|\Psi_{i}\rangle\rightarrow|\Psi\rangle$
"PARENT" HAMILTONIANS

\[ H \Psi = E_0 \Psi \]

- Local: \( H = \sum_n h_n \)
- Frustration-free: \( h_n \Psi = 0 \)
- Degeneracy: \( g \)

\[ |\Psi_1\rangle \rightarrow |\Psi\rangle \]
PROJECTED-ENTANGLED PAIR STATES

$|\Psi\rangle$

$|\Psi_1\rangle = \sum A^i_{\alpha\beta\gamma\delta} |i; \alpha, \beta, \chi, \delta\rangle$

Tensor network

Easy to handle

PEPS give a natural playground to investigate many-body systems
Bulk-boundary correspondence in lattice systems at T=0:
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Related work: Dubail, Read, Rezayi
Qi, Katsura, and Ludwig
Chen, Gu, Wen
Spins on a lattice in 2D at zero temperature:

- Many-body state: $| \Psi \rangle$
- Parent Hamiltonian (local)
  \[
  H | \Psi \rangle = E_0 | \Psi \rangle
  \]
- Reduced state in region A:
  \[
  \rho_A = \text{tr}_B [ | \Psi \rangle \langle \Psi | ]
  \]
**Area law:** (Srednicky 93):

\[ S(\rho_A) \propto N_{\partial A} \]

\# degrees of freedom \( \propto \) \# particles at boundary

**Entanglement spectrum:** (Li and Haldane, 2008; Peschel, Kitaev and Preskill):

\[ \rho_A = e^{-H_A} \quad \sigma(H_A) \]

The low energy sector has the same structure as that for a lower dimensional theory (edge states)
THE (REDUCED) STATE OF THE BULK CAN BE DESCRIBED BY A LOWER DIMENSIONAL THEORY

THAT THEORY IS SOMEHOW RELATED TO THE BOUNDARY OF THE REGION

\[ \rho_A = \text{tr}[|\Psi\rangle\langle\Psi|] \]
QUESTIONS:

- What is that theory? Where does it act?
- Is the Hamiltonian $H_A$ local?
- What are the symmetries of $H_A$, and how are they related to those of $\Psi$?
- How does the topological character of $\Psi$ manifest itself?
- What happens at quantum phase transitions?
- Is there any relation to a dynamical Hamiltonian?
- What is the relation to chiral edges for topological insulators?
PROJECTED-ENTANGLED PAIR STATES

- Approximate well ground states (of gapped phases)
- Fulfill the area law
- There exist numerical techniques

PEPS give a natural playground to investigate this subject
PROJECTED ENTANGLED-PAIR STATES
BULK-BOUNDARY CORRESPONDENCE

IC, Poilblanc, Schuch, and Verstraete, PRB 83, 245134 (2011)

\[ |\Psi\rangle \]

\[ |\Psi_i\rangle \]

\[ \begin{array}{c}
  |A\rangle \\
  |B\rangle \\
\end{array} \]

\[ |\Psi_{Aa}\rangle \]

\[ |\Psi_{Bb}\rangle \]

\[ \sigma_a = tr_A (|\Psi_{Aa}\rangle \langle \Psi_{Aa}|) \]

\[ \sigma_b = tr_B (|\Psi_{Bb}\rangle \langle \Psi_{Bb}|) \]

\[ \sigma_{\tilde{A}} = \sqrt{\sigma_a^T \sigma_b \sigma_a^T} \]

\[ \rho_A = tr_B (|\Psi\rangle \langle \Psi|) = U \sigma_{\tilde{A}} U^\dagger \]
PROJECTED ENTANGLED-PAIR STATES
BULK-BOUNDARY CORRESPONDENCE

\[ |\Psi\rangle \]

\[ |\Psi_i\rangle \]

\[ \rho_A \]

\[ \sigma_{\partial A} = U^\dagger \rho_A U \]
The theory corresponds to the auxiliary particles living in the boundary

- Isommetry between the spins in the bulk and the auxiliary ones in the boundary

\[ \sigma_{\partial A} = U^\dagger \rho_A U \]

- It "compresses" the degrees of freedom
- Implies area law
- Allows to determine expectation values in the boundary

\[ x_{\partial A} = U^\dagger X_A U \quad \Rightarrow \quad \text{tr}(\sigma_{\partial A} x_{\partial A}) = \text{tr}(\rho_A X_A) \]
The theory corresponds to the auxiliary particles living in the boundary

Isommetry between the spins in the bulk and the auxiliary ones in the boundary

**BOUNDARY HAMILTONIAN**

\[ \sigma_{\partial A} = e^{-H_{\partial A}} \]

- Has the same entanglement spectrum \( \sigma(H_{\partial A}) = \sigma(H_A) \)
- It can be easily determined (exactly or approximately)
What can we say starting from the boundary Hamiltonian? (beyond the entanglement spectrum)

\[ \sigma_{\partial A} = e^{-H_{\partial A}} \]

Results:

Symmetries: The boundary Hamiltonian inherits the symmetries

\[ u_g \, |\Psi\rangle = e^{i\theta} \, |\Psi\rangle \quad \Rightarrow \quad U_g H_{\partial A} U_g^\dagger = H_{\partial A} \]

Locality:
- For gapped systems, it is local
- For critical systems, it becomes non-local

Quantum phase transitions:
- They are reflected in the boundary Hamiltonian

\[ \sigma_{\partial A} = e^{-H_{\partial A}} \]
Gapped topological phases in 2D

PROPERTIES
- Boundary state
- Boundary Hamiltonian

EXAMPLES
- Toric code (Kitaev)
- RVB states
- Phase transitions

\[ | \Psi \rangle \]
\[ \rho_A \]
\[ \sigma_{\partial A} = e^{-H_{\partial A}} \]
Topological properties are reflected in symmetries of the virtual particles:

\[ |\Psi_1\rangle \]

\[ |\Psi_2\rangle = |\Psi_{Nv,Nh}\rangle \]

\[ \nu_g |\Psi_1\rangle = |\Psi_1\rangle \quad \rightarrow \quad \nu'_g |\Psi_2\rangle = |\Psi_2\rangle \quad \rightarrow \quad \ldots \quad \rightarrow \quad u_g |\Psi_{Nv,Nh}\rangle = |\Psi_{Nv,Nh}\rangle \]

\[ g \in G \]
Results:

The boundary theory develops an extra symmetry

\[ \sigma_{\partial A} = U_g \sigma_{\partial A} = \sigma_{\partial A} U_g^\dagger \]

- In general, the boundary operator is block diagonal \( \sigma_{\partial A} = \sigma_{\partial A}^1 \oplus \sigma_{\partial A}^2 \oplus \ldots \)
- The projector, \( P_i \), on each subspace is highly non-local

The boundary Hamiltonian splits

\[ H_{\partial A} = H_{\partial A}^{\text{topo}} + H_{\partial A}^{\text{non-universal}} \]

- \( H_{\partial A}^{\text{topo}} \) is universal (only depends on the boundary conditions): \( H_{\partial A}^{\text{topo}} = \bigoplus c_i P_i \)
- \( H_{\partial A}^{\text{non-universal}} \) is local and depends on the details of the state (but not on the boundary conditions)

Phase transition

- \( H_{\partial A}^{\text{non-universal}} \) becomes non-local
- It can eventually compensate the universal part \( H_{\partial A}^{\text{topo}} \)
Chiral topological models:
T. Wahl, S. Yang, H.H. Tu (MPQ), N. Schuch (Aachen)

See also:
Dubail and Read

\[ |\Psi_N\rangle = |\Psi_{N,2}\rangle \]

\[ \langle \phi | \Psi \rangle = \Psi' \]

Fermionic mode Majorana

\[ |\phi\rangle = (1 + icd) |\Omega\rangle \]
Topological properties are reflected in symmetries of the virtual particles:

\[ |\Psi_1\rangle = \ldots |\Psi_{Nv,Nh}\rangle \]

\[ S_1 |\Psi_1\rangle = 0 \rightarrow S_2 |\Psi_2\rangle = 0 \rightarrow \ldots \rightarrow S_N |\Psi_{Nv,Nh}\rangle = 0 \]
**strings**

\[ |\Psi_\square\rangle = |\Psi_{\bar{A}a}\rangle + |\Psi_{\bar{B}b}\rangle \]

\[ |\Psi_\square\rangle = \langle \phi_{ab} | S | \Psi_{\bar{A}a}\rangle | \Psi_{\bar{B}b}\rangle \]

\[ S = \sum x_{n,\alpha} c_{n,\alpha} \]
States with strings along contractible regions vanish

Strings wrapping up the cylinder can be deformed and moved

Strings in topological models: degeneracy, anyons, brading,
CHIRAL FERMIONIC QUASI-FREE PEPS
CASE STUDY 1
GAUSSIAN STATE

Fiducial state:

\[ |\Psi_1\rangle = (1 + a^{\dagger}b^{\dagger})|\Omega\rangle \]

\[ b = (c_L - ic_R)e^{i\pi/4} + c_U - ic_D \]

Is a topological superconductor (class D, \( p+ip \))

Schnyder et al, Kitaev, Altland and Zirnbauer

„Gauge“ symmetry:

\[ d |\Psi_1\rangle = 0 \]

\[ d = (c_L - ic_R)e^{i\pi/4} + c_U - ic_D \]
Case Study 1
Gaussian State

Boundary Theory

- Chiral modes
- Right boundary is entangled to the left boundary
- Zeroth Renyi entropy: topological correction:
  \[
  S_0(N_v) = aN_v - \log(2)
  \]
- This is a consequence of the symmetry
  \[
  \left[ \sum x_{n,\alpha} c_{n,\alpha}^L + \sum y_{n,\alpha} c_{n,\alpha}^R \right]|\Phi_{N_v}\rangle = 0 \quad \rightarrow \quad d\sigma_{LR} = \sigma_{LR} d = 0
  \]
CASE STUDY 1
GAUSSIAN STATE

PARENT HAMILTONIAN: LOCAL

- Degenerate on the torus
- Continuous spectrum (therm. Limit)
- Power-law correlation functions
CASE STUDY 1
GAUSSIAN STATE

PARENT HAMILTONIAN: LOCAL

- Degenerate on the torus
- Continuous spectrum (therm. Limit)
- Power-law correlation functions

It is at a phase transition
CASE STUDY 1
GAUSSIAN STATE

PARENT HAMILTONIAN: GAPPED

- Flat band Hamiltonian
- Long-range hoppings

\[ h_{n,m} \approx \frac{1}{|n - m|^3} \]

- Robust against local perturbations
- Chern number

\[ C = \frac{1}{4\pi} \int_{BZ} d^2k \hat{\mathbf{d}}(k) \cdot \left( \frac{\partial \hat{\mathbf{d}}(k)}{\partial k_x} \times \frac{\partial \hat{\mathbf{d}}(k)}{\partial k_y} \right) \]

It agrees with the boundary theory
SUMMARY

- Family of FGPEPS
- Smallest bond dimension: one majorana „mode“ per bond
- Chiral:
  - Chiral edge modes
  - Gapped Parent Hamiltonian (1/r^3 hopping)
  - Robust against perturbations
  - Chern superconductor
  - c=1/2 and symmetry class D (like $p+ip$ superconductor)

- Topological at a phase transition:
  - Gapless local Hamiltonian
  - Degeneracy depends on topology
  - String operators describe the states
  - Left and right boundaries are entangled
CHIRAL FERMIONIC INTERACTING PEPS
CASE STUDY 2
INTERACTING CHIRAL STATES

IDEA

\[ |\Psi\rangle = \prod_n P_n^{Gutz} |\Phi,\Phi\rangle \]

\( p+ip \) superconductor
Gutzwiller projector

\[ \rightarrow \quad \text{Top QFT} \quad SO(2)_1 \]

Four primary fields with \( h=0,1/8,1/8,1/2 \)

- Replace \( p+ip \) by the chiral FGPEPS:
  - Two copies: two Majorana =1 Fermion mode per bound
  - The Gutzwiller projector does not change the bond dimension
CASE STUDY 2
INTERACTING CHIRAL STATES

SYMMETRIES

- The state develops a new "gauge" symmetry:

\[ (-1)^{\sum_{a} x_{a}^\dagger x_{a}} |\Psi_{1}\rangle = |\Psi_{1}\rangle \]

|\Psi_{1}\rangle \quad \text{spin 1/2} \\
fermionic mode

- The symmetry is inherited for larger regions:
  It corresponds to a flux, akin to the toric code.
CASE STUDY 2
INTERACTING CHIRAL STATES

BOUNDARY THEORY

Entanglement Entropy:
CASE STUDY 2
INTERACTING CHIRAL STATES

BOUNDARY THEORY

- Entanglement spectrum:
  - There are four sectors (MES)
  - Degeneracy according to $SO(2)_I$
CASE STUDY 2
INTERACTING CHIRAL STATES

BOUNDARY THEORY

Entanglement spectrum:

Consistent with \( h=0,1/8,1/8,1/2 \)
TRANSFER MATRIX

- Gap in the transfer matrix gives correlation length:

Consistent with infinite correlation length
- PEPS can describe chiral phases
- Non-interacting and interacting systems
- Can they describe ground states of gapped local Hamiltonians?

Finite temperature: