An Application of Quantum Groups: A q-Deformed Standard Model

or

And Now for Something Completely Different...

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WHY DREDGE UP THIS OLD STUFF NOW?

- QGs and HAs have continued to turn up in several areas of physics, not least of which is condensed matter physics...

- The Standard Model is currently being pushed to the limit by the LHC in CERN, so the importance of beyond-the-SM physics can only increase in the next few years...

OUTLINE

- Review of Hopf algebras (HAs) and quantum groups (QGs): definitions and notation

- Recasting familiar “classical” ideas in the language of HAs and QGs: Lie algebras and gauge theories

- Construction of a toy $SU_q(2)$ gauge theory as a deformed version of the Standard Model (SM)

- Agreement and disagreement with undeformed SM

WHY DEFORM WHAT AIN'T BROKE? (YET)

- *Practicality*: deformation parameters may give alternate ways of – for example – introducing a cut-off in renormalisation or a lattice size.

- *New physics*: special relativity and quantum mechanics are deformed versions of Newtonian mechanics (with deformation parameters c and $\hbar$); who's to say there aren't more deformed theories out there?

- *Fun*: why not? At the very least, it'll be good exercise in seeing how QGs and HAs might play a role in other theories.

HOPF ALGEBRAS

[E. Abe, Hopf Algebras (Cambridge University Press, 1977)]

A HA is a unital associative algebra $\mathcal{U}$ over a field k with coproduct (or comultiplication) $\Delta : \mathcal{U} \to \mathcal{U} \otimes \mathcal{U}$, counit $\epsilon : \mathcal{U} \to k$ and antipode $S : \mathcal{U} \to \mathcal{U}$ satisfying

$$\begin{aligned}
(\Delta \otimes \mathrm{id})\Delta(x) &= (\mathrm{id} \otimes \Delta)\Delta(x) \\
\Delta(xy) &= \Delta(x)\Delta(y) \\
(\epsilon \otimes \mathrm{id})\Delta(x) &= (\mathrm{id} \otimes \epsilon)\Delta(x) = x \\
\epsilon(xy) &= \epsilon(x)\epsilon(y) \\
\cdot(S \otimes \mathrm{id})\Delta(x) &= \cdot(\mathrm{id} \otimes S)\Delta(x) = 1\epsilon(x)
\end{aligned}$$

*-HA: includes involution $\theta : \mathcal{U} \to \mathcal{U}$

$$\begin{aligned}
\theta^2(x) &= x \\
\theta(xy) &= \theta(y)\theta(x) \\
\theta(1) &= 1 \\
\Delta(\theta(x)) &= (\theta \otimes \theta)(\Delta(x)) \\
\epsilon(\theta(x)) &= \epsilon(x)^* \\
\theta(S(\theta(x))) &= S^{-1}(x)
\end{aligned}$$

(* is the conjugation in k)

SWEEDLER NOTATION

[M. E. Sweedler, Hopf Algebras (Benjamin Press, 1969)]

$\Delta(x)$ is generally a sum of elements in $\mathcal{U} \otimes \mathcal{U}$, but sum is suppressed and we write

$$\Delta(x) \quad = \quad \sum_i x^i_{(1)} \otimes x^i_{(2)} = x_{(1)} \otimes x_{(2)}$$

So

$$\begin{aligned} (\Delta \otimes \mathrm{id})\Delta(x) &= \Delta\left(x_{(1)}\right) \otimes x_{(2)} \\ &= \left(x_{(1)}\right)_{(1)} \otimes \left(x_{(1)}\right)_{(2)} \otimes x_{(2)} \end{aligned}$$

and

$$\begin{aligned} (\mathrm{id} \otimes \Delta)\Delta(x) &= x_{(1)} \otimes \Delta\left(x_{(2)}\right) \\ &= x_{(1)} \otimes \left(x_{(2)}\right)_{(1)} \otimes \left(x_{(2)}\right)_{(2)} \end{aligned}$$

Coassociativity $(\Delta \otimes \mathrm{id})\Delta(x) = (\mathrm{id} \otimes \Delta)\Delta(x)$ gives both as

$$x_{(1)} \otimes x_{(2)} \otimes x_{(3)}$$

(like $(ab)c = a(bc) = abc$). Similarly,

$$\cdot\,(S \otimes \mathrm{id})\,\Delta(x) = \epsilon(x)1 \quad \rightarrow \quad S\left(x_{(1)}\right) x_{(2)} = \epsilon(x)1$$

QUASITRIANGULAR HOPF ALGEBRAS

A QHA is a HA $\mathcal{U}$ together with an invertible element, the universal R-matrix, $\mathcal{R} = r_\alpha \otimes r^\alpha \in \mathcal{U} \otimes \mathcal{U}$ satisfying

$$\begin{aligned}(\Delta \otimes \mathrm{id})(\mathcal{R}) &= \mathcal{R}_{13}\mathcal{R}_{23} \\ (\mathrm{id} \otimes \Delta)(\mathcal{R}) &= \mathcal{R}_{12}\mathcal{R}_{23} \\ (\sigma \circ \Delta)(x) &= \mathcal{R}\Delta(x)\mathcal{R}^{-1}\end{aligned}$$

where $\sigma(x \otimes y) = y \otimes x$, and

$$\begin{aligned}\mathcal{R}_{12} &= r_\alpha \otimes r^\alpha \otimes 1, \\ \mathcal{R}_{13} &= r_\alpha \otimes 1 \otimes r^\alpha, \\ \mathcal{R}_{23} &= 1 \otimes r_\alpha \otimes r^\alpha.\end{aligned}$$

$\mathcal{R}$ satisfies the Yang-Baxter equation (YBE)

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$$

We can construct the special element $u \in \mathcal{U}$ via

$$u = \cdot(S \otimes \mathrm{id})(\mathcal{R}_{21}) = S(r^\alpha)r_\alpha$$

which has the following properties:

$$\begin{aligned}u^{-1} &= r^\alpha S^2(r_\alpha) \\ S^2(x) &= uxu^{-1} \\ [uS(u)]\,x &= x\,[uS(u)]\end{aligned}$$

EXAMPLE: A CLASSICAL LIE ALGEBRA

If g is a “classical” Lie algebra with generators $\{T_A\}$, then the universal enveloping algebra $U(g)$ is a quasi-triangular Hopf algebra with

$$\begin{aligned}\Delta(T_A) &= T_A \otimes 1 + 1 \otimes T_A \\ \epsilon(T_A) &= 0 \\ S(T_A) &= -T_A \\ \mathcal{R} &= 1 \otimes 1\end{aligned}$$

If the hermitian adjoint is defined on g, then $U(g)$ is a *-Hopf algebra with

$$\theta(T_A) = T_A^\dagger$$

DUAL PAIRING OF HOPF ALGEBRAS

Two HAs $\mathcal{U}$ and $\mathcal{A}$ over the same field k are dually paired if there is a nondegenerate inner product $\langle \, , \, \rangle : \mathcal{U} \otimes \mathcal{A} \to k$ such that

$$
\begin{aligned}
\langle xy, a \rangle &= \langle x \otimes y, \Delta(a) \rangle \\
\langle 1, a \rangle &= \epsilon(a) \\
\langle \Delta(x), a \otimes b \rangle &= \langle x, ab \rangle \\
\epsilon(x) &= \langle x, 1 \rangle \\
\langle S(x), a \rangle &= \langle x, S(a) \rangle \\
\langle \theta(x), a \rangle &= \langle x, \theta(S(a)) \rangle^{*}
\end{aligned}
$$

$x, y \in \mathcal{U}$, $a, b \in \mathcal{A}$

REPRESENTATIONS OF HOPF ALGEBRAS

A faithful linear representation $\rho : \mathcal{U} \to M(N,k)$ of a HA can be used to dually pair $\mathcal{U}$ with another HA $\mathcal{A}$, generated by the N^2 elements $\left\{A^i{}_j\right\}$, via

$$\rho^i{}_j(x) = \left\langle x, A^i{}_j \right\rangle$$

so

$$\begin{aligned} \rho(xy) = \rho(x)\rho(y) &\Rightarrow \Delta(A^i{}_j) = A^i{}_k \otimes A^k{}_j \\ \rho(1) = I &\Rightarrow \epsilon(A^i{}_j) = \delta^i{}_j \\ \rho\left(S\left(x_{(1)}\right)x_{(2)}\right) = \epsilon(x)I &\Rightarrow S(A^i{}_j) = (A^{-1})^i{}_j \end{aligned}$$

The multiplication in $\mathcal{A}$ is determined by the comultiplication in $\mathcal{U}$, but little can be said of that without more info.

Which leads us to...

QUANTUM GROUPS

[V. G. Drinfel'd, *Proc. Int. Cong. Math., Berkeley* (Berkeley, 1986) 798
S. L. Woronowicz, *Commun. Math. Phys.* **111** (1987) 613]

A quantum group (QG) is a HA $\mathcal{A}$ generated by the elements $A^i{}_j$ is dually paired with a quasitriangular HA $\mathcal{U}$ by means of a representation ρ.

The $N^2 \times N^2$ numerical R-matrix is the universal R-matrix in this representation:

$$R^{ij}{}_{k\ell} \quad = \quad \left\langle \mathcal{R}, A^i{}_k \otimes A^j{}_\ell \right\rangle$$

The dual pairing between $\mathcal{U}$ and $\mathcal{A}$ gives the commutation relations between the generators of $\mathcal{A}$ as

$$R^{ij}{}_{mn} A^m{}_k A^n{}_\ell \quad = \quad A^j{}_n A^i{}_m R^{mn}{}_{k\ell}$$

or

$$RA_1A_2 = A_2A_1R$$

The numerical version of the YBE is

$$R_{12}R_{13}R_{23} \quad = \quad R_{23}R_{13}R_{12}$$

QUANTUM LIE ALGEBRAS

[D. Bernard, *Prog. Theor. Phys. Suppl.* **102** (1990) 49]

A (left) action of $\mathcal{U}$ on itself, the adjoint action, is defined as

$$x \triangleright y \quad = \quad x_{(1)} y S\left(x_{(2)}\right)$$

It satisfies

$$(xy)\triangleright z = x\triangleright(y\triangleright z), \qquad x\triangleright(yz) = (x_{(1)}\triangleright y)(x_{(2)}\triangleright z)$$
$$x\triangleright 1 = \epsilon(x)1, \qquad 1\triangleright x = x$$

When $\mathcal{U}$ is the UEA of a "classical" Lie algebra, then

$$\begin{aligned} T_A \triangleright T_B &= T_A \cdot T_B \cdot 1 + 1 \cdot T_B \cdot S\left(T_A\right) \\ &= T_A T_B - T_B T_A = [T_A, T_B] \end{aligned}$$

so $\triangleright$ generalises the commutator.

The projectors

$$P_1(x) = \epsilon(x)1, \qquad P_0(x) = x - \epsilon(x)1$$

decompose $\mathcal{U}$ into $k1 \oplus \mathcal{U}_0$. $\mathcal{U}$ is a quantum Lie algebra (QLA) if

(a) $\mathcal{U}_0$ is finitely generated by n elements $\{T_1, T_2, \ldots, T_n\}$
(b) $\mathcal{U}_0 \triangleright \mathcal{U}_0 \subseteq \mathcal{U}_0$

If $\mathcal{U}$ is a quasitriangular HA whose universal R-matrix depends on a parameter λ such that $\mathcal{R} \to 1 \otimes 1$ as $\lambda \to 0$ and there is a dually paired QG $\mathcal{A}$, then $\mathcal{U}$ is a QLA generated by the elements of the matrix

$$X^i{}_j \quad = \quad \frac{1}{\lambda}\left\langle 1 \otimes 1 - \mathcal{R}_{21}\mathcal{R}, A^i{}_j \otimes \mathrm{id} \right\rangle$$

[P. Schupp, PW, B. Zumino, *Lett. Math. Phys.* **25** (1992) 139]

The deformation parameter q is usually defined via $\lambda = q - q^{-1}$, with $q \to 1$ giving the “classical limit”.

THE KILLING METRIC

There is also an invariant trace for such QLAs, defined by

$$\mathrm{tr}_\rho(x) \quad = \quad \mathrm{tr}\left[\rho(u)\rho(x)\right]$$

such that

$$\mathrm{tr}_\rho(y\triangleright x) \quad = \quad \epsilon(y)\mathrm{tr}_\rho(x)$$

which vanishes if $y \in \mathcal{U}_0$. This means that the Killing form

$$\eta^{(\rho)}(x, y) \quad = \quad \mathrm{tr}_\rho(xy)$$

is invariant under the adjoint action of $\mathcal{U}_0$:

$$\eta^{(\rho)}\left(z_{(1)}\triangleright x, z_{(2)}\triangleright y\right) \quad = \quad \epsilon(z)\eta^{(\rho)}(x, y) = 0$$

and we may define a $\mathcal{U}_0$-invariant Killing metric

$$\eta^{(\rho)}_{AB} \quad = \quad \mathrm{tr}_\rho\left(T_A T_B\right)$$

[PW, arXiv:q-alg/9505027]

DEFORMED GAUGE THEORIES

Mathematically, gauge theories are described in terms of fibre bundles...

- Fibre $\mathcal{F}$: where the matter fields live.

- Connection Γ: how we move between fibres; the gauge fields.

- Structure group $\mathcal{A}$: the group of transformations on the fields.

- Base space $\mathcal{M}$: the manifold on which the fields live.

We wish to generalise the structure group to a HA, and so the others must be generalised as well.

THE FIBRE AND STRUCTURE GROUP

Take $\mathcal{F}$ to be a unital associative *-algebra (with involution $^{-}$) and $\mathcal{A}$ a *-Hopf algebra which acts on $\mathcal{F}$ via a linear homomorphism $L : \mathcal{F} \to \mathcal{A} \otimes \mathcal{F}$ as

$$L(\psi) = \psi^{(1)'} \otimes \psi^{(2)}$$

satisfying

$$\begin{aligned}
\psi^{(1)'} \otimes L\left(\psi^{(2)}\right) &= \Delta\left(\psi^{(1)'}\right) \otimes \psi^{(2)} \\
\epsilon\left(\psi^{(1)'}\right)\psi^{(2)} &= \psi \\
L\left(\overline{\psi}\right) &= \theta\left(\psi^{(1)'}\right) \otimes \overline{\psi^{(2)}} \\
L(1) &= 1 \otimes 1
\end{aligned}$$

THE EXTERIOR DERIVATIVES AND CONNECTION

Suppose d and δ are exterior derivatives on $\mathcal{F}$ and $\mathcal{A}$ respectively. The coaction of $\mathcal{A}$ on differential forms on $\mathcal{F}$ is given recursively by

$$L(\mathrm{d}\psi) = \delta\psi^{(1)'} \otimes \psi^{(2)} + (-1)^{\left|\psi^{(1)'}\right|}\psi^{(1)'} \otimes \mathrm{d}\psi^{(2)}$$

A connection is a linear map taking p-forms on $\mathcal{A}$ to $(p+1)$-forms on $\mathcal{F}$ satisfying

$$\begin{aligned} \Gamma(1) &= 0 \\ \Gamma(\delta\alpha) &= -\mathrm{d}\Gamma(\alpha) \\ L(\Gamma(\alpha)) &= (-1)^{\left|\alpha_{(1)}\right|+\left|\alpha_{(3)}\right|\left(\left|\alpha_{(2)}\right|+1\right)}\alpha_{(1)}S\left(\alpha_{(3)}\right)\otimes\Gamma\left(\alpha_{(2)}\right) \\ &\quad -\delta\alpha_{(1)}S\left(\alpha_{(2)}\right)\otimes 1 \end{aligned}$$

The FIELD STRENGTH AND COVARIANT DERIVATIVE

The field strength is given by

$$F(\alpha) = \mathsf{d}\Gamma(\alpha) + (-1)^{|\alpha_{(1)}|}\Gamma\left(\alpha_{(1)}\right) \wedge \Gamma\left(\alpha_{(2)}\right)$$

Thus,

$$L(F(\alpha)) = (-1)^{|\alpha_{(2)}||\alpha_{(3)}|}\alpha_{(1)} S\left(\alpha_{(3)}\right) \otimes F\left(\alpha_{(2)}\right).$$

The covariant derivative D of a p-form ψ on $\mathcal{F}$ is

$$\mathsf{D}\psi = \mathsf{d}\psi + \Gamma\left(\psi^{(1)'}\right) \wedge \psi^{(2)},$$

Thus,

$$\mathsf{D}^2\psi = F\left(\psi^{(1)'}\right) \wedge \psi^{(2)}$$

and

$$L(\mathsf{D}\psi) = (-1)^{\left|\psi^{(1)'}\right|}\psi^{(1)'} \otimes \mathsf{D}\psi^{(2)}$$

QUANTUM STRUCTURE GROUP

Let $\mathcal{U}$ and $\mathcal{A}$ be a QLA and its associated QG under a representation ρ. If ψ^i a form living in this rep,

$$L\left(\psi^i\right) \quad = \quad A^i{}_j \otimes \psi^j;$$

With $\Gamma^i{}_j := \Gamma\left(A^i{}_j\right)$,

$$L\left(\Gamma^i{}_j\right) \quad = \quad A^i{}_k S\left(A^\ell{}_j\right) \otimes \Gamma^k{}_\ell - \delta A^i{}_k S\left(A^k{}_j\right) \otimes 1,$$

and

$$\mathrm{D}\psi^i = \mathrm{d}\psi^i + \Gamma^i{}_j \wedge \psi^j \quad \mapsto \quad A^i{}_j \otimes \mathrm{D}\psi^j.$$

The field strength $F^i{}_j := \mathrm{d}\Gamma^i{}_j + \Gamma^i{}_k \wedge \Gamma^k{}_j$ transforms as

$$L\left(F^i{}_j\right) \quad = \quad A^i{}_k S\left(A^\ell{}_j\right) \otimes F^k{}_\ell.$$

Classically, the above correspond to

$$\begin{aligned} \psi &\mapsto A\psi \\ \Gamma &\mapsto A\Gamma A^{-1} - \delta A A^{-1} \\ \mathrm{D}\psi &\mapsto A\mathrm{D}\psi \\ F &\mapsto AFA^{-1} \end{aligned}$$

COMMUTATION RELATIONS

The components $\{\Gamma^A\}$ are 1-forms given by

$$\Gamma(a) \;=\; \Gamma^A \langle T_A, a \rangle$$

The field strength is then

$$F(a) \;=\; \mathrm{d}\Gamma^A \langle T_A, a \rangle + \Gamma^A \wedge \Gamma^B \langle T_A T_B, a \rangle \,.$$

Classically,

$$\Gamma^A \wedge \Gamma^B T_A T_B \;=\; \frac{1}{2}\Gamma^A \wedge \Gamma^B [T_A, T_B]$$

so that $F = F^A T_A$. Here, we *require* that F takes this form, and so $\Gamma\Gamma$ commutation relations are determined.

Classically, the Bianchi identity $\mathrm{D}F = 0$ must hold. Requiring this in the deformed case as well gives $\Gamma\mathrm{d}\Gamma$ and $\mathrm{d}\Gamma\mathrm{d}\Gamma$ commutation relations.

THE BASE SPACE

How we treat the base space isn't obvious...

Noncommutative geometry?

[T. Brzeziński, S. Majid, *Commun. Math. Phys.* **157** (1993) 591

A. Connes, J. Lott, *Nucl. Phys. Proc. Supp.* **18B** (1991) 89]

Sheaf theory?

[M. J. Pflaum, *Commun. Math. Phys.* **166** (1994) 279]

Our approach: *assume* the existence of a quadratic form $\langle\ |\ \rangle$ taking two p-forms on $\mathcal{F}$ to k such that:

1. $\langle\phi|\,\psi\rangle^* = \langle\bar{\psi}\big|\,\bar{\phi}\rangle$;
2. $\langle\phi|\,\psi\rangle \mapsto \phi^{(1)'}\psi^{(1)'}\left\langle\phi^{(2)}\middle|\,\psi^{(2)}\right\rangle$ under the action of L (*not* necessarily symmetric);
3. $\langle\phi|\,\psi\rangle \to \int_{\mathcal{M}}\phi\wedge\star\psi$ in the undeformed limit.

$SU_q(2)$

The numerical R-matrix for the q-deformed version of $SU(2)$ is

$$R = q^{-\frac{1}{2}} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}.$$

with $q \in \mathbb{R}$ and $\lambda = q - q^{-1}$

[L. D. Faddeev, N. Yu. Reshetikhin, L. A. Takhtadzhyan, *Leningrad Math. J.* **1** (1990) 193]

If the generators of the QG $SU_q(2)$ are the elements of the matrix

$$U = \begin{pmatrix} a & b \\ -\frac{1}{q}\bar{b} & \bar{a} \end{pmatrix}$$

Then $RU_1U_2 = U_2U_1R$ gives

$$\begin{aligned} ab &= qba & a\bar{b} &= q\bar{b}a \\ b\bar{b} &= \bar{b}b & b\bar{a} &= q\bar{a}b \\ \bar{b}\bar{a} &= q\bar{a}\bar{b} & a\bar{a} &= \bar{a}a - \frac{\lambda}{q}b\bar{b} \end{aligned}$$

with $a\bar{a} + b\bar{b} = 1$.

The other HA operations are

$$
\begin{aligned}
\Delta(U) &= \begin{pmatrix} a\otimes a + b\otimes \bar{b} & a\otimes b + b\otimes \bar{a} \\ \bar{b}\otimes a + \bar{a}\otimes \bar{b} & \bar{b}\otimes b + \bar{a}\otimes \bar{a} \end{pmatrix} \\
\epsilon(U) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
\theta(U) = S(U) &= \begin{pmatrix} \bar{a} & -q^{-1}b \\ \bar{b} & a \end{pmatrix}
\end{aligned}
$$

THE QLA $U_q(su(2))$

Generated by T_1, T_+, T_- and T_2 defined by

$$X = \begin{pmatrix} T_1 & T_+ \\ T_- & T_2 \end{pmatrix} = \frac{1}{\lambda}\langle 1 \otimes 1 - \mathcal{R}_{21}\mathcal{R}, U \otimes \mathrm{id} \rangle$$

If we define

$$T_0 = T_1 + \frac{1}{q^2}T_2, \qquad T_3 = \frac{q^2}{1+q^2}(T_1 - T_2)$$

then the adjoint actions are

$$\begin{aligned} T_A \triangleright T_0 &= 0, \qquad T_0 \triangleright T_a = -\lambda\,[2]\,T_a \\ T_3 \triangleright T_3 &= -\lambda T_3, \qquad T_\pm \triangleright T_\mp = \pm\frac{[2]}{q}T_3, \\ T_3 \triangleright T_\pm &= \pm q^{\mp 1}T_\pm, \qquad T_\pm \triangleright T_3 = \mp q^{\pm 1}T_\pm \end{aligned}$$

where $A = 0, +, -, 3$, $a = +, -, 3$ and the "quantum number" $[n]$ is

$$[n] := \frac{1 - q^{-2n}}{1 - q^{-2}}.$$

Or, as "commutation relations", T_0 is central and

$$\begin{aligned} q^{\mp 1}T_3T_\pm - q^{\pm 1}T_\pm T_3 &= \pm\left(1 - \frac{\lambda}{[2]}T_0\right)T_\pm, \\ T_+T_- - T_-T_+ &= \frac{[2]}{q}\left(1 - \frac{\lambda}{[2]}T_0\right)T_3 + \frac{\lambda\,[2]}{q}T_3^2 \end{aligned}$$

The generators are linearly independent, but related quadratically by

$$\left(1 - \frac{\lambda}{[2]} T_0\right)^2 = 1 + q^2 \lambda^2 J^2,$$

where

$$J^2 = \frac{1}{q^2\,[2]} \left(q^2 T_+ T_- + T_- T_+ + [2]\, T_3^2\right)$$

REPRESENTATIONS

"Trivial":

$$\underline{\mathrm{tv}}'(T_a) = 0 \qquad \underline{\mathrm{tv}}'(T_0) = \frac{2\,[2]}{\lambda}$$

Fundamental:

$$\underline{\mathrm{fn}}(T_0) = -\frac{\lambda}{q}\left[\frac{1}{2}\right]\left[\frac{3}{2}\right]\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \underline{\mathrm{fn}}(T_3) = \frac{1}{[2]}\begin{pmatrix} -1 & 0 \\ 0 & \frac{1}{q^2} \end{pmatrix},$$

$$\underline{\mathrm{fn}}(T_+) = \begin{pmatrix} 0 & 0 \\ -\frac{1}{q} & 0 \end{pmatrix}, \qquad \underline{\mathrm{fn}}(T_-) = \begin{pmatrix} 0 & -\frac{1}{q} \\ 0 & 0 \end{pmatrix}$$

Adjoint:

$$\underline{\mathrm{ad}}(T_0) = -\lambda\,[2]\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \underline{\mathrm{ad}}(T_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{q} & 0 & 0 \\ 0 & 0 & -q & 0 \\ 0 & 0 & 0 & -\lambda \end{pmatrix},$$

$$\underline{\mathrm{ad}}(T_+) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -q\,[2] \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{q} & 0 \end{pmatrix}, \qquad \underline{\mathrm{ad}}(T_-) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{[2]}{q} \\ 0 & -\frac{1}{q} & 0 & 0 \end{pmatrix}.$$

$$\underline{\mathrm{ad}}(u) = \frac{1}{q^4}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q^2 & 0 & 0 \\ 0 & 0 & \frac{1}{q^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\eta_{AB}^{(\underline{\mathrm{ad}})} = \frac{[4]}{q^3}\begin{pmatrix} \frac{q\lambda^2[2]^2[3]}{[4]} & 0 & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & \frac{1}{q} & 0 & 0 \\ 0 & 0 & 0 & \frac{q}{[2]} \end{pmatrix}.$$

CONNECTION COMMUTATION RELATIONS

From $F = F^A T_A$:

$$
\begin{aligned}
\Gamma^0 \wedge \Gamma^0 = \Gamma^\pm \wedge \Gamma^\pm &= 0, \\
\Gamma^\pm \wedge \Gamma^3 + q^{\pm 2} \Gamma^3 \wedge \Gamma^\pm &= 0, \\
\Gamma^\pm \wedge \Gamma^0 + \Gamma^0 \wedge \Gamma^\pm &= \pm \frac{q^{\pm 1} \lambda}{[2]} \Gamma^3 \wedge \Gamma^\pm, \\
\Gamma^+ \wedge \Gamma^- + \Gamma^- \wedge \Gamma^+ &= 0 \\
\Gamma^0 \wedge \Gamma^3 + \Gamma^3 \wedge \Gamma^0 &= -\frac{\lambda}{q} \Gamma^- \wedge \Gamma^+ \\
\Gamma^3 \wedge \Gamma^3 &= \frac{\lambda\,[2]}{q} \Gamma^- \wedge \Gamma^+
\end{aligned}
$$

From $\mathrm{D}F = 0$:

$$
\begin{aligned}
\mathrm{d}\Gamma^0 \wedge \Gamma^A &= \Gamma^A \wedge \mathrm{d}\Gamma^0 \\
\mathrm{d}\Gamma^\pm \wedge \Gamma^\pm &= \Gamma^\pm \wedge \mathrm{d}\Gamma^\pm \\
\mathrm{d}\Gamma^\pm \wedge \Gamma^\mp - \Gamma^\mp \wedge \mathrm{d}\Gamma^\pm &= \pm q\lambda\Gamma^0 \wedge \mathrm{d}\Gamma^3 \pm \frac{q\lambda}{[2]}\Gamma^3 \wedge \mathrm{d}\Gamma^3 \\
&\quad \mp\lambda\,[2]\,\Gamma^0 \wedge \Gamma^- \wedge \Gamma^+ \\
\mathrm{d}\Gamma^\pm \wedge \Gamma^3 - \Gamma^3 \wedge \mathrm{d}\Gamma^\pm &= \mp q^{\pm 1}\lambda\Gamma^\pm \wedge \mathrm{d}\Gamma^3 \\
&\quad \mp q^{\pm 1}\lambda\,[2]\,\Gamma^0 \wedge \mathrm{d}\Gamma^\pm \\
&\quad - q^{\pm 2}\lambda\,[2]\,\Gamma^0 \wedge \Gamma^3 \wedge \Gamma^\pm \\
\mathrm{d}\Gamma^\pm \wedge \Gamma^0 - \left(1+\lambda^2\right)\Gamma^0 \wedge \mathrm{d}\Gamma^\pm &= \mp\frac{q^{\mp 1}\lambda}{[2]}\Gamma^3 \wedge \mathrm{d}\Gamma^\pm \\
&\quad \pm\frac{q^{\pm 1}\lambda}{[2]}\Gamma^\pm \wedge \mathrm{d}\Gamma^3 \\
&\quad \pm q^{\pm 1}\lambda^2\Gamma^0 \wedge \Gamma^3 \wedge \Gamma^\pm \\
\mathrm{d}\Gamma^3 \wedge \Gamma^\pm - \Gamma^\pm \wedge \mathrm{d}\Gamma^3 &= \pm q^{\mp 1}\lambda\Gamma^3 \wedge \mathrm{d}\Gamma^\pm \\
&\quad \pm q^{\mp 1}\lambda\,[2]\,\Gamma^0 \wedge \mathrm{d}\Gamma^\pm \\
&\quad + \lambda\,[2]\,\Gamma^0 \wedge \Gamma^3 \wedge \Gamma^\pm \\
\mathrm{d}\Gamma^3 \wedge \Gamma^3 - \left(1-\lambda^2\right)\Gamma^3 \wedge \mathrm{d}\Gamma^3 &= \frac{\lambda\,[2]}{q}\Gamma^+ \wedge \mathrm{d}\Gamma^- \\
&\quad - \frac{\lambda\,[2]}{q}\Gamma^- \wedge \mathrm{d}\Gamma^+ \\
&\quad - \lambda^2\,[2]\,\Gamma^0 \wedge \mathrm{d}\Gamma^3 \\
&\quad + \frac{\lambda^2\,[2]^2}{q}\Gamma^0 \wedge \Gamma^- \wedge \Gamma^+
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{d}\Gamma^3 \wedge \Gamma^0 - \left(1 + \lambda^2\right)\Gamma^0 \wedge \mathrm{d}\Gamma^3 &= \frac{\lambda}{q}\Gamma^- \wedge \mathrm{d}\Gamma^+ - \frac{\lambda}{q}\Gamma^+ \wedge \mathrm{d}\Gamma^- \\
&\quad + \frac{\lambda^2}{[2]}\Gamma^3 \wedge \mathrm{d}\Gamma^3 \\
&\quad - \frac{\lambda^2\,[2]}{q}\Gamma^0 \wedge \Gamma^- \wedge \Gamma^+ \\
\mathrm{d}\Gamma^3 \wedge \mathrm{d}\Gamma^\pm - q^{\pm 2}\mathrm{d}\Gamma^\pm \wedge \mathrm{d}\Gamma^3 &= \pm q^{\pm 1}\lambda\,[2]\,\mathrm{d}\Gamma^0 \wedge \mathrm{d}\Gamma^\pm \\
&\quad + q^{\pm 2}\lambda\,[2]\,\Gamma^3 \wedge \Gamma^\pm \wedge \mathrm{d}\Gamma^0 \\
&\quad - q^{\pm 2}\lambda\,[2]\,\Gamma^0 \wedge \Gamma^\pm \wedge \mathrm{d}\Gamma^3 \\
&\quad + \lambda\,[2]\,\Gamma^0 \wedge \Gamma^3 \wedge \mathrm{d}\Gamma^\pm \\
\mathrm{d}\Gamma^+ \wedge \mathrm{d}\Gamma^- - \mathrm{d}\Gamma^- \wedge \mathrm{d}\Gamma^+ &= q\lambda\mathrm{d}\Gamma^0 \wedge \mathrm{d}\Gamma^3 + \frac{q\lambda}{[2]}\mathrm{d}\Gamma^3 \wedge \mathrm{d}\Gamma^3 \\
&\quad + \lambda\,[2]\,\Gamma^0 \wedge \Gamma^+ \wedge \mathrm{d}\Gamma^- \\
&\quad - \lambda\,[2]\,\Gamma^0 \wedge \Gamma^- \wedge \mathrm{d}\Gamma^+ \\
&\quad - \lambda\,[2]\,\Gamma^- \wedge \Gamma^+ \wedge \mathrm{d}\Gamma^0 \\
&\quad - q\lambda^2\Gamma^0 \wedge \Gamma^3 \wedge \mathrm{d}\Gamma^3
\end{aligned}
$$

FIELD STRENGTH COMMUTATION RELATIONS

$$F^0 = \mathrm{d}\Gamma^0$$

$$\begin{aligned} F^\pm &= \mathrm{d}\Gamma^\pm \pm q^{\pm 1}\Gamma^3 \wedge \Gamma^\pm, \\ F^3 &= \mathrm{d}\Gamma^3 - \frac{[2]}{q}\Gamma^- \wedge \Gamma^+ \end{aligned}$$

so

$$\begin{aligned} F^3 \wedge F^\pm - q^{\pm 2} F^\pm \wedge F^3 &= \pm q^{\pm 1}\lambda\,[2]\,F^0 \wedge F^\pm, \\ F^+ \wedge F^- - F^- \wedge F^+ &= q\lambda F^0 \wedge F^3 + \frac{q\lambda}{[2]} F^3 \wedge F^3, \\ F^0 \wedge F^A &= F^A \wedge F^0. \end{aligned}$$

Γ^0, Γ^3, F^0 and F^3 are all antihermitian, and

$$\left(\Gamma^\pm\right)^\dagger = -\Gamma^\mp \qquad \left(F^\pm\right)^\dagger = -F^\mp$$

A q-DEFORMED STANDARD MODEL

Now we put everything we've developed so far into action:

(Pun intended.)

Using the quadratic form on $\mathcal{M}$, the Killing metric in the adjoint representation, the field strength F and introducing the coupling κ, we get the $SU_q(2)$-symmetric action

$$\begin{aligned} S_{\mathrm{YM}} &= -\frac{1}{2\kappa^2}\eta^{\underline{(\mathrm{ad})}}_{AB}\left\langle F^A \middle| F^B \right\rangle \\ &= -\frac{[4]}{2\kappa^2 q^2}\left\{ \left\langle F^+ \middle| F^- \right\rangle + \frac{1}{q^2}\left\langle F^- \middle| F^+ \right\rangle + \frac{1}{[2]}\left\langle F^3 \middle| F^3 \right\rangle \right. \\ &\quad \left. + \frac{\lambda^2\,[2]^2\,[3]}{[4]}\left\langle F^0 \middle| F^0 \right\rangle \right\}. \end{aligned}$$

THE YANG-MILLS ACTION

Define the four 1-forms $W^\pm$, W^3 and B and the coupling constant g by

$$\Gamma^\pm = -\frac{ig\sqrt{2}}{[2]}W^\pm, \quad \Gamma^3 = -igW^3, \quad \Gamma^0 = -\frac{ig}{\lambda}\sqrt{\frac{[4]}{[2]^3\,[3]}}B$$

$$g = q\kappa\sqrt{\frac{[2]}{[4]}}$$

Then

$$\begin{aligned}
S_{\mathrm{YM}} \;=\; & \frac{1}{[2]}\left\langle \mathrm{d}W^+ \middle| \mathrm{d}W^- \right\rangle + \frac{1}{q^2\,[2]}\left\langle \mathrm{d}W^- \middle| \mathrm{d}W^+ \right\rangle \\
& + \frac{1}{2}\left\langle \mathrm{d}W^3 \middle| \mathrm{d}W^3 \right\rangle + \frac{1}{2}\left\langle \mathrm{d}B \middle| \mathrm{d}B \right\rangle \\
& + \frac{ig}{q\,[2]}\Big(\left\langle \mathrm{d}W^+ \middle| W^3 \wedge W^- \right\rangle - \left\langle \mathrm{d}W^- \middle| W^3 \wedge W^+ \right\rangle \\
& + \left\langle \mathrm{d}W^3 \middle| W^- \wedge W^+ \right\rangle + \frac{1}{q^2}\left\langle W^3 \wedge W^- \middle| \mathrm{d}W^+ \right\rangle \\
& - q^2\left\langle W^3 \wedge W^+ \middle| \mathrm{d}W^- \right\rangle + \left\langle W^- \wedge W^+ \middle| \mathrm{d}W^3 \right\rangle \Big) \\
& + \frac{g^2}{q\,[2]}\Big(\left\langle W^3 \wedge W^+ \middle| W^3 \wedge W^- \right\rangle \\
& + \frac{1}{q^2}\left\langle W^3 \wedge W^- \middle| W^3 \wedge W^+ \right\rangle \\
& - \frac{2}{q\,[2]^2}\left\langle W^- \wedge W^+ \middle| W^- \wedge W^+ \right\rangle \Big)
\end{aligned}$$

THE HIGGS MECHANISM

The Higgs field is introduced as a complex doublet Φ^i living in the fundamental rep of $SU_q(2)$:

$$\Phi = \begin{pmatrix} \phi^- \\ \phi^0 \end{pmatrix}, \qquad \Phi^\dagger = \begin{pmatrix} \phi^+ & \bar{\phi}^0 \end{pmatrix}.$$

Under the QG action, these transform respectively as

$$\Phi^i \mapsto U^i{}_j \otimes \Phi^j, \qquad \Phi^\dagger_i \mapsto S\left(U^j{}_i\right) \otimes \Phi^\dagger_j$$

Noncommutativity of the elements of U requires non-commutativity of the elements of Φ:

$$\phi^0\phi^\pm = \frac{1}{q}\phi^\pm\phi^0, \qquad \bar{\phi}^0\phi^\pm = q\phi^\pm\bar{\phi}^0$$

$$\phi^+\phi^- = \phi^-\phi^+, \qquad \bar{\phi}^0\phi^0 = \phi^0\bar{\phi}^0 - \frac{\lambda}{q}\phi^+\phi^-$$

$\Phi^\dagger\Phi = \overline{\Phi^i}\Phi^i \equiv \bar{\phi}^0\phi^0 + \phi^+\phi^-$ is central and invariant, so we take the Higgs action to be

$$S_{\mathrm{H}} \;=\; \left\langle (\mathrm{D}\Phi)^\dagger \middle| \mathrm{D}\Phi \right\rangle - V\left(\Phi^\dagger\Phi\right)$$

THE Z-BOSON AND THE PHOTON

Φ lives in the fundamental, so

$$\begin{aligned}
\mathrm{D}\phi^- &= \mathrm{d}\phi^- + \frac{ig}{q\,[2]}\left(\sqrt{\frac{[4]}{[2]\,[3]}}\left[\frac{1}{2}\right]\left[\frac{3}{2}\right]B + qW^3\right)\phi^- \\
&\quad + \frac{ig\sqrt{2}}{q\,[2]}W^-\phi^0, \\
\mathrm{D}\phi^0 &= \mathrm{d}\phi^0 + \frac{ig}{q\,[2]}\left(\sqrt{\frac{[4]}{[2]\,[3]}}\left[\frac{1}{2}\right]\left[\frac{3}{2}\right]B - \frac{1}{q}W^3\right)\phi^0 \\
&\quad + \frac{ig\sqrt{2}}{q\,[2]}W^+\phi^-
\end{aligned}$$

If we define new fields Z and A by

$$W^3 = \cos\theta_W Z + \sin\theta_W A, \qquad B = -\sin\theta_W Z + \cos\theta_W A,$$

where

$$\tan\theta_W = q\sqrt{\frac{[4]}{[2]\,[3]}}\left[\frac{1}{2}\right]\left[\frac{3}{2}\right].$$

then there is no $A - \phi^0$ term and

$$\begin{aligned}
\mathrm{D}\phi^- &= \mathrm{d}\phi^- + \frac{ig}{\cos\theta_W}\left(\frac{1}{[2]} - \sin^2\theta_W\right)Z\phi^- + \frac{ig\sqrt{2}}{q\,[2]}W^-\phi^0 \\
&\quad + ig\sin\theta_W A\phi^- \\
\mathrm{D}\phi^0 &= \mathrm{d}\phi^0 - \frac{ig}{q^2\,[2]\cos\theta_W}Z\phi^0 + \frac{ig\sqrt{2}}{q\,[2]}W^+\phi^-
\end{aligned}$$

GAUGE BOSON MASSES

Assume the potential V has a minimum (and vanishes) when $\Phi^\dagger\Phi = v^2/2$. Take

$$\left\langle \phi^\pm \right\rangle = 0, \qquad \left\langle \phi^0 \right\rangle = \left\langle \bar{\phi}^0 \right\rangle = \frac{v}{\sqrt{2}}$$

The masses of the gauge fields are found by evaluating S_{H} at $\langle\Phi\rangle$:

$$\begin{aligned} S_{\mathrm{H}}|_{\langle\Phi\rangle} &= m_W^2 \left\langle W^+ \middle| W^- \right\rangle + \frac{1}{2} m_Z^2 \langle Z | Z \rangle + \frac{1}{2} m_A^2 \langle A | A \rangle \\ &= \frac{g^2 v^2}{q^2 [2]^2} \left\langle W^+ \middle| W^- \right\rangle + \frac{g^2 v^2}{2q^4 [2]^2 \cos^2\theta_{\mathrm{W}}} \langle Z | Z \rangle . \end{aligned}$$

so

$$m_A = 0, \qquad m_W = \frac{gv}{q\,[2]} = q m_Z \cos\theta_{\mathrm{W}}$$

AND NOW, SOME ACTUAL PHYSICS...

The chosen value for $\tan\theta_{\rm W}$ has two consequences:

1. A is massless and thus we may identify it with the photon.
2. The $A-\phi^{-}$ coupling is $-g\sin\theta_W$; call it the electron charge $-e$.

If we assume we live at or very near $q = 1$, then we find

$$\sin^2\theta_{\rm W} = \frac{3}{11} \approx 0.273, \qquad g \approx 0.580$$

The experimental value for $\sin^2\theta_{\rm W}$ is 0.2319, within 20% of the above.

If we take the experimental value of $m_Z = 91.187$ GeV, then at $q = 1$,

$$m_W = 77.76 \text{ GeV}, \qquad v = 268 \text{ GeV}$$

The first is within 3% of the actual mass of 80.22 GeV.

SYMMETRY BREAKING & ELECTRIC CHARGES

Define two new fields with vanishing VEV:

$$H = \sqrt{2}\left[\frac{1}{2}\right]\left(\bar{\phi}^0 + \frac{1}{q}\phi^0\right) - v, \qquad \phi = \frac{\sqrt{2}}{iq}\left[\frac{1}{2}\right]\left(\phi^0 - \bar{\phi}^0\right)$$

These obey the commutation relations

$$\begin{aligned} H\phi^{\pm} &= \phi^{\pm}H + i(1-q)\phi^{\pm}\phi \\ H\phi &= \phi H + 2i\left(1 - \frac{1}{q}\right)\phi^{+}\phi^{-} \\ \phi\phi^{\pm} &= \left(q + \frac{1}{q} - 1\right)\phi^{\pm}\phi + i\left(1 - \frac{1}{q}\right)\phi^{\pm}H + i\left(1 - \frac{1}{q}\right)v\phi^{\pm} \end{aligned}$$

The linear term in the last of the above is linear in the fields and breaks the $SU_q(2)$ symmetry.

However, if z is the sole generator of a HA such that

$$\Delta(z) = z \otimes z, \quad \epsilon(z) = 1, \quad S(z) = \theta(z) = z^{-1}$$

then

$$H \mapsto 1 \otimes H, \quad \phi \mapsto 1 \otimes \phi, \quad \phi^{\pm} \mapsto z^{\pm 1} \otimes \phi^{\pm}$$

is a left coaction that leaves the commutation relations invariant. This is the HA obtained from the *classical* $U(1)$.

Define a new derivative D$'$ by subtracting off the VEV of the Higgs:

$$\begin{aligned} \mathsf{D}'\phi^- &= \mathsf{D}\phi^- - \frac{igv}{q\,[2]}W^-, \\ \mathsf{D}'\left(\phi^0 - \frac{1}{\sqrt{2}}v\right) &= \mathsf{D}\phi^0 + \frac{igv}{q^2\sqrt{2}\,[2]\cos\theta_{\mathrm{W}}}Z. \end{aligned}$$

D$'$ is a covariant derivative if $z = e^{ie\chi}$ and

$$W^\pm \mapsto e^{\pm ie\chi} \otimes W^\pm, \quad Z \mapsto 1 \otimes Z, \quad A \mapsto 1 \otimes A + \delta\chi \otimes 1,$$

which are the gauge transformations for a classical $U(1)$ with gauge field A.

The central element in $U_q(su(2))$ generating the unbroken $u(1)$ algebra is the charge operator

$$Q = \frac{q}{\lambda\,[2]\left[\frac{1}{2}\right]\left[\frac{3}{2}\right]}T_0 + T_3$$

and so the covariant derivative of a field ψ living in rep ρ is

$$\begin{aligned} \mathsf{D}'\psi &= \mathsf{d}\psi - \frac{ig\sqrt{2}}{[2]}\left[W^+\rho\,(T_+) + W^-\rho\,(T_-)\right]\psi \\ &\quad - \frac{ig}{\cos\theta_{\mathrm{W}}}Z\left[\rho\,(T_3) - \sin^2\theta_{\mathrm{W}}\rho(Q)\right]\psi - ig\sin\theta_{\mathrm{W}}A\rho(Q)\psi. \end{aligned}$$

LEPTONS

Let Ψ^i be a (left-handed) lepton doublet living in the fundamental

$$\Psi = \begin{pmatrix} \psi \\ \nu \end{pmatrix}, \qquad \bar{\Psi} = \begin{pmatrix} \bar{\psi} & \bar{\nu} \end{pmatrix}$$

with anticommutation relations

$$\begin{aligned} \psi\nu &= -\frac{1}{q}\nu\psi, \qquad & \psi\bar{\nu} &= -q\bar{\nu}\psi \\ \bar{\psi}\nu &= -\frac{1}{q}\nu\psi, \qquad & \bar{\psi}\bar{\nu} &= -q\bar{\nu}\psi \\ \psi^2 = \nu^2 &= & \bar{\psi}^2 = \bar{\nu}^2 &= 0 \end{aligned}$$

$Q = \mathrm{diag}(-1, 0)$ in this rep, so we may identify ψ with the electron ($Q = -1$) and ν ($Q = 0$) with the electron neutrino.

Taking $\not{D}'$ as the covariant derivative on fermions, then

$$\begin{aligned} S_{\mathrm{F}} &= \langle\bar{\psi}|\, i\not{\partial}\psi\rangle + \langle\bar{\nu}|\, i\not{\partial}\nu\rangle \\ &\quad -g\sin\theta_{\mathrm{W}}\langle\bar{\psi}|\, \not{A}\psi\rangle - \frac{g\sqrt{2}}{q\,[2]}\left(\left\langle\bar{\psi}\right|\, \not{W}^{-}\nu\right\rangle + \left\langle\bar{\nu}\right|\, \not{W}^{+}\psi\right\rangle\right) \\ &\quad +\frac{g}{\cos\theta_{\mathrm{W}}}\left[\left(-\frac{1}{[2]} + \sin^2\theta_{\mathrm{W}}\right)\langle\bar{\psi}|\, \not{Z}\psi\rangle + \frac{1}{q^2\,[2]}\langle\bar{\nu}|\, \not{Z}\nu\rangle\right] \end{aligned}$$

In the low-energy theory, the $W\nu\psi$ coupling will result in a four-fermion interaction with the Fermi coupling constant $G_{\mathrm{F}} = \frac{g^2}{q^2[2]^2\sqrt{2}m_W^2}$. In the $q \to 1$ limit $G_{\mathrm{F}} \to 0.983 \times 10^{-5}$ GeV^{-2}, about 16% away from the experimental value 1.16639×10^{-5} GeV^{-2}.

SO FAR, SO GOOD. BUT...

PROBLEM #1: Where are the right-handed leptons?

We've incorporated leptons that live in the fundamental rep; at $q = 1$, these transform as $SU(2)$ fields, since $T_0 = 0$, and so become left-handed.

Right-handed leptons are $SU(2)$ singlets, but carry $U(1)$ hypercharge, so must be in a rep where $T_{\pm,3}$ vanish but T_0 does not: the "trivial" rep.

Thus, if χ is a fermion living in this "trivial" rep $\underline{\mathrm{tv}}'$, its contribution to the action is

$$\begin{aligned}
\left\langle \bar{\chi} \middle| i\underline{\mathrm{tv}}'\left(\not{D}'\right)\chi \right\rangle &= \langle \bar{\chi} | i\not{\partial}\chi\rangle + \frac{2i\,[2]}{\lambda}\left\langle \bar{\chi} \middle| \not{\Gamma}^0 \chi \right\rangle \\
&= \langle \bar{\chi} | i\not{\partial}\chi\rangle - \frac{g}{\cos\theta_{\mathrm{W}}}\left(\frac{2\sin^2\theta_{\mathrm{W}}}{q\lambda^2\left[\frac{1}{2}\right]\left[\frac{3}{2}\right]}\right)\langle \bar{\chi} | \not{Z}\chi\rangle \\
&\quad + g\sin\theta_{\mathrm{W}}\left(\frac{2}{q\lambda^2\left[\frac{1}{2}\right]\left[\frac{3}{2}\right]}\right)\langle \bar{\chi} | \not{A}\chi\rangle
\end{aligned}$$

As desired, it couples to the A and Z but not $W^{\pm}$, but the $q = 1$ limit does not exist. And all other reps of $SU_q(2)$ will be in a nontrivial rep of $SU(2)$ at $q = 1$, so it seems there are no chiral leptons in this theory.

PROBLEM #2: Weird electric charges!

Recall

$$Q = \frac{q}{\lambda\,[2]\left[\frac{1}{2}\right]\left[\frac{3}{2}\right]} T_0 + T_3, \qquad \left(1 - \frac{\lambda}{[2]} T_0\right)^2 = 1 + q^2\lambda^2 J^2,$$

Eliminating T_0 and taking the $q \to 1$ limit gives an $SU_q(2)$ analogue of the Gell-Mann-Nishijima relation ($Q = T_3 + Y/2$):

$$Q = T_3 - \frac{2}{3} J^2$$

A state in the isospin-j rep with T_3-component m will have charge $m - 2j(j+1)/3$. So

j	Q
0	0
$\frac{1}{2}$	-1,0
1	$-\frac{7}{3}, -\frac{4}{3}, -\frac{1}{3}$
$\frac{3}{2}$	-4,-3,-2,-1
2	-6,-5,-4,-3,-2
⋮	⋮

The appearance of 3 in the denominator is intriguing; note that if we amend the formula slightly to be

$$Q = m - \frac{2}{3} j(j+1) - S + 1$$

and let (u, d, s) be an $SU_q(2)$ triplet with $S = 0$ for the u and d and -1 for the s, then we get charges $2/3$, $-1/3$ and $-1/3$...

CONCLUSIONS

PROS:

- A consistent way of extending the structure group, fibre and connection of a fibre bundle to include HA structure

- An $SU_q(2)$-invariant action that includes gauge fields, Higgs bosons and left-handed leptons and agrees with the undeformed action at $q = 1$

- Values for $\sin^2\theta_{\mathrm{W}}$, m_W and G_F which are within 20% of experimental values, and predicted values for the Higgs VEV and $SU_q(2)$ coupling constant.

- Correct electric charges for the left-handed leptons after the QG symmetry is broken to $U(1)$

CONS:

- Unclear picture of what the base space is in the $q \neq 1$ case

- Problems incorporating right-handed leptons into the theory

- Electric charges take on bizarre values