Double Affine Hecke Algebras and Nonsymmetric Macdonald Polynomials

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September 5th, 2011

Outline

- Algebra structure
- Representing D.A.H.A - why?
- Importance of ordering
- Macdonald polynomials
- Models

Motivation

- Representation theory is a powerful tool as it reduces problems in abstract algebra to problems in linear algebra.

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- Double Affine Hecke Algebra, (D.A.H.A) gives broader view.
- Gives reasons to believe D.A.H.A is this missing link.

The Braid Group B_n

- Invertible generators $\{T_i; i = 1, .., n-1\}$
- Relations:

$$\begin{aligned} T_i T_j &= T_j T_i \quad |i-j| \geq 2 \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} \end{aligned}$$

<u>note:</u> This is just Yang Baxter equation on braids.

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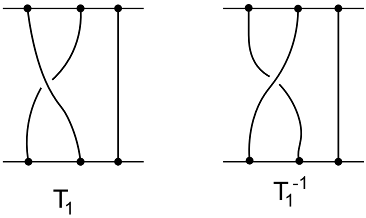
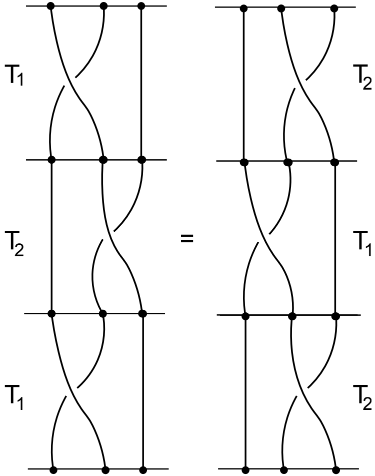
- Pictorially we have:



The Braid Group B_n

- Furthermore:



note: We see that the Yang Baxter equation on braids is the third Reidemeister move.

Hecke Algebra H_n

- Associate to B_n a Hecke Algebra H_n. (over some field $\mathcal{K}$)
- If each T_i also satisfies the following skein relation:

$$(T_i - t^{1/2})(T_i + t^{-1/2}) = 0 \qquad t \in \mathcal{K}$$

- This gives explicit form for inverse:

$$T_i^{-1} = T_i - (t^{1/2} - t^{-1/2}) \qquad t \in \mathcal{K}$$

Affine Hecke Algebra $\mathcal{A}_n$

- Extend H_n to an Affine Hecke Algebra $\mathcal{A}_n$.
- Append to it n invertible operators Y_i.
- Relations:

$$\begin{aligned} Y_i Y_j &= Y_j Y_i \\ T_i Y_j &= Y_j T_i \qquad j \neq i, i+1 \\ T_i^{-1} Y_i T_i^{-1} &= Y_{i+1} \end{aligned}$$

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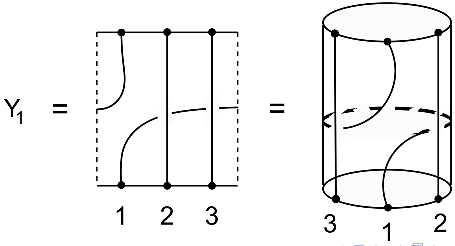

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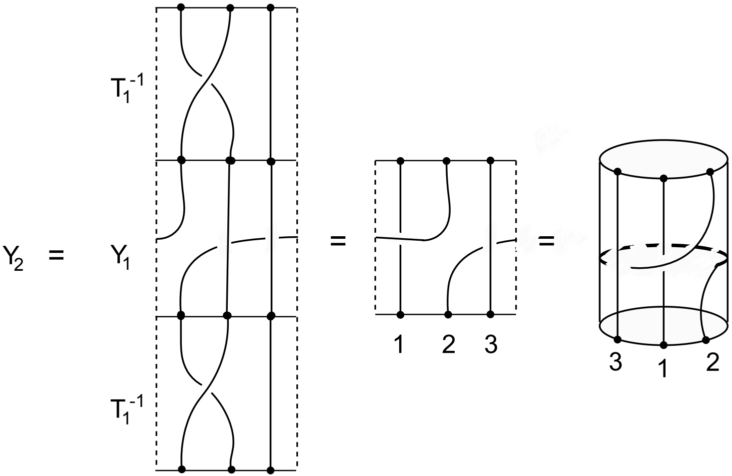
- For example Y_2:



Double Affine Hecke Algebra $\mathcal{D}_n$

- Further extend $\mathcal{A}_n$ to a Double Affine Hecke Algebra $\mathcal{D}_n$.
- Append to it n invertible operators X_i.
- Relations:

$$\begin{aligned} X_iX_j &= X_jX_i \\ T_iX_j &= X_jT_i \qquad j \neq i, i+1 \\ T_iX_iT_i &= X_{i+1} \end{aligned}$$

Double Affine Hecke Algebra $\mathcal{D}_n$

- Furthermore:

$$
\begin{aligned}
T_1^2 &= Y_2^{-1} X_1 Y_2 X_1^{-1} \\
Y_i \tilde{X} &= q \tilde{X} Y_i \qquad \text{where } \tilde{X} = \prod_{i=1}^{n} X_i \ , q \in \mathcal{K} \\
X_i \tilde{Y} &= q^{-1} \tilde{Y} X_i \qquad \text{where } \tilde{Y} = \prod_{i=1}^{n} Y_i \ , q \in \mathcal{K}
\end{aligned}
$$

Representing $\mathcal{D}_n$

- Look at representation U of $\mathcal{D}_n$ on the ring of n variable Laurent polynomials.

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- U is irreducible and Y- semisimple.
- Y_i therefore simultaneously diagonalisable on U.
- Nonsymmetric Macdonald polynomial is a monic simultaneous eigenvector of Y_i.
- Nonsymmetric Macdonald polynomials form a basis of U.

Representing $\mathcal{D}_n$

- Polynomial map given by:

$$\begin{aligned} X_i &\longmapsto x_i \\ T_i &\longmapsto t^{1/2}s_i + \frac{(t^{1/2} - t^{-1/2})x_{i+1}}{x_i - x_{i+1}}(s_i - 1) \\ Y_i &\longmapsto T_i T_{i+1}\ldots\ldots T_{n-1}\omega T_1^{-1} T_2^{-1}\ldots\ldots T_{i-1}^{-1} \end{aligned}$$

- s_i permutes the variables x_i and x_{i+1}.
- $\omega = s_{n-1}...s_1\tau_1$ with $\tau_i x_j = q^{\delta_{ij}}x_j$.
  Namely, $\omega f(x_1x_2..x_n) = f(qx_nx_1x_2..x_{n-1})$
  for any f $\in$ U.

Examples (n=3 case)

- $X_1(x_1) = x_1^2$
- $X_2(x_1) = x_2 x_1$
- $T_1(x_2) = t^{1/2} x_1 + (t^{1/2} - t^{-1/2}) x_2$
- $T_2(x_2) = t^{-1/2} x_3$
- $Y_1(x_1) = qtx_1 + q(t-1)x_2 + q(t-1)x_3$
- $Y_3(x_2) = t^{-1} x_2 + q(1-t)x_3$
- $Y_2(x_3^2) = t^{-1} x_3^2 + (t^{-1} - 1)x_2 x_3$

Ordering

- Define an ordering $\succ$ such that Y_i is triangular.
- Definition $\succ$:

$$\lambda \succ \mu \Leftrightarrow (\lambda^+ > \mu^+) \text{ or } (\lambda^+ = \mu^+ \text{ and } \lambda > \mu)$$

- Here $>$ is the dominance ordering:

$$\lambda \geq \mu \Leftrightarrow \sum_{j=1}^{l} \lambda_j \geq \sum_{j=1}^{l} \mu_j \text{ for any } 1 \leq l \leq n.$$

- For example under this ordering $x^\lambda \succ x^\mu$ when $\lambda = (2,0,0)$ and $\mu = (1,1,0)$.

$$x_1^2 x_2^0 x_3^0 \succ x_1^1 x_2^1 x_3^0$$

Ordering

- Action of Y_i on any polynomial now given by:

$$Y_i(x^\lambda) = t^{\rho(\lambda)_i} q^{\lambda_i} x^\lambda + \sum_{\mu \prec \lambda} c_{\lambda,\mu} x^\mu$$

- Recall:

$$Y_1(x_1) = qtx_1 + q(t-1)x_2 + q(t-1)x_3$$
$$\Rightarrow \rho(\lambda)_i = 1$$
$$Y_3(x_2) = t^{-1}x_2 + q(1-t)x_3$$
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- Since Y_i is triangular under $\succ$, the action of Y_i on polynomials gives rise to triangular matrices.
- Easily diagonalisable!

Matrix Representation (n=3 case)

Here are the matrices corresponding to the action of Y_1, Y_2 and Y_3 on degree zero and degree one polynomials with the basis $\{1, x_1, x_2, x_3\}$.

$$Y_1 = \begin{bmatrix} t & 0 & 0 & 0 \\ 0 & qt & 0 & 0 \\ 0 & q(t-1) & 1 & 0 \\ 0 & q(t-1) & 0 & 1 \end{bmatrix} \quad Y_3 = \begin{bmatrix} t^{-1} & 0 & 0 & 0 \\ 0 & t^{-1} & 0 & 0 \\ 0 & 0 & t^{-1} & 0 \\ 0 & q(t^{-1}-1) & q(1-t) & qt \end{bmatrix}$$

$$Y_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q(1-t) & qt & 0 \\ 0 & q(2-t-t^{-1}) & q(t-1) & t^{-1} \end{bmatrix}$$

Nonsymmetric Macdonald Polynomials (n=3 case)

- Monic eigenvectors satisfying all of the above matrices easily obtained.
- These are nonsymmetric Macdonald polynomials E_λ.
- For the above we obtain:

$$
\begin{aligned}
E_{(0,0,0)} &= 1 \\
E_{(1,0,0)} &= x_1 + \frac{q(t-1)}{qt-1}x_2 + \frac{q(t-1)}{qt-1}x_3 \\
E_{(0,1,0)} &= x_2 + \frac{q(t-1)}{qt-t^{-1}}x_3 \\
E_{(0,0,1)} &= x_3
\end{aligned}
$$

Nonsymmetric Macdonald Polynomials (n=3 case)

- For degree two polynomials:

$$E_{(2,0,0)} = x_1^2 + \frac{q^2(t-1)}{q^2t-1}x_2^2 + \frac{q^2(t-1)}{q^2t-1}x_3^2 + \frac{q(t-1)(q+1)}{q^2t-1}x_1x_2 + \frac{q(t-1)(q+1)}{q^2t-1}x_1x_3 + \frac{q^2(t-1)^2(q+1)}{(qt-1)(q^2t-1)}x_2x_3$$

$$E_{(0,2,0)} = x_2^2 + \frac{q^2t(t-1)}{q^2t^2-1}x_1x_2 + \frac{q^2t(t-1)^2}{(q^2t^2-1)(qt-1)}x_1x_3 + \frac{(q^3t^2+q^2t^2-q^2t-1)(t-1)}{(q^2t^2-1)(qt-1)}x_2x_3$$

$$E_{(0,0,2)} = x_3^2 + \frac{t-1}{(qt-1)}x_1x_3 + \frac{t-1}{(qt-1)}x_2x_3$$

Applications

- Examples where these D.A.H.A polynomials are deformed Q.H.E wave functions.

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  In fact the vanishing conditions obeyed by the polynomials are the q-deformed vanishing conditions of the Q.H.E wavefunctions.
- Also shows that the polynomials are related to components of loop model ground states.
- Related to other integrable models?

Acknowledgements

- Jiri Vala
- Paul Watts
- Vincent Pasquier

IRCSET
Investing in People and Ideas
Irish Research Council for Science, Engineering and Technology

References

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