Topological Qubit Design and Leakage

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Ollscoil na hÉireann Má Nuad

Overview

- Designing topological qubits, the braid group, leakage errors.
- Designing the optimal qubit/qudit.
- Possible leakage-free two-qubit gates.
- Summary, future work.

Topological Qubit Design and Leakage:

- **New Journal of Physics** - 2011 New J. Phys. **13** 075006
- **ArXiv** - arXiv:1102.5029v1

Topological Quantum Computation[1]

- Collection of anyons.

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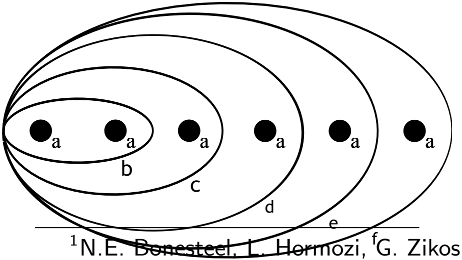
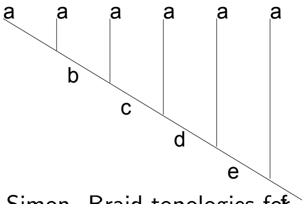
- Hilbert space is fusion space of anyons → no tensor product structure.





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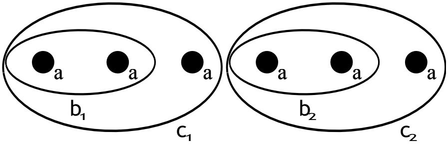
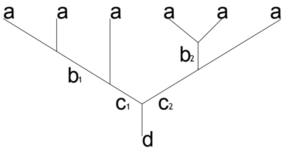
Topological Quantum Computation[1]

- Collection of anyons.
- Hilbert space is fusion space of anyons $\rightarrow$ no tensor product structure.
- Associate qubits with small groups of anyons.
- Computational space is the tensor product of qubit spaces.





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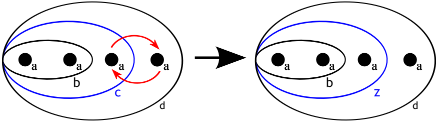
Leakage

Example:

- 3-anyon qubit composed of Fibonacci anyons, γ_i.
- Fibonacci anyon fusion rules:
  1. $1 \otimes 1 = 1$
  2. $\gamma \otimes 1 = \gamma$
  3. $\gamma \otimes \gamma = 1 \oplus \gamma$
- Set overall charge to be γ: $((\bullet\ \bullet)\ \bullet)_\gamma$.
- γ_1 and γ_2 can fuse to 1 or γ:
  - $|0\rangle_L = ((\bullet\ \bullet)_1\ \bullet)_\gamma$
  - $|1\rangle_L = ((\bullet\ \bullet)_\gamma\ \bullet)_\gamma$
- But could have: $|N.C.\rangle = ((\bullet\ \bullet)_\gamma\ \bullet)_1$.

Leakage

Braiding only changes the topological charge at the point where the braiding takes place.



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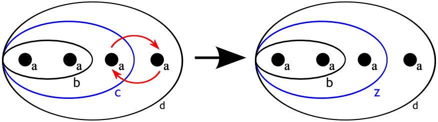
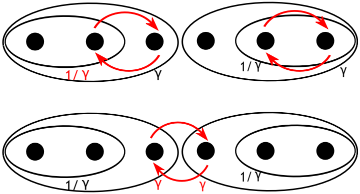


Two qubit gate: Only braiding between the qubits can change overall charge of qubits.



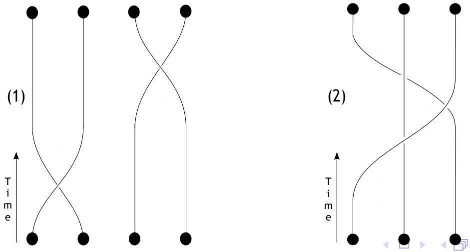
The Braid Group, B_n

- Describes all possible braids on n particles.
- Generators: τ_i - swaps particle i with particle $i+1$
- Relations:

$$\tau_i \tau_j = \tau_j \tau_i \ [\text{if } |i-j| \geq 2] \quad (1)$$

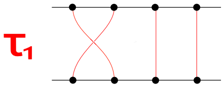
$$\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} \quad (2)$$

- Generators of the braid group are conjugate to each other.



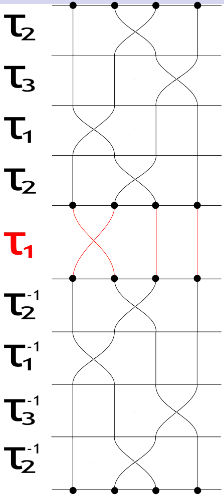
Conjugacy of Braid Generators

- Start with τ_1



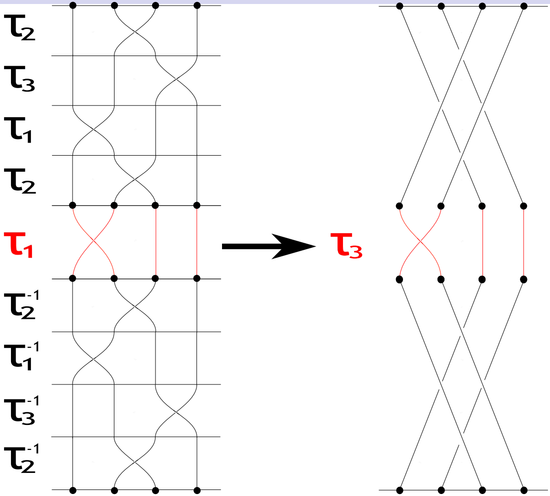
Conjugacy of Braid Generators

- Start with τ_1
- Conjugate by $T = (\tau_2 \tau_3 \tau_1 \tau_2)$



Conjugacy of Braid Generators

- Start with τ_1
- Conjugate by $T = (\tau_2 \tau_3 \tau_1 \tau_2)$
- $T \tau_1 T^{-1} = \tau_3$



Optimal Qubit

If it is experimentally hard to manipulate many anyons:

- The less anyons we need the better.

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Three-anyon qubit $\rightarrow$ can be universal[2]

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Optimal Qubit

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Two-anyon qubit → not universal
Three-anyon qubit → can be universal[2]
Four-anyon qubit → can be universal, trivial topological charge

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Optimal Qubit

If it is experimentally easy to manipulate many anyons:

- More anyons per qubit means less reliance on two-qubit gates.
- Possible to avoid leakage.

Q: What is the highest number of anyons a qubit can contain.

Qubits as Braid Group Representations

- Obtain a representation of B_n, $n =$ number of anyons in qubit.
- Hilbert space of a qubit is two dimensional.
- Logic operations on qubits are unitary.
- Represent the braid generators as matrices in U(2).
- Divide representation by determinant $\rightarrow$ SU(2).

Representation of Qubit

Abelian vs. non-Abelian:

Abelian Group

Representation is the sum of 1 dimensional representations $\rightarrow$ non-universal.

Non-Abelian Group

Representation is not reducible into 1 dimensional representations $\rightarrow$ can be universal.

Find largest n for which 2-d representations of B_n can be non-Abelian.

Maximum Number of Anyons

- By relation (1), all τ_{odd} can be simultaneously diagonalised.
- By relation (2), if two neighbouring generators commute they are equal.

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Consider $n = 5$:

- 4 generators: τ_1, τ_3 are diagonal.
- τ_4 commutes with $\tau_1 \Rightarrow \tau_4$ is diagonal (unless τ_1 is a multiple of $\mathbb{I}$).
- τ_4 now commutes with $\tau_3 \rightarrow$ neighbours commute $\Rightarrow$ Abelian.

$\Rightarrow$ **Not true for n $<$ 5.**

Maximum Number of Anyons

For Qubits

We must have n $<$ 5.

If n $\geq$ 5:

- Uninteresting 1-qubit braiding.

OR

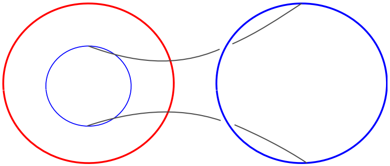
- Universality but with leakage.

OR

- Higher dimensional computational space.

Maximal Number of Anyons for Qudits, d > 2

- For $d = 2$, have shown $n < 5$.
- Would like to find similar result for $d > 2$.
- Won't explicitly use the braid group, only arguments which apply to any exchange group.



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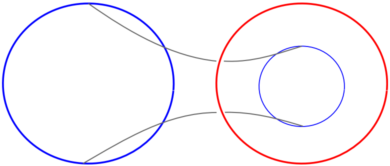

Maximal Number of Anyons for Qudits, d $>$ 2

Assumed Properties:

1. Generators which don't involve the same object commute.
2. The group is represented unitarily.
3. Generators are conjugate to each other, also any adjacent pair is conjugate to any other adjacent pair.

Upper Limit on n

- Consider: d-dimensional representation, χ, of some exchange group on n excitations.
- Choose basis: σ_{odd} are simultaneously diagonalisable.
- For $n \geq 5$ no two σ_{odd} can be equal:
  - Take $n = 5$ and let $\sigma_1 = \sigma_3$.
  - σ_4 and σ_1 commute (physically separated).
  - $\sigma_3 = \sigma_1 \Rightarrow \sigma_4$ and σ_3 commute $\Rightarrow$ Abelian.

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  - $\sigma_3 = \sigma_1 \Rightarrow \sigma_4$ and σ_3 commute $\Rightarrow$ Abelian.
- Each σ_{odd} must have a different arrangement of the eigenvalues.
- **Upper limit on n is 2d! + 1**

Actually...

Upper limit on n is actually much lower:

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1. All σ_{even} must commute with each other.
2. Each σ_i must commute with all σ_j where $j \neq i \pm 1$.
3. If $n \geq 5$ and 2 σ_{odd} have same form $\Rightarrow$ Abelian, e.g.:
    - σ_3 has eigenvalue arrangement $(\alpha, \alpha, \beta, \gamma, \beta)$.
    - σ_5 has eigenvalue arrangement $(\beta, \beta, \alpha, \gamma, \alpha)$.
    - σ_2 is of some form that commutes with σ_5.
    - Therefore σ_2 commutes with $\sigma_3 \Rightarrow$ Abelian.

Results

$N(d)$: Largest n for which the representation of the exchange group(dimension d) is non-Abelian.

Formanek[5] Result

For the braid group B_n: $\mathbf{N(d) = d + 2}$

- We have replicated this result for general exchange group for $d = 2, 3$.
- No complete proof yet for arbitrary d.

[5] E. Formanek. Braid group representations of low degree. London Math. Soc., 2(s3- 73):279322, 1996.

Results

But...

- Have shown upper limit of n exists for any exchange group.
- This upper limit depends strongly on the number and multiplicity of eigenvalues of the generators.
- The maximum number of anyons per qudit is related to the number of topological charges of the chosen anyon model.

Leakage in Two-Qudit Gates

- Possible to avoid leakage in single qubit gates.
- Two-qudit gates $\rightarrow$ topological charge of qudits can be changed.
- Is it possible to completely avoid leakage?

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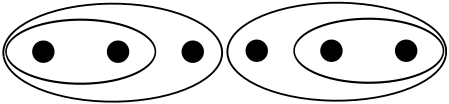
1. Have system of n anyons, n_1 in qubit 1 and n_2 in qubits 2; $n = n_1 + n_2$.
2. We know $\rho_{1/2}$; representations of $B_{n_{1/2}}$ of dimension $d_{1/2}$. We want to find ρ, a representation of B_n of dimension d.
3. Leakage-free braiding $\Rightarrow \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and $d = d_1 d_2$.

Leakage in Two-Qudit Gates

Find a representation ρ such that:

- Braids that braid within first qubit: $\rho(\tau_i) = \rho_1(\tau_i) \otimes \mathbb{I}_{d_2}$.
- Braids that braid within second qubit: $\rho(\tau_i) = \mathbb{I}_{d_1} \otimes \rho_2(\tau_i)$.
- Use braid relations to obtain form of braid which swaps anyons between qubits, τ_{q_1,q_2}.
- If a matrix exists to represent τ_{q_1,q_2} $\rightarrow$ Leakage-free braiding.

Three-Anyon Qubits



- ρ_1 and ρ_2 are 2-dimensional representations of B_3.

$$\rho(\tau_{1/2}) = \rho_1(\tau_{1/2}) \otimes \mathbb{I}_2$$
$$\rho(\tau_{4/5}) = \mathbb{I}_2 \otimes \rho_2(\tau_{1/2})$$

- $\rho(\tau_3)$ exchanges anyons between qubits.

Three-Anyon Qubits

- τ_3 can be represented by matrix provided eigenvalues are 8^{th} roots of unity.
- Leakage-free two-qubit braiding! $\rightarrow$ All braids are leakage free.

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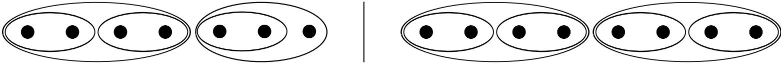
Problem:

- These eigenvalues correspond to representations of Ising-like anyon models $\rightarrow$ non-universal.
- Leakage-free but non-universal two-qubit braiding.

Other Types of Qubits



- $2 \times$ 4-anyon qubits: 4-dimensional representation of B_8.
- 3-anyon qubit & 4-anyons qubit: 4-dimensional representation of B_7.

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- $2 \times$ 4-anyon qubits: 4-dimensional representation of B_8.
- 3-anyon qubit & 4-anyons qubit: 4-dimensional representation of B_7.
- **But:** $d \geq n - 2 \rightarrow 4 \not\geq 8 - 2 = 6$, $4 \not\geq 7 - 2 = 5$.
- Non-Abelian representations of these gates will not exist.
- Also no leakage-free, non-Abelian two-qutrit/qutrit-qubit gates.

Summary

- Optimal qubit: Should be composed of 3 or 4 anyons.
- Optimal qudit:
  - For braiding: Should be composed of $\leq d + 2$ anyons.
  - In general: Number of "anyons" relates to number of topological charges in model
- Leakage-free 2-qudit gates: Not possible without sacrificing universality.

Future Work

- Find a solution to the problem of how many "anyons" a qudit can contain in general.
- Examine possibilities of leakage-free subgroups.
  - Need only one two-qubit entangling braid to be leakage-free $\rightarrow$ full universal quantum computation.