Kitaev honeycomb lattice model: from A to B and beyond

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Outline

Toric code and Kitaev honeycomb lattice model

- introduction
- relation between both models
- vorticity
- loop symmetries
- Abelian phase: summary of results

Exact solution of the Kitaev honeycomb model

- map onto spin-hardcore boson system
- Jordan-Wigner fermionization
- adding magnetic field
- ground state as BCS state with explicit vacuum

Further developments

- ground states on torus
- ground state degeneracy in Abelian and non-Abelian phase

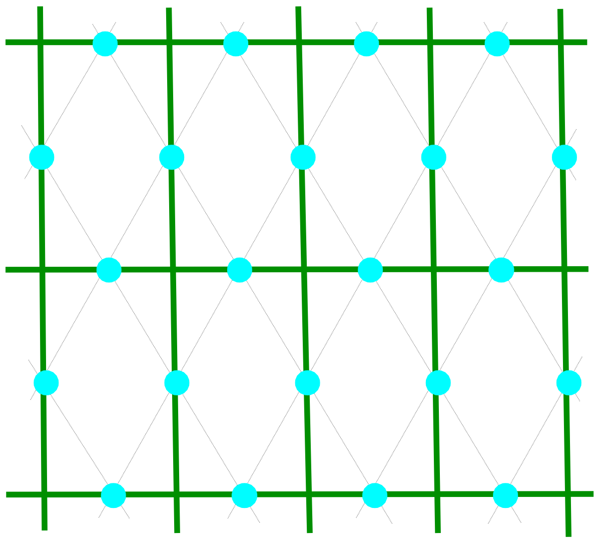
From A …

toric code
and
Kitaev honeycomb lattice model

Toric code

Toric code

- spin 1/2 particles on the edges of a square lattice (green)

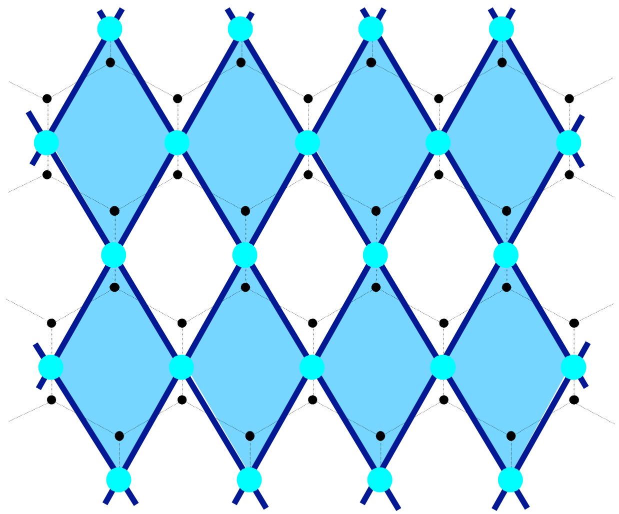


$$H_{TC} = -J_{eff}\left(\Sigma_{star}\, Q_s + \Sigma_{plaquettes}\, Q_p\right)$$

A.Y.Kitaev, *Fault-tolerant quantum computation by anyons*,
Ann. Phys. 303, 2 (2003).

Unitarily equivalent toric code

- spin 1/2 particles on the vertices of a square lattice (blue)
- connects naturally with the Kitaev honeycomb model



$$H_{TC} = -J_{eff}\, \Sigma_q\, Q_q$$

$$Q_q = \tau^y_{left(q)} \tau^y_{right(q)} \tau^z_{up(q)} \tau^z_{up(q)}$$

Pauli matrices

Toric code

Hamiltonian $H_{TC} = -J_{eff} \sum_p Q_p$ $\qquad Q_p = \tau^z_p \tau^y_{p+n_x} \tau^y_{p+n_y} \tau^z_{p+n_x+n_y}$

"Symmetries" $[H_{TC}, Q_p] = 0$ $\qquad [Q_p, Q_q] = 0$

Eigensvalues the operators $\boldsymbol{Q}_{\mathbf{p}}$ have eigenvalues $Q_p = \pm 1$; for all p we have $\{Q_p\}$

$|\{Q_p\}>$ is characterized completely by the eigenvalues Q_p: $\boldsymbol{Q}_{\mathbf{p}} |\{Q_p\}> = Q_p |\{Q_p\}> \ \forall$ p

Ground state is stabilized by $\mathbf{Q_p}$ for all p

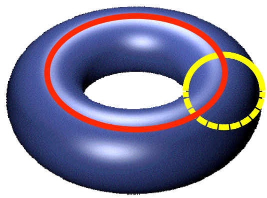
$\boldsymbol{Q}_{\mathbf{p}} |\{Q_p\}> = |\{Q_p\}>$

$|\{Q_p\}>_{TC} = |\{Q_p = 1\}$ for all plaquettes>

On **torus**, we have $\prod Q_p = 1$, and

two additional homologically nontrivial symmetries

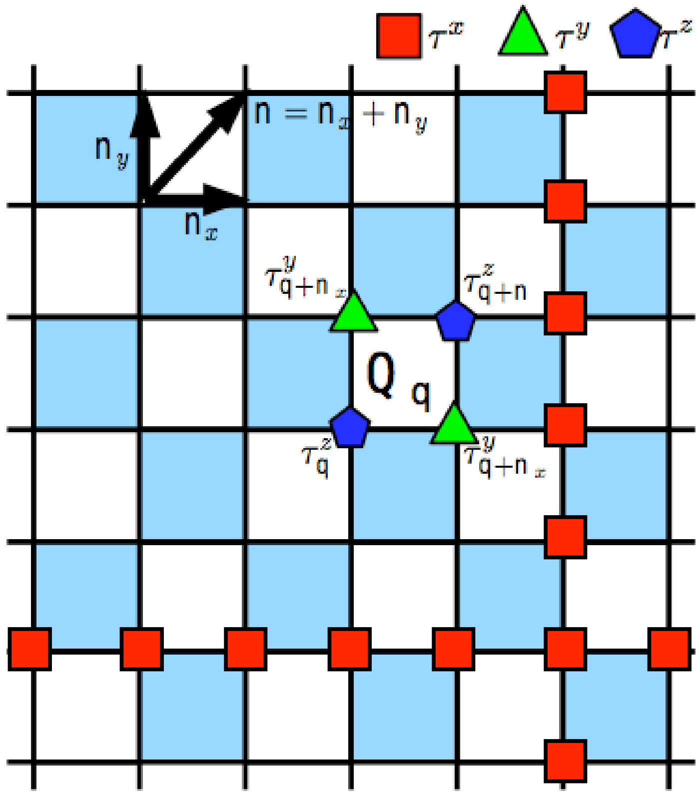
$|\{Q_p\}, l_x, l_y>_{TC}$



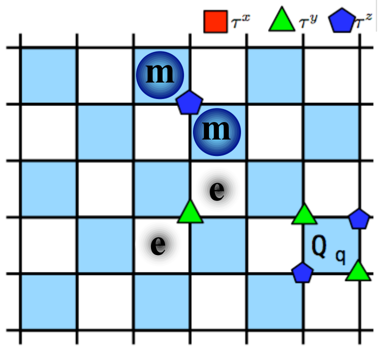
The energy does not depend on

the eigenvalues of the homologically nontrivial symmetries;

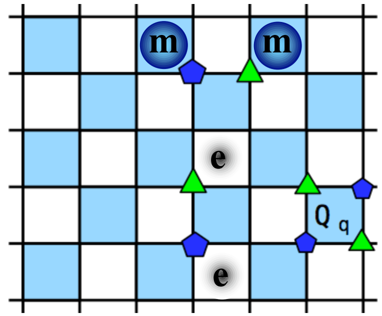
this implies **four-fold ground state degeneracy**.



Quasiparticles



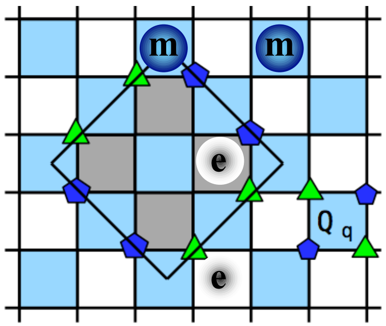
Toric code **quasiparticle excitations,** Q_p=-1, are
- **“magnetic”** (living on blue plaquettes) or **“electric”** (white plaquettes),
- are created in pairs by acting on the ground state with Pauli operators.

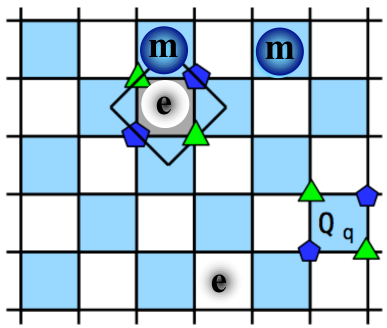


Operator $C_{L,m}$ to move a single “magnetic” excitation in a contractible loop L is the product of all “electric” plaquette operators enclosed by the loop (and vice versa).

If the initial state $|\{Q_p\}>$ contains an “electric” excitation then moving a magnetic excitation around it returns the initial state with the phase changed by -1 implying that:

- “magnetic” and “electric” particles are relative **semions**
- “e-m” composite is a **fermion**





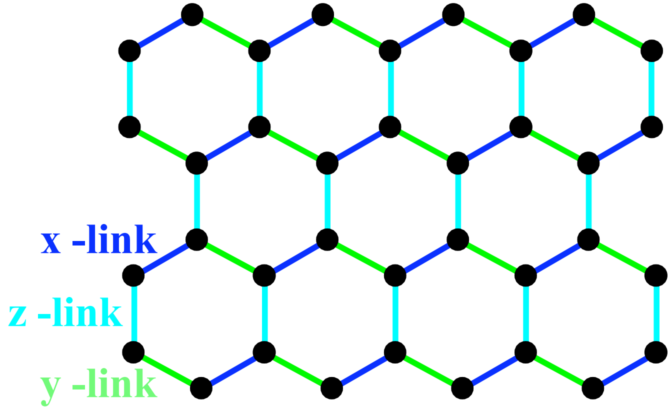
- “e-m” fermion behaves as semion when braided with an “e” or “m” particle

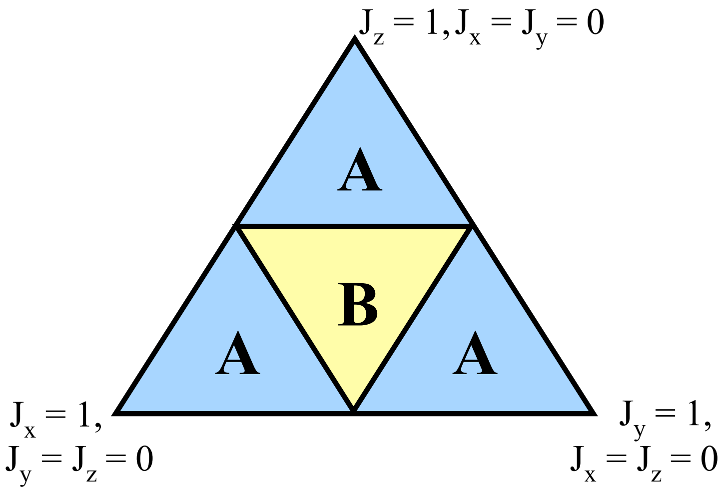
Kitaev honeycomb lattice model

$$H_0 = J_x \sum_{i,j \; x\text{-link}} \sigma^x_i \sigma^x_j + J_y \sum_{i,j \; y\text{-link}} \sigma^y_i \sigma^y_j + J_z \sum_{i,j \; z\text{-link}} \sigma^z_i \sigma^z_j$$

$$= \sum_\alpha J_\alpha \sum_{i,j} \sigma^\alpha_i \sigma^\alpha_j = \sum_\alpha J_\alpha \sum_{i,j} K^\alpha_{ij}$$

α-link:





Phase diagram:

- phase A - can be mapped perturbatively onto the toric code;
- phase B - gapless.

Adding magnetic field:

$$H = H_0 + H_1 = H_0 + \sum_i \sum_{\alpha=x,y,z} B_\alpha \sigma_{\alpha,i}$$

- parity and time-reversal symmetry are broken
- phase B acquires a gap and becomes non-abelian topological phase of Ising type

The leading P and T breaking term in perturbation theory occurs at the third order:

$$H_1 = -\kappa \sum_{\boldsymbol{q}} \sum_{l=1}^{6} P(\boldsymbol{q})^{(l)} \qquad \sum_{l=1}^{6} P(\boldsymbol{q})^{(l)} = \sigma_1^x \sigma_6^y \sigma_5^z + \sigma_2^z \sigma_3^y \sigma_4^x + \sigma_1^y \sigma_2^x \sigma_3^z + \sigma_4^y \sigma_5^x \sigma_6^z + \sigma_3^x \sigma_4^z \sigma_5^y + \sigma_2^y \sigma_1^z \sigma_6^x$$

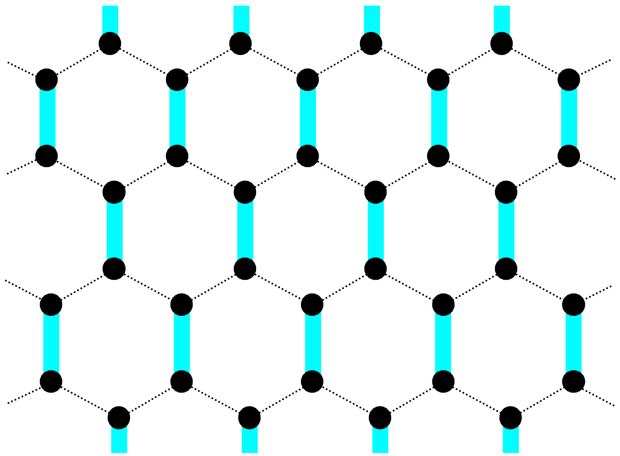
A.Y.Kitaev, Ann. Phys. 321, 2 (2006).

Mapping abelian phase onto toric code

$$J_z >> J_y, J_x$$

$$H_D \quad = \quad -J_z \sum_{z-links} \sigma_j^z \sigma_k^z,$$ "dimers"

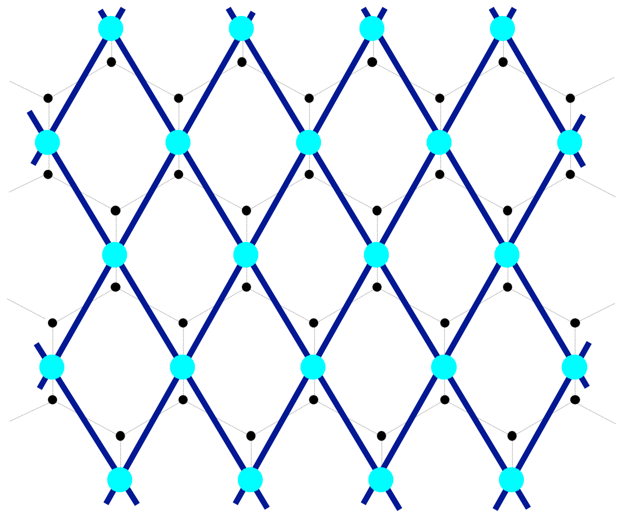
$$V \quad = \quad -J_x \sum_{x-links} \sigma_j^x \sigma_k^x - J_y \sum_{y-links} \sigma_j^y \sigma_k^y$$



Effective spins

- are formed by ferromagnetic ground states of $-J_z\sigma_j^z\sigma_k^z$

$$|\uparrow\rangle_{eff} = |\uparrow\uparrow\rangle \quad |\downarrow\rangle_{eff} = |\downarrow\downarrow\rangle$$



A.Y.Kitaev, *Fault-tolerant quantum computation by anyons*,
Ann. Phys. 303, 2 (2003).

Mapping abelian phase onto toric code

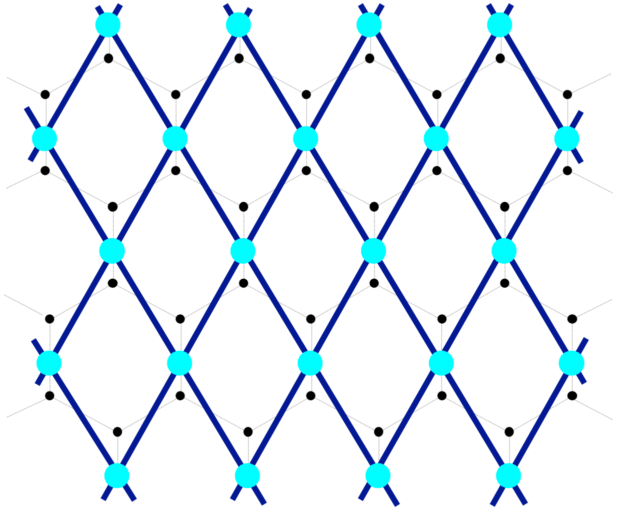
Effective Hamiltonian (no magnetic field)
first non-constant term of perturbation theory occurs on the 4th order

$$H_{\text{eff}} = -\frac{J_x^2 J_y^2}{16|J_z|^3} \sum_p Q_p,$$

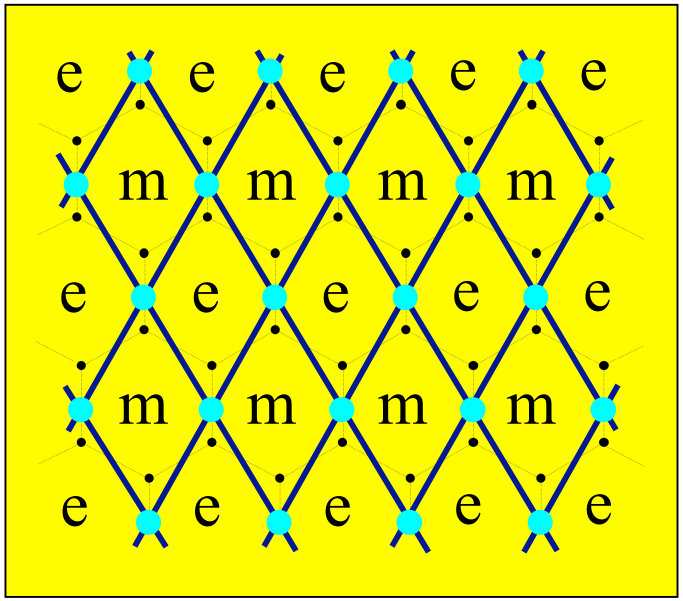
toric code

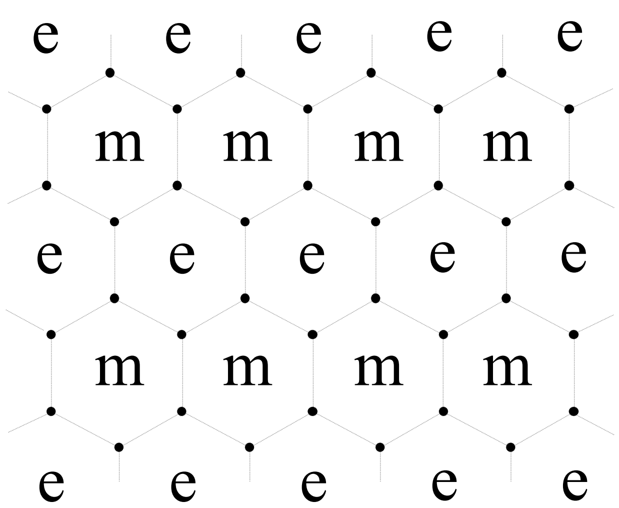
$$Q_p = \sigma^y_{\text{left}(p)} \sigma^y_{\text{right}(p)} \sigma^z_{\text{up}(p)} \sigma^z_{\text{down}(p)}$$

defined on the square lattice with effective spins on the vertices

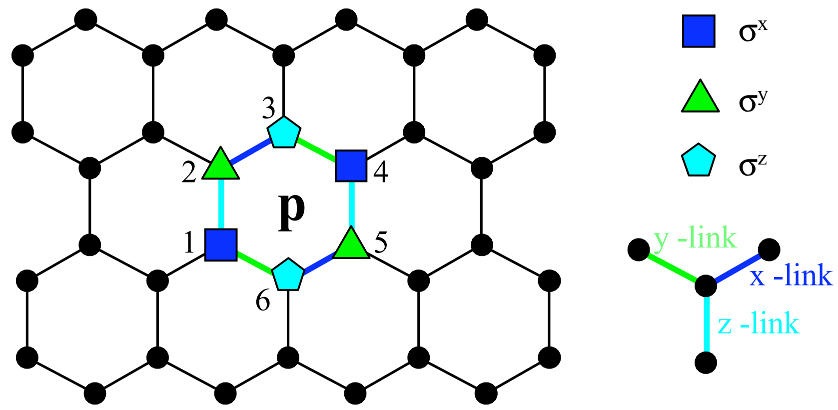


Toric code quasiparticles and vortices of the honeycomb lattice model





Vortex operators in the honeycomb model



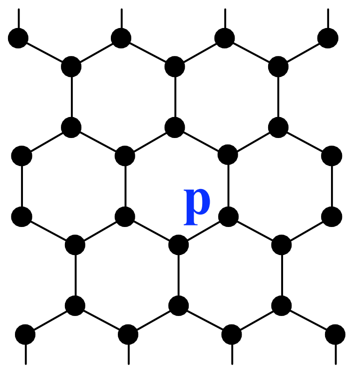
$$W_p = \sigma^x_1 \sigma^y_2 \sigma^z_3 \sigma^x_4 \sigma^y_5 \sigma^z_6 =$$

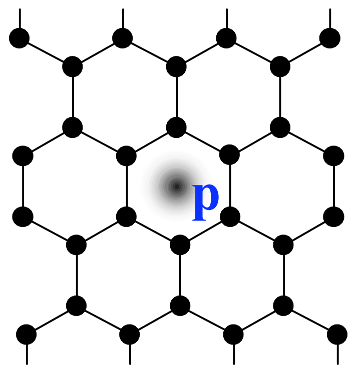
$$= K^z_{1,2} K^x_{2,3} K^y_{3,4} K^z_{4,5} K^x_{5,6} K^y_{6,1}$$

$$[H_0, W_p] = 0 \qquad (K^\alpha_{k,k+1})^2 = 1 \qquad K^\beta_{k+1,k+2} K^\alpha_{k,k+1} = - K^\alpha_{k,k+1} K^\beta_{k+1,k+2}$$

$$H_0 |n\rangle = E_n |n\rangle$$

$$w_p = \langle n|W_p|n\rangle = +1 \qquad\qquad w_p = \langle n|W_p|n\rangle = -1$$





Vortex sectors

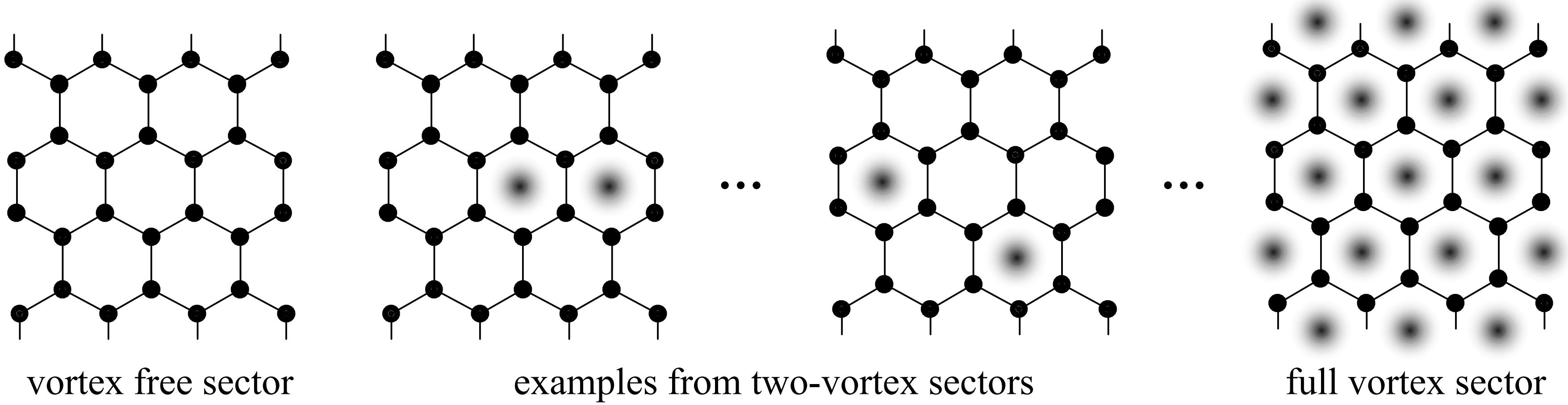
Each energy eigenstate |n> is characterized by some vortex configuration

$$\{w_p = \langle n|W_p|n\rangle = \pm 1\} \text{ for all plaquettes p}$$

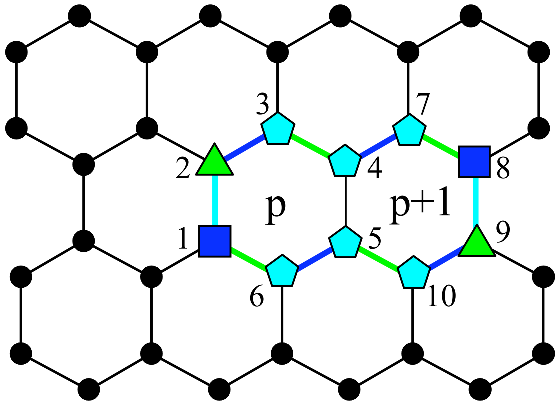
also the vortices are always excited in pairs,
i.e. even-vortex configurations are relevant on closed surfaces or infinite plane,

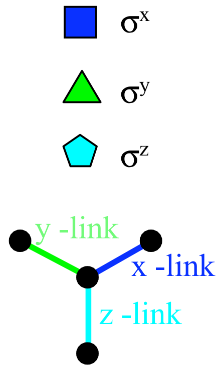
the Hilbert space splits into vortex sectors, i.e. subspaces of the system with a particular configuration of vortices

$$L = \bigoplus_{w_{1,\dots\dots,}w_m} L_{w_{1,\dots,}w_m}$$



Products of vortex operators





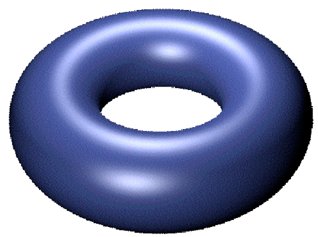
(we used $(K^{\alpha}_{k,k+1})^2 = 1$)

Products of vortex operators generate closed loops

$$K_{i,j}^{\alpha(1)} K_{j,k}^{\alpha(2)} \ldots K_{p,q}^{\alpha(M-1)} K_{q,i}^{\alpha(M)}$$

On a torus, this gives the condition

$$\Pi_p W_p = 1$$



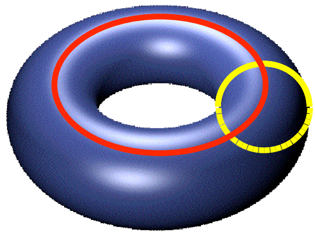
Loop symmetries on torus

For a system of N spins on a torus (i.e. a system with N/2 plaquettes), $\Pi_p W_p = 1$ implies that there are N/2-1 independent vortex quantum numbers $\{w_1, \ldots, w_{N/2-1}\}$.

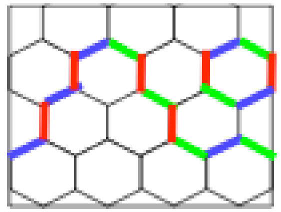
Loops on the torus $K_{i,j}^{\alpha(1)} K_{j,k}^{\alpha(2)} \ldots K_{p,q}^{\alpha(M-1)} K_{q,i}^{\alpha(M)}$

- all homologically trivial loops are generated by plaquette operators

- in addition, two distinct homologically nontrivial loops are needed to generate the full loop symmetry group (the third nontrivial loop is a product of these two).



The full loop symmetry of the torus is the abelian group with N/2+1 independent generators of the order 2 ($\mathrm{loop}^2 = I$), i.e. $Z_2^{N/2+1}$.



All loop symmetries can be written as

$$C_{(k,l)} = G_k F_l(W_1, W_2, \ldots, W_{N-1})$$

where k is from {0,1,2,3} and $G_0 = I$, and G_1, G_2, G_3 are arbitrarily chosen symmetries from the three nontrivial homology classes, and F_l, with l from $\{1, \ldots, 2^{N/2-1}\}$, run through all monomials in the W_p operators.

Results on the Abelian phase

1) The symmetry structure of the system is manifested in the effective Hamiltonian obtained using the Brillouin-Wigner perturbation theory. The longer loops occur at the higher order of the perturbation expansion:

$$H_{eff} = \sum_{i=0}^{3} \sum_{j=1}^{2^{N/2-2}} c_{i,j} G_i(z,y) F_j(Q_1, Q_2, \ldots\ldots Q_{N/2-2})$$

trivial
$W_p \longrightarrow Q_p$

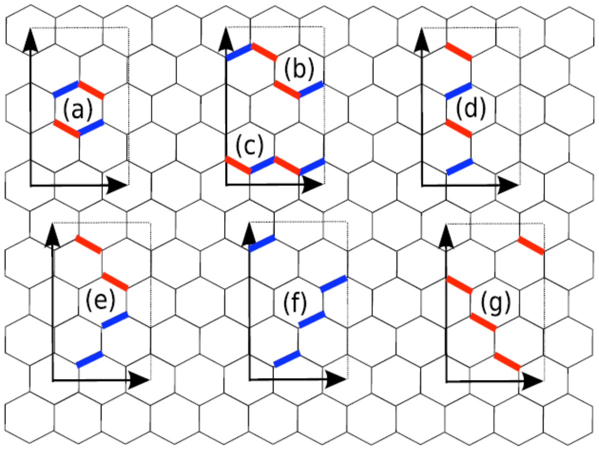
nontrivial
- reflects topology

G. Kells, A. T. Bolukbasi, V. Lahtinen, J. K. Slingerland, J. K. Pachos and J. Vala,
Topological degeneracy and vortex manipulation in the Kitaev honeycomb model,
Phys. Rev. Lett. **101**, 240404 (2008).

2) Fermions of the Abelian phase can be moved efficiently using the K strings from the symmetries.

A. T. Bolukbasi, et al., in preparation.

3) The symmetry structure of the effective Hamiltonian allows to classify all finite size effect, intrisic to the system of sizes <36 spins: for example N=16 spins.



$$\begin{aligned} H^{(4)} &= -\frac{J_x^2 J_y^2}{16|J_z|} \sum^{8} (Q_n + R_n - 5A_n) \\ &- \frac{J_x^2 J_y^2}{16|J_z|} \sum^{4} (Z_n + 5Y_n) \\ &- \frac{5}{16|J_z|} (J_x^4 \sum_{n=1}^{2} X_n + J_y^4 \sum_{n=3}^{4} X_n) \end{aligned}$$

G. Kells, N. Moran and J. Vala,
Finite size effects in the Kitaev honeycomb lattice model on torus, J. Stat. Mech. – Th. Exp., (2009) P03006

… to B …

exact solution

of the Kitaev honeycomb lattice model

Effective spins and hardcore bosons

New perspective:

spin-hardcore boson representation

$$|\uparrow_{\blacksquare}\uparrow_{\square}\rangle = |\Uparrow, 0\rangle, \quad |\downarrow_{\blacksquare}\downarrow_{\square}\rangle = |\Downarrow, 0\rangle$$
$$|\uparrow_{\blacksquare}\downarrow_{\square}\rangle = |\Uparrow, 1\rangle, \quad |\downarrow_{\blacksquare}\uparrow_{\square}\rangle = |\Downarrow, 1\rangle$$

Schmidt, Dusuel, and Vidal (2008)

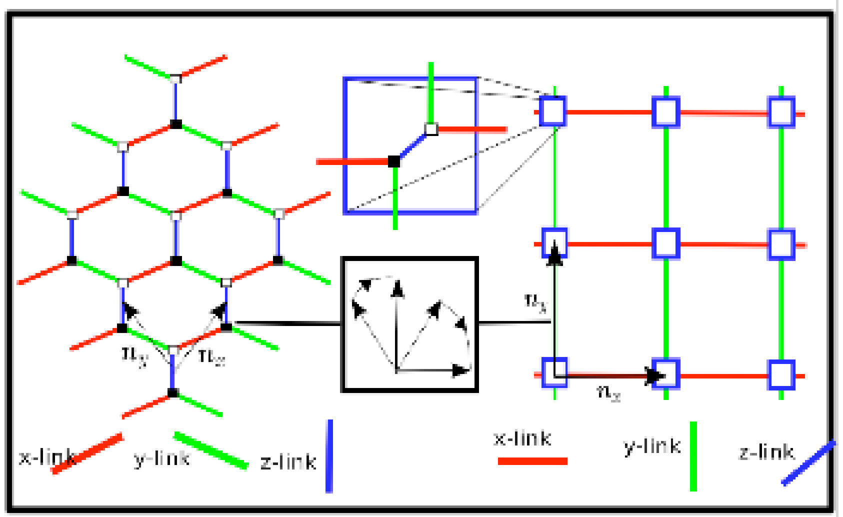
Pauli operators:

$$\sigma^x_{q,\blacksquare} = \tau^x_q(b^\dagger_q + b_q) \quad , \quad \sigma^x_{q,\square} = b^\dagger_q + b_q,$$
$$\sigma^y_{q,\blacksquare} = \tau^y_q(b^\dagger_q + b_q) \quad , \quad \sigma^y_{q,\square} = i\,\tau^z_q(b^\dagger_q - b_q),$$
$$\sigma^z_{q,\blacksquare} = \tau^z_q \quad , \quad \sigma^z_{q,\square} = \tau^z_q(I - 2b^\dagger_q b_q),$$

Vortex and plaquette operators:

$$W_q = (I - 2N_q)(I - 2N_{q+n_y})Q_q$$

$$N_q = b^\dagger_q b_q \qquad Q_q = \tau^z_q \; \tau^y_{q+n_x} \tau^y_{q+n_y} \tau^z_{q+n}$$



In the A_z-phase, $J_z >> J_x, J_y$, the bosons are energetically suppressed, thus at low energy

$$|\{W_q\}, 0> = |\{Q_q\}>$$

the low-energy perturbative Hamiltonian equals to toric code $H_{TC} = -J_{\text{eff}} \sum_q Q_q \otimes I$

This allows to write down an orthonormal basis of the full system in terms of the **toric code stabilizers**:

$$|\{W_q\}, \{q\}>$$

where $\{W_q\}$ lists all honeycomb plaquette operators and $\{q\}$ lists the position vectors of any occupied bosonic modes. On a torus, the homologically nontrivial symmetries must be added

$$|\{W_q\}, l_0^{(x)}, l_0^{(y)}, \{q\}>$$

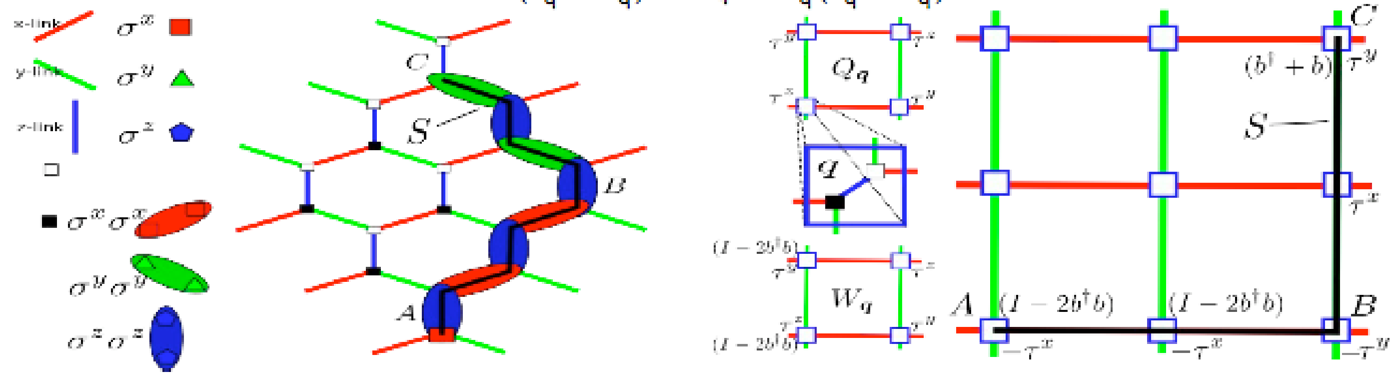
G. Kells, et al., **arXiv:0903.5211 (2009)**

Jordan-Wigner transformation

Bosonic and effective spin Hamiltonian can be written in terms of fermions and vortices by applying a Jordan-Wigner transformation

$$
\begin{aligned}
\mathsf{H} &= -J_x \sum_{\mathsf{q}} (b^\dagger_{\mathsf{q}} + b_{\mathsf{q}}) \tau^x_{\mathsf{q}+\mathsf{n}_x} (b^\dagger_{\mathsf{q}+\mathsf{n}_x} + b_{\mathsf{q}+\mathsf{n}_x}) \\
&- J_y \sum_{\mathsf{q}} i\tau^z_{\mathsf{q}} (b^\dagger_{\mathsf{q}} - b_{\mathsf{q}}) \tau^y_{\mathsf{q}+\mathsf{n}_y} (b^\dagger_{\mathsf{q}+\mathsf{n}_y} + b_{\mathsf{q}+\mathsf{n}_y}) \\
&- J_z \sum_{\mathsf{q}} (I - 2b^\dagger_{\mathsf{q}} b_{\mathsf{q}}).
\end{aligned}
$$

$$(c^\dagger_{\mathsf{q}} + c_{\mathsf{q}}) = S_{\mathsf{q}} = S'_{\mathsf{q}}(b^\dagger_{\mathsf{q}} + b_{\mathsf{q}})$$



$$
\begin{aligned}
\mathsf{H} &= J_x \sum_{\mathsf{q}} \mathsf{X}_{\mathsf{q}} (c^\dagger_{\mathsf{q}} - c_{\mathsf{q}})(c^\dagger_{\mathsf{q}+\mathsf{n}_x} + c_{\mathsf{q}+\mathsf{n}_x}) \\
&+ J_y \sum_{\mathsf{q}} \mathsf{Y}_{\mathsf{q}} (c^\dagger_{\mathsf{q}} - c_{\mathsf{q}})(c^\dagger_{\mathsf{q}+\mathsf{n}_y} + c_{\mathsf{q}+\mathsf{n}_y}) \\
&+ J_z \sum_{\mathsf{q}} (2c^\dagger_{\mathsf{q}} c_{\mathsf{q}} - I),
\end{aligned}
$$

where on a plane

$$\mathsf{Y}_{\mathsf{q}} = I \qquad \mathsf{X}_{q_x,q_y} \equiv \prod_{q'_y=0}^{q_y-1} \mathsf{W}_{q_x,q'_y}$$

Importantly, presence of a fermion indicates an anti-ferromagnetic configuration of z-link

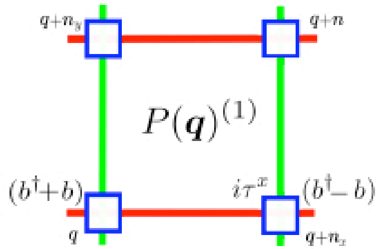
Magnetic field

- breaks parity and time-reversal symmetry
- opens a gap in phase B and turns it into non-abelian topological phase of Ising type

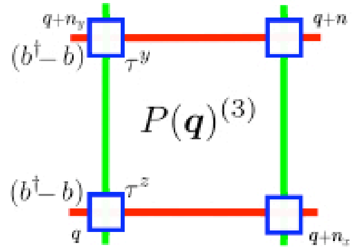
$$H_1 = -\kappa \sum_{\boldsymbol{q}} \sum_{l=1}^{6} P(\boldsymbol{q})^{(l)}$$

$$\sum_{l=1}^{6} P(\boldsymbol{q})^{(l)} = \sigma_1^x\sigma_6^y\sigma_5^z + \sigma_2^z\sigma_3^y\sigma_4^x + \sigma_1^y\sigma_2^x\sigma_3^z + \sigma_4^y\sigma_5^x\sigma_6^z + \sigma_3^x\sigma_4^z\sigma_5^y + \sigma_2^y\sigma_1^z\sigma_6^x$$

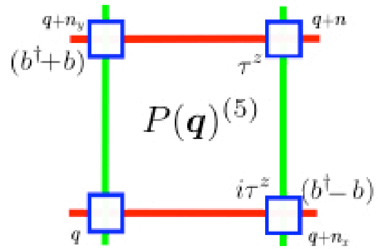
- H_1 commutes with the plaquette operators, so stabilizer formalism can still be used



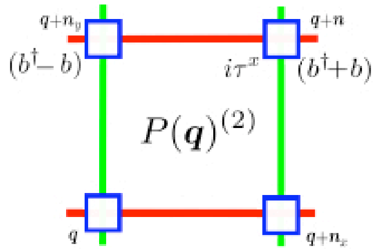
$-i\boldsymbol{X}_{q_x,q_y}(c^\dagger_{\boldsymbol{q}} - c_{\boldsymbol{q}})(c^\dagger_{\boldsymbol{q}+\boldsymbol{n}_x} - c_{\boldsymbol{q}+\boldsymbol{n}_x})$



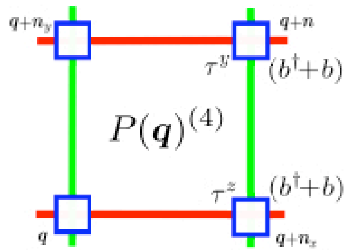
$i\boldsymbol{Y}_{q_x,q_y}(c^\dagger_{\boldsymbol{q}} - c_{\boldsymbol{q}})(c^\dagger_{\boldsymbol{q}+\boldsymbol{n}_y} - c_{\boldsymbol{q}+\boldsymbol{n}_y})$



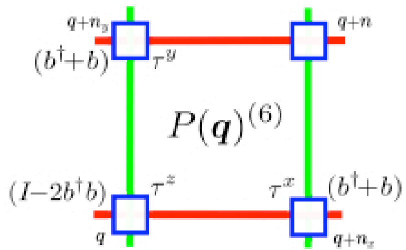
$i\boldsymbol{X}_{q_x,q_y+1}(c^\dagger_{\boldsymbol{q}+\boldsymbol{n}_y} - c_{\boldsymbol{q}+\boldsymbol{n}_y})(c^\dagger_{\boldsymbol{q}+\boldsymbol{n}_x} - c_{\boldsymbol{q}+\boldsymbol{n}_x})$



$-i\boldsymbol{X}_{q_x,q_y+1}(c^\dagger_{\boldsymbol{q}+\boldsymbol{n}_y} + c_{\boldsymbol{q}+\boldsymbol{n}_y})(c^\dagger_{\boldsymbol{q}+\boldsymbol{n}} + c_{\boldsymbol{q}+\boldsymbol{n}})$



$i\boldsymbol{Y}_{q_x+1,q_y}(c^\dagger_{\boldsymbol{q}+\boldsymbol{n}_x} + c_{\boldsymbol{q}+\boldsymbol{n}_x})(c^\dagger_{\boldsymbol{q}+\boldsymbol{n}} + c_{\boldsymbol{q}+\boldsymbol{n}})$



$i\boldsymbol{X}_{q_x,q_y}(c^\dagger_{\boldsymbol{q}+\boldsymbol{n}_y} + c_{\boldsymbol{q}+\boldsymbol{n}_y})(c^\dagger_{\boldsymbol{q}+\boldsymbol{n}_x} + c_{\boldsymbol{q}+\boldsymbol{n}_x})$

Vortex-free sector

Transformation to the momentum representation

$$c_{\boldsymbol{q}} = M^{-1/2} \sum c_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{q}}$$

$$H = \sum_{\boldsymbol{k}} \left[\xi_{\boldsymbol{k}} c_{\boldsymbol{k}}^\dagger c_{\boldsymbol{k}} + \frac{1}{2}(\Delta c_{\boldsymbol{k}}^\dagger c_{-\boldsymbol{k}}^\dagger + \Delta^* c_{-\boldsymbol{k}} c_{\boldsymbol{k}}) \right] - M J_z$$

$$\begin{aligned} \xi_{\boldsymbol{k}} &= \varepsilon_{\boldsymbol{k}} - \mu \\ \Delta_{\boldsymbol{k}} &= \alpha_{\boldsymbol{k}} + i\beta_{\boldsymbol{k}} \\ \mu &= -2J_z \\ \varepsilon_{\boldsymbol{k}} &= 2J_x \cos(k_x) + 2J_y \cos(k_y) \\ \alpha_{\boldsymbol{k}} &= 4\kappa(\sin(k_x) - \sin(k_y) - \sin(k_x - k_y)) \\ \beta_{\boldsymbol{k}} &= 2J_x \sin(k_x) + 2J_y \sin(k_y). \end{aligned}$$

The effect of the magnetic field is contained fully in the α_k term.

The Hamiltonian can be diagonalized by Bogoliubov transformation:

$$\gamma_{\boldsymbol{k}} = u_{\boldsymbol{k}} c_{\boldsymbol{k}} - v_{\boldsymbol{k}} c_{-\boldsymbol{k}}^\dagger \qquad |u_{\boldsymbol{k}}|^2 + |v_{\boldsymbol{k}}|^2 = 1$$

resulting in the BCS Hamiltonian

$$\mathsf{H} = \sum_{n=1}^{M} E_n(\gamma_n^\dagger \gamma_n - 1/2)$$

$$\begin{aligned} E_{\boldsymbol{k}} &= \sqrt{\xi_{\boldsymbol{k}}^2 + |\Delta_{\boldsymbol{k}}|^2} \\ u_{\boldsymbol{k}} &= \sqrt{1/2(1 + \xi_{\boldsymbol{k}}/E_{\boldsymbol{k}})} \\ v_{\boldsymbol{k}} &= i\sqrt{1/2(1 - \xi_{\boldsymbol{k}}/E_{\boldsymbol{k}})} \end{aligned}$$

the ground state is BCS state with the **vacuum** given here **explicitly** in terms of **toric code stabilizers**

$$|gs\rangle_{HC} = \prod \left(u_{\mathsf{k}} + v_{\mathsf{k}} c_{\mathsf{k}}^\dagger c_{-\mathsf{k}}^\dagger \right) | \{Q_{\mathsf{q}}\}, \{\emptyset\} \rangle \rightarrow |\{1,1,\ldots,1\},\{0\}\rangle$$

Other vortex sectors on torus

To address an arbitrary vortex configuration we rewrite the general Hamiltonian

$$\mathsf{H} = \frac{1}{2}\sum_{\mathsf{qq'}} \begin{bmatrix} c^\dagger_{\mathsf{q}} & c_{\mathsf{q}} \end{bmatrix} \begin{bmatrix} \xi_{\mathsf{qq'}} & \Delta_{\mathsf{qq'}} \\ \Delta^\dagger_{\mathsf{qq'}} & -\xi^T_{\mathsf{qq'}} \end{bmatrix} \begin{bmatrix} c_{\mathsf{q'}} \\ c^\dagger_{\mathsf{q'}} \end{bmatrix}$$

To specify a particular vortex sector, the operators X_q and Y_q are replaced by their eigenvalues in that sector; for example for H_0 we obtain

$$\begin{aligned} \xi_{\boldsymbol{qq'}} &= 2J_z\delta_{\boldsymbol{q},\boldsymbol{q'}} + J_x\boldsymbol{X_q}(\delta_{\boldsymbol{q},\boldsymbol{q'}-\boldsymbol{n}_x} + \delta_{\boldsymbol{q}-\boldsymbol{n}_x,\boldsymbol{q'}}) \\ &+ J_y\boldsymbol{Y_q}(\delta_{\boldsymbol{q},\boldsymbol{q'}-\boldsymbol{n}_y} + \delta_{\boldsymbol{q}-\boldsymbol{n}_y,\boldsymbol{q'}}) \\ \Delta_{\boldsymbol{qq'}} &= J_x\boldsymbol{X_q}(\delta_{\boldsymbol{q},\boldsymbol{q'}-\boldsymbol{n}_x} - \delta_{\boldsymbol{q}-\boldsymbol{n}_x,\boldsymbol{q'}}) \\ &+ J_y\boldsymbol{Y_q}(\delta_{\boldsymbol{q},\boldsymbol{q'}-\boldsymbol{n}_y} - \delta_{\boldsymbol{q}-\boldsymbol{n}_y,\boldsymbol{q'}}). \end{aligned}$$

On **torus,** these terms include **periodicity**, i.e. the terms connecting the sites $(0, q_y)$ and $(N_x - 1, q_y)$, and $(q_x, 0)$ and $(q_x, N_y - 1)$, and thus the **homologically nontrivial symmetries**

$$\begin{aligned} X_{q_x,q_y} &= \textstyle\prod_{q'_y=0}^{q_y-1} W_{q_x,q'_y} && (q_y \neq 0 \text{ and } q_x \neq N_x - 1) \\ X_{q_x,q_y} &= 1 && (q_y = 0 \text{ and } q_x \neq N_x - 1) \\ X_{q_x,q_y} &= -l_0^{(x)} \textstyle\prod_{q'_y=0}^{q_y-1} W_{q_x,q_y} && (q_y \neq 0 \text{ and } q_x = N_x - 1) \\ X_{q_y,q_x} &= -l_0^{(x)} && (q_y = 0 \text{ and } q_x = N_x - 1) \end{aligned}$$

$$\begin{aligned} Y_{q_x,q_y} &= 1 && (q_y \neq N_y - 1) \\ Y_{q_x,q_y} &= -l^{(y)}_{q_x} && (q_y = N_y - 1) \end{aligned}$$

$$l^{(y)}_{q_x} = l^{(y)}_0 \textstyle\prod_{q_y=0}^{N_{y-1}} \prod_{q'_x=0}^{q_x-1} W_{q'_x,q_y}$$

In order to include the magnetic field H_1 we have to add also

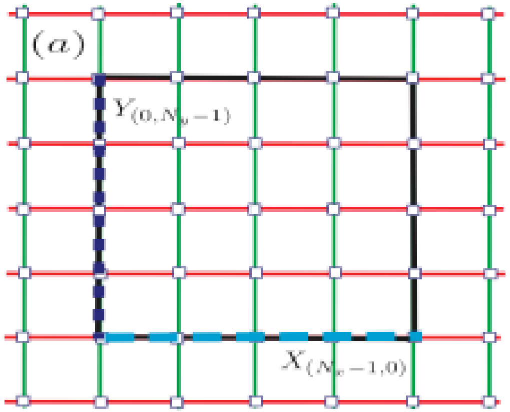
$$\begin{aligned} X_{q_x,q_y+1} &= -l^{(y)}_{q_x+1} && (q_x \neq N_x - 1) \\ X_{q_x,q_y+1} &= l_0^{(x)} l_0^{(y)} && (q_x = N_x - 1), \end{aligned}$$

$$\begin{aligned} X_{q_x,q_y} &= l^{(y)}_{q_x} \textstyle\prod_{q'_y=0}^{q_y-1} W_{q_x,q'_y} && (q_x \neq N_x - 1) \\ X_{q_x,q_y} &= l_0^{(x)} l_0^{(y)} W_{q_x,q_y} && (q_x = N_x - 1). \end{aligned}$$

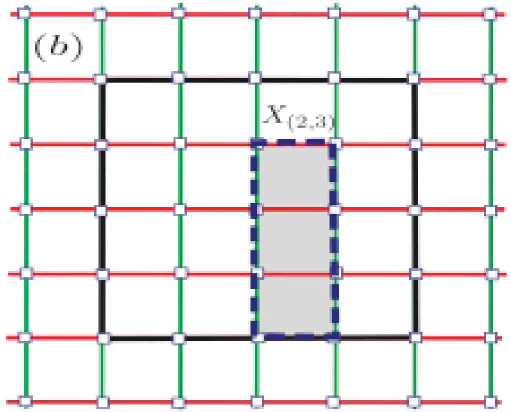
Role of symmetries

On a torus, the system has N/2+1 loop symmetry generators from which all other loop symmetries can be obtained. We can specify a particular sector of the Hamiltonian by specifying the eigenvalues of the N/2-1 plaquette symmetries and 2 homologically nontrivial symmetries.

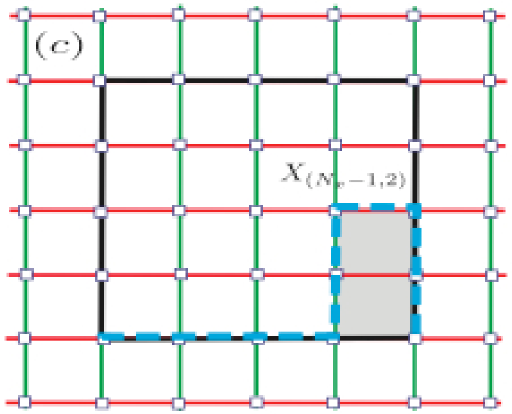
$$Y_{(0,N_{y-1})} = -l_0^{(y)},\ X_{(N_{x-1},0)} = -l_0^{(x)}$$



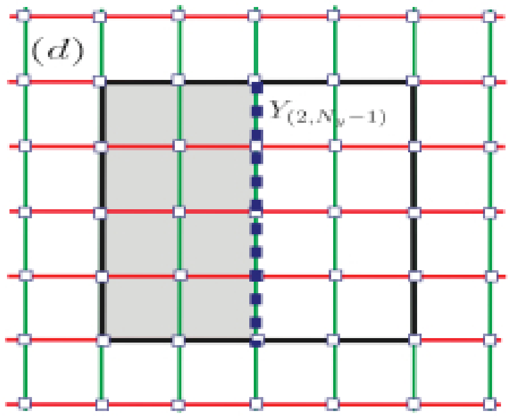
$$X_{(2,3)} = \prod_{q_y=0}^{2} W_{(2,q_y)}$$



$$X_{(N_{x-1},2)} = -l_0^{(x)} \prod_{q_y=0}^{1} W_{(N_{x-1},q_y)}$$



$$Y_{(2,N_{y-1})} = -l_0^{(y)} \prod_{q_x=0}^{1} \prod_{q_y=0}^{N_y-1} W_{(q_x,q_y)}$$



Fermionization on torus

The general Hamiltonian for an arbitrary vortex configuration

$$\mathsf{H} = \frac{1}{2}\sum_{\mathsf{qq'}} \left[\begin{array}{cc} c^\dagger_{\mathsf{q}} & c_{\mathsf{q}} \end{array} \right] \left[\begin{array}{cc} \xi_{\mathsf{qq'}} & \Delta_{\mathsf{qq'}} \\ \Delta^\dagger_{\mathsf{qq'}} & -\xi^T_{\mathsf{qq'}} \end{array} \right] \left[\begin{array}{c} c_{\mathsf{q'}} \\ c^\dagger_{\mathsf{q'}} \end{array} \right]$$

presents the Bogoliubov-de Gennes eigenvalue problem

$$\left[\begin{array}{cc} \xi & \Delta \\ \Delta^\dagger & -\xi^T \end{array} \right] = \left[\begin{array}{cc} U & V^* \\ V & U^* \end{array} \right] \left[\begin{array}{cc} E & 0 \\ 0 & -E \end{array} \right] \left[\begin{array}{cc} U & V^* \\ V & U^* \end{array} \right]^\dagger$$

The system thus reduces to free fermion Hamiltonian

$$\mathsf{H} = \sum_{n=1}^{M} E_n(\gamma^\dagger_n \gamma_n - 1/2)$$

with quasiparticle excitations

$$\left[\begin{array}{cc} \gamma^\dagger_1, ..., \gamma^\dagger_M, & \gamma_1, ..., \gamma_M \end{array} \right] = \left[\begin{array}{cc} c^\dagger_1, ..., c^\dagger_M, & c_1, ..., c_M \end{array} \right] \left[\begin{array}{cc} U & V^* \\ V & U^* \end{array} \right]$$

and the eigenstates

$$|\, gs\rangle_{HC} = \prod_{\mathsf{k}} \left(u_{\mathsf{k}} + v_{\mathsf{k}} c^\dagger_{\mathsf{k}} c^\dagger_{-\mathsf{k}} \right) |\, \{Q_{\mathsf{q}}\}, l_x^{(0)}, l_y^{(0)}, \{\emptyset\}\rangle$$

Fermionization on torus: momentum representation

In the momentum representation

$$H = \sum_{k_x, k_y} E_k (\gamma_k^\dagger \gamma_k - \frac{1}{2})$$

The allowed values of momentum k_α in the various homology sectors on torus are given as

$$k_\alpha = \theta_\alpha + 2\pi n_\alpha / N_\alpha$$

$$n_\alpha = 0, 1, \ldots, N_\alpha - 1$$

where the four topological sectors (in vortex free sector)

$$(l_0^{(x)} l_0^{(y)}) = (\pm 1, \pm 1)$$

correspond to

$$\theta_\alpha = \frac{l_0^{(\alpha)} + 1}{2} \frac{\pi}{N_\alpha}$$

The configuration

$$(l_0^{(x)} l_0^{(y)}) = (-1, -1)$$

is fully periodic, permitting the momenta (π, π) exactly.

Non-Abelian phase on torus - vanishing of one BCS state

In the fully symmetric configuration $(l_0^{(x)} l_0^{(y)}) = (-1, -1)$ where momentum π appears exactly,

passing the phase transition to the non-Abelian phase leads has the following consequences:

- $\Delta_{\pi,\pi} = 0$
- $\xi_{\pi,\pi} / E_{\pi,\pi} = -1$

  (the sign flips from +1 at transition, $J_z = J_x + J_y$)

$$\begin{aligned} \xi_{\boldsymbol{k}} &= \varepsilon_{\boldsymbol{k}} - \mu \\ \Delta_{\boldsymbol{k}} &= \alpha_{\boldsymbol{k}} + i\beta_{\boldsymbol{k}} \\ \mu &= -2J_z \\ \varepsilon_{\boldsymbol{k}} &= 2J_x \cos(k_x) + 2J_y \cos(k_y) \\ \alpha_{\boldsymbol{k}} &= 4\kappa(\sin(k_x) - \sin(k_y) - \sin(k_x - k_y)) \\ \beta_{\boldsymbol{k}} &= 2J_x \sin(k_x) + 2J_y \sin(k_y). \end{aligned}$$

$$\begin{aligned} E_{\boldsymbol{k}} &= \sqrt{\xi_{\boldsymbol{k}}^2 + |\Delta_{\boldsymbol{k}}|^2} \\ u_{\boldsymbol{k}} &= \sqrt{1/2(1 + \xi_{\boldsymbol{k}}/E_{\boldsymbol{k}})} \\ v_{\boldsymbol{k}} &= i\sqrt{1/2(1 - \xi_{\boldsymbol{k}}/E_{\boldsymbol{k}})} \end{aligned}$$

implying that

- $u_{\pi,\pi} = 0$
- $v_{\pi,\pi} = i$

This cause **one** of four BCS state on torus

$$|\, gs\rangle_{HC} = \prod_{\mathsf{k}} \left(u_{\mathsf{k}} + v_{\mathsf{k}} c_{\mathsf{k}}^\dagger c_{-\mathsf{k}}^\dagger \right) |\, \{Q_{\mathsf{q}}\}, l_x^{(0)}, l_y^{(0)}, \{\emptyset\}\rangle$$

to **vanish** as $c^+_{\pi,\pi}\, c^+_{-\pi,-\pi} = (c^+_{\pi,\pi})^2 = 0$

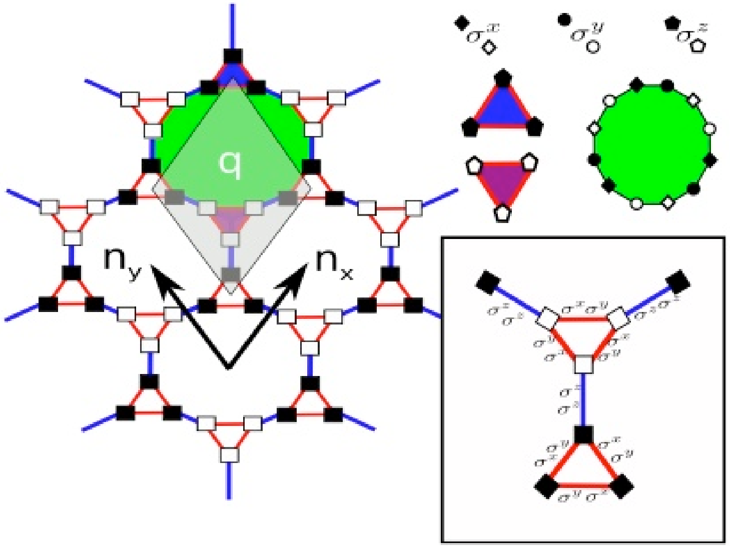
The ground state of the system in the non-Abelian phase on a torus is three-fold degenerate as expected for the Ising theory.

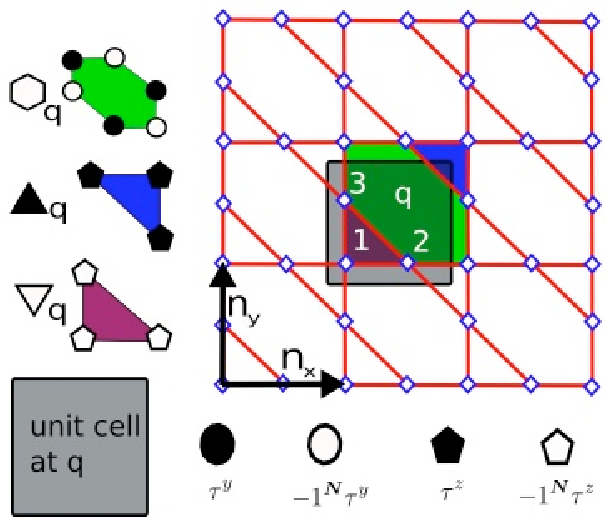
… and beyond

Yao-Kivelson model

$$H \;=\; H_z + H_w + H_b \;=\; -J' \sum_{\square-\blacksquare} \sigma^z \sigma^z - J \sum_{\square-\square} \sigma^y \sigma^x - J \sum_{\blacksquare-\blacksquare} \sigma^x \sigma^y$$

$$|\uparrow_{\blacksquare}\uparrow_{\square}\rangle \;=\; |\Uparrow, 0\rangle, \quad |\downarrow_{\blacksquare}\downarrow_{\square}\rangle = |\Downarrow, 0\rangle$$
$$|\uparrow_{\blacksquare}\downarrow_{\square}\rangle \;=\; |\Uparrow, 1\rangle, \quad |\downarrow_{\blacksquare}\uparrow_{\square}\rangle = |\Downarrow, 1\rangle$$



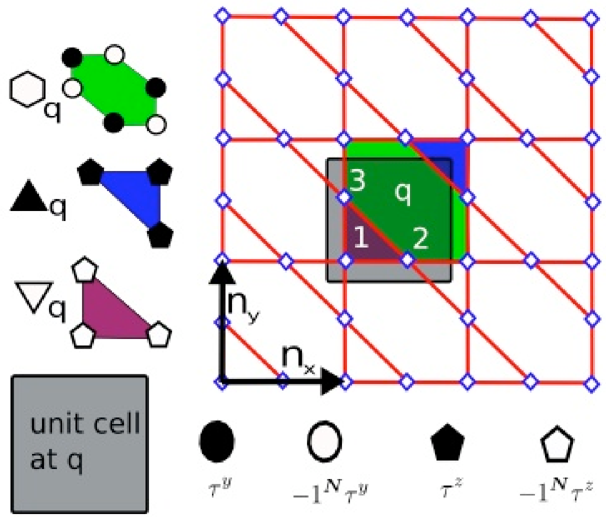


$J' >> J$

$$H_{\text{eff}} = -J^{(6)}_{\text{eff}} \sum_{\boldsymbol{q}} \text{⬡}_{\boldsymbol{q}} - J^{(8)}_{\text{eff}} \sum_{\boldsymbol{q}} (\blacktriangle_{\boldsymbol{q}_i} \text{⬡}_{\boldsymbol{q}_j} \triangledown_{\boldsymbol{q}_k})$$

$$\text{⬡}_{\boldsymbol{q}} = \prod \tau^y_{\boldsymbol{q}}$$
$$\triangledown_{\boldsymbol{q}} = \prod_{\square} \tau^z_{\boldsymbol{q}}$$
$$\blacktriangle_{\boldsymbol{q}} = \prod_{\blacksquare} \tau^z_{\boldsymbol{q}}$$

Yao-Kivelson model



$$\sigma^x_{\boldsymbol{q},\blacksquare} = \tau^x_{\boldsymbol{q}}(b^\dagger_{\boldsymbol{q}} + b_{\boldsymbol{q}}) \quad , \quad \sigma^x_{\boldsymbol{q},\square} = b^\dagger_{\boldsymbol{q}} + b_{\boldsymbol{q}},$$
$$\sigma^y_{\boldsymbol{q},\blacksquare} = \tau^y_{\boldsymbol{q}}(b^\dagger_{\boldsymbol{q}} + b_{\boldsymbol{q}}) \quad , \quad \sigma^y_{\boldsymbol{q},\square} = i\,\tau^z_{\boldsymbol{q}}(b^\dagger_{\boldsymbol{q}} - b_{\boldsymbol{q}}),$$
$$\sigma^z_{\boldsymbol{q},\blacksquare} = \tau^z_{\boldsymbol{q}} \quad , \quad \sigma^z_{\boldsymbol{q},\square} = \tau^z_{\boldsymbol{q}}(I - 2b^\dagger_{\boldsymbol{q}} b_{\boldsymbol{q}})$$

$$H_z = -J' \sum_{\boldsymbol{q},n} (I - 2b^\dagger_{\boldsymbol{q},n} b_{\boldsymbol{q},n})$$

$$H_w = -J \sum_{\boldsymbol{q},n} i\tau^z_{\boldsymbol{q},n}(b^\dagger_{\boldsymbol{q},n} - b_{\boldsymbol{q},n})(b^\dagger_{\boldsymbol{q},n+1} + b_{\boldsymbol{q},n+1})$$

$$H_b = -J \sum_{\boldsymbol{q}} [\tau^x_{\boldsymbol{q},1}(b^\dagger_{\boldsymbol{q},1} + b_{\boldsymbol{q},1})\tau^y_{\boldsymbol{q}\downarrow,3}(b^\dagger_{\boldsymbol{q}\downarrow,3} + b_{\boldsymbol{q}\downarrow,3})$$
$$+ \tau^x_{\boldsymbol{q},3}(b^\dagger_{\boldsymbol{q},3} + b_{\boldsymbol{q},3})\tau^y_{\boldsymbol{q}\nwarrow,2}(b^\dagger_{\boldsymbol{q}\nwarrow,2} + b_{\boldsymbol{q}\nwarrow,2}) + \tau^x_{\boldsymbol{q},2}(b^\dagger_{\boldsymbol{q},2} + b_{\boldsymbol{q},2})\tau^y_{\boldsymbol{q}\rightarrow,1}(b^\dagger_{\boldsymbol{q}\rightarrow,1} + b_{\boldsymbol{q}\rightarrow,1})]$$

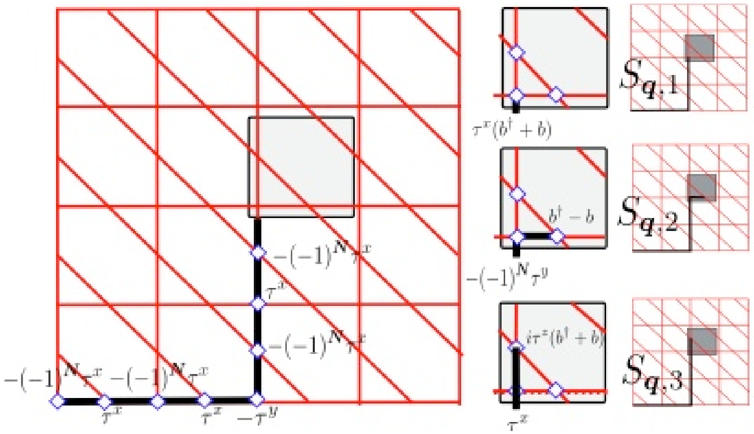
Plaquette operators

$$\hexagon_{\boldsymbol{q}} = (-1)^{b^\dagger_{q,3}b_{q,3} + b^\dagger_{q\rightarrow,1}b_{q\rightarrow,1} + b^\dagger_{q\uparrow,2}b_{q\uparrow,2}}\ \tau^y_{\boldsymbol{q},3}\ \tau^y_{\boldsymbol{q},2}\ \tau^y_{\boldsymbol{q}\rightarrow,1}\tau^y_{\boldsymbol{q}\rightarrow,3}\tau^y_{\boldsymbol{q}\uparrow,2}\tau^y_{\boldsymbol{q}\uparrow,1}$$

$$\blacktriangle_{\boldsymbol{q}} = \tau^z_{\boldsymbol{q},1\nearrow}\tau^z_{\boldsymbol{q}\uparrow,2}\tau^z_{\boldsymbol{q}\rightarrow,3}$$

$$\triangledown_{\boldsymbol{q}} = \prod_{n=1}^{3} (-1)^{b^\dagger_{q,n}b_{q,n}}\tau^z_{\boldsymbol{q},n}$$

Yao-Kivelson model



$$c^\dagger_{\boldsymbol{q},n} = b^\dagger_{\boldsymbol{q},n} S'_{\boldsymbol{q},n}$$

$$c_{\boldsymbol{q},n} = b_{\boldsymbol{q},n} S'_{\boldsymbol{q},n}$$

$$\{c^\dagger_{\boldsymbol{q},m}, c_{\boldsymbol{q}',m}\} = \delta_{\boldsymbol{q}\boldsymbol{q}'}\delta_{n,m}$$

$$\{c^\dagger_{\boldsymbol{q},n}, c^\dagger_{\boldsymbol{q}',m}\} = 0, \{c_{\boldsymbol{q},n}, c_{\boldsymbol{q}',m}\} = 0$$

$$H_z = J' \sum_{\boldsymbol{q},n} (2c^\dagger_{\boldsymbol{q},n} c_{\boldsymbol{q},n} - I)$$

$$\begin{aligned} H_w = J\sum\nolimits_{\boldsymbol{q}} [(c^\dagger_{\boldsymbol{q},1} - c_{\boldsymbol{q},1})(c^\dagger_{\boldsymbol{q},2} + c_{\boldsymbol{q},2}) \\ -i\nabla_{\boldsymbol{q}}((c^\dagger_{\boldsymbol{q},2} + c_{\boldsymbol{q},2})((c^\dagger_{\boldsymbol{q},3} + c_{\boldsymbol{q},3}) \\ +(c^\dagger_{\boldsymbol{q},1} - c_{\boldsymbol{q},1})(c^\dagger_{\boldsymbol{q},3} + c_{\boldsymbol{q},3})]. \end{aligned}$$

$$\begin{aligned} H_b = J\sum\nolimits_{\boldsymbol{q}} [(c^\dagger_{\boldsymbol{q},3} - c_{\boldsymbol{q},3})(c^\dagger_{\boldsymbol{q}\uparrow,1} + c_{\boldsymbol{q}\uparrow,1}) \\ -i\Box_{\boldsymbol{q}\downarrow}\blacktriangle_{\boldsymbol{q}\downarrow}((c^\dagger_{\boldsymbol{q},2} - c_{\boldsymbol{q},2})((c^\dagger_{\boldsymbol{q}\searrow,3} - c_{\boldsymbol{q}\searrow,3}) \\ +\Box_{\boldsymbol{q}\downarrow}(c^\dagger_{\boldsymbol{q},2} - c_{\boldsymbol{q},2})(c^\dagger_{\boldsymbol{q}\rightarrow,1} + c_{\boldsymbol{q}\rightarrow,1})]. \end{aligned}$$

$$c_{\boldsymbol{q},n} = M^{-1/2}\sum c_{\boldsymbol{k},n} e^{i\boldsymbol{k}\cdot\boldsymbol{q}}$$

$$H = \frac{1}{2}\sum_{\boldsymbol{k}n\boldsymbol{k}'m} \begin{bmatrix} c^\dagger_{\boldsymbol{k}n} & c_{\boldsymbol{k}n} \end{bmatrix} \begin{bmatrix} \xi_{\boldsymbol{k}n\boldsymbol{k}'m} & \Delta_{\boldsymbol{k}n\boldsymbol{k}'m} \\ \Delta^\dagger_{\boldsymbol{k}n\boldsymbol{k}'m} & \bar{\xi}_{\boldsymbol{k}n\boldsymbol{k}'m} \end{bmatrix} \begin{bmatrix} c_{\boldsymbol{k}'m} \\ c^\dagger_{\boldsymbol{k}'m} \end{bmatrix}$$

$$\begin{bmatrix} \xi_{\boldsymbol{k}n\boldsymbol{k}'m} & \Delta_{\boldsymbol{k}n\boldsymbol{k}'m} \\ \Delta^\dagger_{n\boldsymbol{k}m\boldsymbol{k}'} & \bar{\xi}_{n\boldsymbol{k}m\boldsymbol{k}'} \end{bmatrix} = \begin{bmatrix} \xi_{nm}\delta_{\boldsymbol{k},\boldsymbol{k}'} & \Delta_{nm}\delta_{\boldsymbol{k},-\boldsymbol{k}'} \\ \Delta^\dagger_{nm}\delta_{\boldsymbol{k},-\boldsymbol{k}'} & \bar{\xi}_{nm}\delta_{\boldsymbol{k},\boldsymbol{k}'} \end{bmatrix}$$

$$\xi_{nm} = \begin{bmatrix} 2J' & J(1-\theta^*_x) & -J(1-\theta^*_y) \\ J(1-\theta_x) & 2J' & iJ(1+\theta_x\theta^*_y) \\ -J(1-\theta_y) & -iJ(1-\theta^*_x\theta_y) & 2J' \end{bmatrix}$$

$$\Delta_{nm} = \begin{bmatrix} 0 & J(1+\theta^*_x) & -J(1+\theta^*_y) \\ -J(1+\theta_x) & 0 & iJ(1-\theta_x\theta^*_y) \\ J(1+\theta_y) & -iJ(1-\theta^*_x\theta_y) & 0 \end{bmatrix}$$

$$|\,\mathrm{gs}\rangle = \prod_{\boldsymbol{k},n} (u_{\boldsymbol{k},n} + v_{\boldsymbol{k},n} a^\dagger_{\boldsymbol{k},n} a^\dagger_{-\boldsymbol{k},n})|\,\{\hexagon\},\{\triangledown\},\{\blacktriangle\},\{\emptyset\}\rangle$$

Yao-Kivelson model

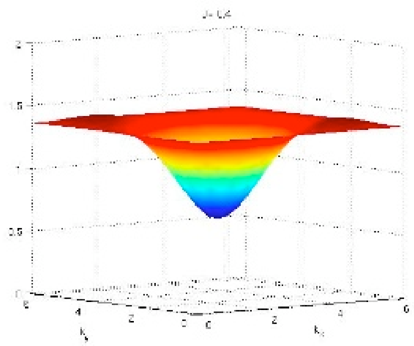
$$H = \frac{1}{2} \sum_{\boldsymbol{k}n\boldsymbol{k}'m} \left[\begin{array}{cc} c^\dagger_{\boldsymbol{k}n} & c_{\boldsymbol{k}n} \end{array} \right] \left[\begin{array}{cc} \xi_{\boldsymbol{k}n\boldsymbol{k}'m} & \Delta_{\boldsymbol{k}n\boldsymbol{k}'m} \\ \Delta^\dagger_{\boldsymbol{k}n\boldsymbol{k}'m} & \bar{\xi}_{\boldsymbol{k}n\boldsymbol{k}'m} \end{array} \right] \left[\begin{array}{c} c_{\boldsymbol{k}'m} \\ c^\dagger_{\boldsymbol{k}'m} \end{array} \right]$$

$$\left[\begin{array}{cc} \xi_{\boldsymbol{k}n\boldsymbol{k}'m} & \Delta_{\boldsymbol{k}n\boldsymbol{k}'m} \\ \Delta^\dagger_{n\boldsymbol{k}m\boldsymbol{k}'} & \bar{\xi}_{n\boldsymbol{k}m\boldsymbol{k}'} \end{array} \right] = \left[\begin{array}{cc} \xi_{nm}\delta_{\boldsymbol{k},\boldsymbol{k}'} & \Delta_{nm}\delta_{\boldsymbol{k},-\boldsymbol{k}'} \\ \Delta^\dagger_{nm}\delta_{\boldsymbol{k},-\boldsymbol{k}'} & \bar{\xi}_{nm}\delta_{\boldsymbol{k},\boldsymbol{k}'} \end{array} \right]$$

$$\xi_{nm} = \left[\begin{array}{ccc} 2J' & J(1-\theta_x^*) & -J(1-\theta_y^*) \\ J(1-\theta_x) & 2J' & iJ(1+\theta_x\theta_y^*) \\ -J(1-\theta_y) & -iJ(1-\theta_x^*\theta_y) & 2J' \end{array} \right] \qquad \Delta_{nm} = \left[\begin{array}{ccc} 0 & J(1+\theta_x^*) & -J(1+\theta_y^*) \\ -J(1+\theta_x) & 0 & iJ(1-\theta_x\theta_y^*) \\ J(1+\theta_y) & -iJ(1-\theta_x^*\theta_y) & 0 \end{array} \right]$$

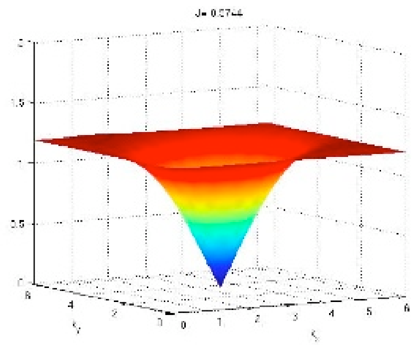
Dispersion relations:

Abelian phase



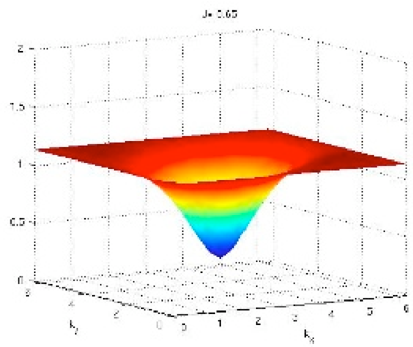
(a)$J' = 1, J = 0.4$

phase transition



(b)$J' = 1, J = 1/\sqrt{(3)}$

Non-abelian phase



(c)$J' = 1, J = 0.65$

Conclusions

Closed expression for the ground state of the Kitaev honeycomb lattice

$$|gs\rangle_{HC} = \prod \left(u_{\mathbf{k}} + v_{\mathbf{k}} c_{\mathbf{k}}^{\dagger} c_{-\mathbf{k}}^{\dagger} \right) |gs\rangle_{TC}$$

Combines two powerful wavefunction descriptions:

- BCS product
- stabilizer formalism

Connection with Hartree-Fock-Bogoliubov theory

Shows relations between the $\mathcal{D}(Z_2)$ abelian phase and the Ising non-Abelian phase

Ground state degeneracy of torus and its change on the phase trasition to the non-Abelian phase

Arbitrary vortex configuration on torus (e.g. vortex interactions and energies in large systems)

Generalization to Yao-Kivelson type models

References:
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