Kitaev honeycomb lattice model: from A to B and beyond

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Outline

Toric code and Kitaev honeycomb lattice model

• introduction
• relation between both models
• vorticity
• loop symmetries
• Abelian phase: summary of results

Exact solution of the Kitaev honeycomb model

• map onto spin-hardcore boson system
• Jordan-Wigner fermionization
• adding magnetic field
• ground state as BCS state with explicit vacuum

Further developments

• ground states on torus
• ground state degeneracy in Abelian and non-Abelian phase
From A ...

toric code

and

Kitaev honeycomb lattice model
Toric code

- spin 1/2 particles on the edges of a square lattice (green)

\[
H_{TC} = -J_{\text{eff}} \left( \sum_{\text{star}} Q_s + \sum_{\text{plaquettes}} Q_p \right)
\]


Unitarily equivalent toric code

- spin 1/2 particles on the vertices of a square lattice (blue)
- connects naturally with the Kitaev honeycomb model

\[
H_{TC} = -J_{\text{eff}} \sum_{q} Q_q
\]

\[
Q_q = \tau_{\text{left}(q)} \tau_{\text{right}(q)} \tau_{\text{up}(q)} \tau_{\text{up}(q)}
\]

Pauli matrices
Toric code

**Hamiltonian**

\[ H_{TC} = -J_{\text{eff}} \sum_p Q_p \]

\[ Q_p = \tau_p^x \tau_{p+n_x}^y \tau_{p+n_y}^y \tau_{p+n_x+n_y}^z \]

**“Symmetries”**

\[ [H_{TC}, Q_p] = 0 \]

\[ [Q_p, Q_q] = 0 \]

**Eigensvalues**

The operators \( Q_p \) have eigenvalues \( Q_p = \pm 1 \); for all \( p \) we have \( \{Q_p\} \)

\[ |\{Q_p\}\rangle \text{ is characterized completely by the eigenvalues } Q_p: Q_p |\{Q_p\}\rangle = Q_p |\{Q_p\}\rangle \quad \forall p \]

**Ground state is stabilized by** \( Q_p \) **for all** \( p \)

\[ Q_p |\{Q_p\}\rangle = |\{Q_p\}\rangle \]

\[ |\{Q_p\}\rangle_{TC} = |\{Q_p = 1\}\rangle \text{ for all plaquettes} \]

On **torus**, we have \( \prod Q_p = 1 \), and two additional homologically nontrivial symmetries

\[ |\{Q_p\}; I_x, I_y\rangle_{TC} \]

The energy does not depend on the eigenvalues of the homologically nontrivial symmetries; this implies **four-fold ground state degeneracy**.
Quasiparticles

Toric code quasiparticle excitations, \( Q_p = -1 \), are
- "magnetic" (living on blue plaquettes) or "electric" (white plaquettes),
- are created in pairs by acting on the ground state with Pauli operators.

Operator \( C_{L,m} \) to move a single “magnetic” excitation in
a contractible loop \( L \) is the product of all “electric” plaquette operators
enclosed by the loop (and vice versa).

If the initial state \( |\{Q_p\}\rangle \) contains an “electric” excitation then moving a magnetic excitation around it
returns the initial state with the phase changed by -1 implying that:
- “magnetic” and “electric” particles are relative semions
- “e-m” composite is a fermion
- “e-m” fermion behaves as semion when braided with an “e” or “m” particle
Kitaev honeycomb lattice model

\[ H_0 = J_x \sum_{x\text{-link}} \sigma^x_i \sigma^x_j + J_y \sum_{y\text{-link}} \sigma^y_i \sigma^y_j + J_z \sum_{z\text{-link}} \sigma^z_i \sigma^z_j \]

\[ = \sum_\alpha J_\alpha \sum_{i,j} \sigma^\alpha_i \sigma^\alpha_j = \sum_\alpha J_\alpha \sum_{i,j} K^\alpha_{ij} \]

\[ \alpha \text{-link: } x\text{-link} \quad \beta \text{-link: } y\text{-link} \quad \gamma \text{-link: } z\text{-link} \]

Phase diagram:

- phase A - can be mapped perturbatively onto the toric code;
- phase B - gapless.

Adding magnetic field:

\[ H = H_0 + H_1 = H_0 + \sum_\alpha \sum_{x,y,z} B_\alpha \sigma_{\alpha,i} \]

- parity and time-reversal symmetry are broken
- phase B acquires a gap and becomes non-abelian topological phase of Ising type

The leading P and T breaking term in perturbation theory occurs at the third order:

\[ H_1 = -\kappa \sum_q \sum_{i=1}^6 P(q)^{(l)} \]

\[ \sum_{l=1}^6 P(q)^{(l)} = \sigma^y_1 \sigma^y_6 \sigma^y_5 + \sigma^y_2 \sigma^y_3 \sigma^y_4 + \sigma^y_1 \sigma^y_2 \sigma^y_3 + \sigma^y_4 \sigma^y_5 \sigma^y_6 + \sigma^y_3 \sigma^y_4 \sigma^y_5 + \sigma^y_2 \sigma^y_3 \sigma^y_4 \]

Mapping abelian phase onto toric code

\[ J_z \gg J_y, J_x \]

\[ H_{\Omega} = -J_z \sum_{z \text{-links}} \sigma_j^z \sigma_k^z, \quad \text{“dimers”} \]

\[ V = -J_x \sum_{x \text{-links}} \sigma_j^x \sigma_k^x - J_y \sum_{y \text{-links}} \sigma_j^y \sigma_k^y \]

Effective spins

- are formed by ferromagnetic ground states of \(-J_z \sigma_j^z \sigma_k^z\)

\[ | \uparrow\rangle_{\text{eff}} = | \uparrow\uparrow\rangle \quad | \downarrow\rangle_{\text{eff}} = | \downarrow\downarrow\rangle \]

Mapping abelian phase onto toric code

Effective Hamiltonian (no magnetic field)
first non-constant term of perturbation theory
occurs on the 4\textsuperscript{th} order

\[
H_{\text{eff}} = -\frac{J_x^2 J_y^2}{16|J_z|^3} \sum_p Q_p,
\]

\[
Q_p = \sigma_{\text{left}(p)}^y \sigma_{\text{right}(p)}^y \sigma_{\text{up}(p)}^z \sigma_{\text{down}(p)}^z
\]
defined on the square lattice with effective spins on the vertices

Toric code quasiparticles and vortices of the honeycomb lattice model
Vortex operators in the honeycomb model

\[ W_p = \sigma^x_1 \sigma^y_2 \sigma^z_3 \sigma^x_4 \sigma^y_5 \sigma^z_6 = \]

\[ = K^z_{1,2} K^x_{2,3} K^y_{3,4} K^z_{4,5} K^x_{5,6} K^y_{6,1} \]

\[ [H_0, W_p] = 0 \quad (K^\alpha_{k,k+1})^2 = 1 \]

\[ H_0 \left| n \right> = E_n \left| n \right> \]

\[ w_p = \left< n \left| W_p \right| n \right> = +1 \]

\[ w_p = \left< n \left| W_p \right| n \right> = -1 \]
Vortex sectors

Each energy eigenstate $|n>$ is characterized by some vortex configuration

$$\{w_p = \langle n|W_p|n> = \pm 1\} \text{ for all plaquettes } p$$

also the vortices are always excited in pairs, i.e. even-vortex configurations are relevant on closed surfaces or infinite plane,

the Hilbert space splits into vortex sectors, i.e. subspaces of the system with a particular configuration of vortices

$$L = \bigoplus_{w_1,\ldots,w_m} L_{w_1,\ldots,w_m}$$

vortex free sector  examples from two-vortex sectors  full vortex sector
Products of vortex operators generate closed loops

\[ K_{i,j}^{\alpha(1)} K_{j,k}^{\alpha(2)} ... K_{p,q}^{\alpha(M-1)} K_{q,i}^{\alpha(M)} \]

On a torus, this gives the condition

\[ \prod_p W_p = I \]
Loop symmetries on torus

For a system of N spins on a torus (i.e. a system with N/2 plaquettes), \( \prod_p W_p = I \) implies that there are N/2-1 independent vortex quantum numbers \( \{w_1, \ldots, w_{N/2-1}\} \).

Loops on the torus

\[ K_{i,j}^{\alpha(1)} K_{j,k}^{\alpha(2)} \ldots K_{p,q}^{\alpha(M-1)} K_{q,i}^{\alpha(M)} \]

- all homologically trivial loops are generated by plaquette operators

- in addition, two distinct homologically nontrivial loops are needed to generate the full loop symmetry group (the third nontrivial loop is a product of these two).

The full loop symmetry of the torus is the abelian group with N/2+1 independent generators of the order 2 (loop\(^2=I\)), i.e. \( \mathbb{Z}_{2^{N/2+1}} \).

All loop symmetries can be written as

\[ C_{(k,l)} = G_k F_l(W_1, W_2, \ldots, W_{N-1}) \]

where k is from \( \{0,1,2,3\} \) and \( G_0 = I \), and \( G_1, G_2, G_3 \) are arbitrarily chosen symmetries from the three nontrivial homology classes, and \( F_l \), with \( l \) from \( \{1, \ldots, 2^{N/2-1}\} \), run through all monomials in the \( W_p \) operators.
Results on the Abelian phase

1) The symmetry structure of the system is manifested in the effective Hamiltonian obtained using the Brillouin-Wigner perturbation theory. The longer loops occur at the higher order of the perturbation expansion:

\[ H_{\text{eff}} = \sum_{i=0}^{3} \sum_{j=1}^{2^{N/2-2}} c_{i,j} G_i(z, y) F_j(Q_1, Q_2, \ldots, Q_{N/2-2}) \]


2) Fermions of the Abelian phase can be moved efficiently using the K strings from the symmetries. A. T. Bolukbasi, et al., in preparation.

3) The symmetry structure of the effective Hamiltonian allows to classify all finite size effect, intrisic to the system of sizes <36 spins: for example N=16 spins.

\[ H^{(4)} = -\frac{J_x^2 J_y^2}{16|J_z|} \sum_{n=1}^{8} (Q_n + R_n - 5A_n) \]

\[ -\frac{J_x^2 J_y^2}{16|J_z|} \sum_{n=1}^{4} (Z_n + 5Y_n) \]

\[ -\frac{5}{16|J_z|} \left( J_x^2 \sum_{n=1}^{2} X_n + J_y^4 \sum_{n=3}^{4} X_n \right) \]

... to B ... 

exact solution
of the Kitaev honeycomb lattice model
Effective spins and hardcore bosons

New perspective:

**spin-hardcore boson representation**

\[
\begin{align*}
|\uparrow\uparrow\rangle &= |\uparrow, 0\rangle, & |\downarrow\downarrow\rangle &= |\downarrow, 0\rangle \\
|\uparrow\downarrow\rangle &= |\uparrow, 1\rangle, & |\downarrow\uparrow\rangle &= |\downarrow, 1\rangle
\end{align*}
\]

Schmidt, Dusuel, and Vidal (2008)

**Pauli operators:**

\[
\begin{align*}
\sigma_x^{q,\square} &= \tau_x^{q} (b_q^+ + b_q), & \sigma_y^{q,\square} &= \tau_y^{q} (b_q^+ - b_q), \\
\sigma_y^{q,\circ} &= i \tau_y^{q} (b_q^+ - b_q), & \sigma_z^{q,\square} &= \tau_z^{q} (I - 2b_q^+ b_q),
\end{align*}
\]

**Vortex and plaquette operators:**

\[
W_q = (I - 2N_q)(I - 2N_{q+n_y})Q_q
\]

\[
N_q = b_q^+ b_q, \quad Q_q = \tau_x^{q} \tau_y^{q+n_x} \tau_y^{q+n_y} \tau_z^{q+n}
\]

In the $A_z$-phase, $J_z >> J_x, J_y$, the bosons are energetically suppressed, thus at low energy

\[
|\{W_q\}, 0\rangle = |\{Q_q\}\rangle
\]

the low-energy perturbative Hamiltonian equals to toric code $H_{TC} = -J_{eff} \sum_q Q_q \otimes I$

This allows to write down an orthonormal basis of the full system in terms of the **toric code stabilizers:**

\[
|\{W_q\}, \{q\}\rangle
\]

where $\{W_q\}$ lists all honeycomb plaquette operators and $\{q\}$ lists the position vectors of any occupied bosonic modes. On a torus, the homologically nontrivial symmetries must be added

\[
|\{W_q\}, l_0^{(x)}, l_0^{(y)}, \{q\}\rangle
\]

Bosonic and effective spin Hamiltonian can be written in terms of fermions and vortices by applying a Jordan-Wigner transformation.

\[ H = -J_x \sum_q \left( b_q^\dagger b_q \right) \tau_q^x \left( b_{q+n_x}^\dagger + b_{q+n_x} \right) \]
\[ + J_y \sum_q \left( i \tau_q^y \left( b_q^\dagger - b_q \right) \tau_q^y \left( b_{q+n_y}^\dagger + b_{q+n_y} \right) \right) \]
\[ + J_z \sum_q \left( I - 2b_q^\dagger b_q \right). \]

Importantly, presence of a fermion indicates an anti-ferromagnetic configuration of z-link.
Magnetic field

- breaks parity and time-reversal symmetry
- opens a gap in phase B and turns it into non-abelian topological phase of Ising type

\[ H_1 = -\kappa \sum_{q} \sum_{l=1}^{6} P(q)^{(l)} \]

\[ \sum_{l=1}^{6} P(q)^{(l)} = \sigma_1^z \sigma_5^x + \sigma_2^z \sigma_3^x \sigma_4^y + \sigma_1^x \sigma_2^y \sigma_3^z + \sigma_4^x \sigma_5^y \sigma_6^z + \sigma_3^x \sigma_4^y \sigma_5^z + \sigma_2^y \sigma_1^y \sigma_6^x \]

- \( H_1 \) commutes with the plaquette operators, so stabilizer formalism can still be used
Vortex-free sector

Transformation to the momentum representation

\[ c_q = M^{-1/2} \sum_k c_k e^{i k \cdot q} \]

\[ H = \sum_k \left[ \xi_k c_k^\dagger c_k + \frac{1}{2} (\Delta c_k^\dagger c_{-k}^\dagger + \Delta^* c_{-k}^\dagger c_k) \right] - MJ_z \]

The effect of the magnetic field is contained fully in the \( \alpha_k \) term.

The Hamiltonian can be diagonalized by Bogoliubov transformation:

\[ \gamma_k = u_k c_k - v_k c_{-k}^\dagger \]

resulting in the BCS Hamiltonian

\[ H = \sum_{n=1}^M E_n (\gamma_n^\dagger \gamma_n - 1/2) \]

\[ E_k = \sqrt{\xi_k^2 + |\Delta_k|^2} \]

\[ u_k = \sqrt{1/2(1 + \xi_k/E_k)} \]

\[ v_k = i\sqrt{1/2(1 - \xi_k/E_k)} \]

the ground state is BCS state with the vacuum given here explicitly in terms of toric code stabilizers

\[ |gs\rangle_{HC} = \prod (u_k + v_k c_k^\dagger c_{-k}^\dagger) | \{Q_q\}, \{0\} \rangle \]

\[ |\{1,1,\ldots,1\},\{0\}\rangle \]
Other vortex sectors on torus

To address an arbitrary vortex configuration we rewrite the general Hamiltonian

\[
H = \frac{1}{2} \sum_{qq'} \left[ \begin{array}{cc} c_q^\dagger & c_q \\ \xi_{qq'} & \Delta_{qq'} \end{array} \right] \left[ \begin{array}{c} c_{q'} \\ \Delta_{qq'}^T \end{array} \right]
\]

To specify a particular vortex sector, the operators \(X_q\) and \(Y_q\) are replaced by their eigenvalues in that sector; for example for \(H_0\) we obtain

\[
\xi_{qq'} = 2J_z \delta_{q, q'} + J_x X_q (\delta_{q, q'} - n_x + \delta_{q - n_x, q'}) \\
+ J_y Y_q (\delta_{q, q'} - n_y + \delta_{q - n_y, q'}) \\
\Delta_{qq'} = J_x X_q (\delta_{q, q'} - n_x - \delta_{q - n_x, q'}) \\
+ J_y Y_q (\delta_{q, q'} - n_y - \delta_{q - n_y, q'}).
\]

On torus, these terms include periodicity, i.e. the terms connecting the sites \((0, q_y)\) and \((N_x - 1, q_y)\), and \((q_x, 0)\) and \((q_x, N_y - 1)\), and thus the homologically nontrivial symmetries

\[
X_{q_x, q_y} = \prod_{q_y'=0}^{q_y-1} W_{q_x, q_y'} (q_y \neq 0 \text{ and } q_x \neq N_x - 1) \quad Y_{q_x, q_y} = 1 (q_y \neq N_y - 1) \\
X_{q_x, q_y} = 1 (q_y = 0 \text{ and } q_x \neq N_x - 1) \quad Y_{q_x, q_y} = -l_q^{(y)} (q_y = N_y - 1) \\
X_{q_x, q_y} = -l_0^{(x)} \prod_{q_y'=0}^{q_y-1} W_{q_x, q_y'} (q_y \neq 0 \text{ and } q_x = N_x - 1) \quad Y_{q_x, q_y} = -l_q^{(y)} \prod_{q_y'=0}^{N_y-1} \prod_{q_x'=0}^{q_x-1} W_{q_x', q_y} \\
X_{q_y, q_x} = -l_0^{(x)} (q_y = 0 \text{ and } q_x = N_x - 1) \\
X_{q_y, q_x} = l_0^{(x)} l_0^{(y)} (q_x = N_x - 1) \quad X_{q_x, q_y} = l_q^{(y)} \prod_{q_y'=0}^{q_y-1} W_{q_x, q_y'} (q_x \neq N_x - 1)
\]

In order to include the magnetic field \(H_1\) we have to add also

\[
X_{q_x, q_y + 1} = -l_q^{(y)} (q_x \neq N_x - 1) \\
X_{q_x, q_y + 1} = l_0^{(x)} l_0^{(y)} (q_x = N_x - 1), \\
X_{q_x, q_y} = l_q^{(y)} \prod_{q_y'=0}^{q_y-1} W_{q_x, q_y'} (q_x \neq N_x - 1) \\
X_{q_x, q_y} = l_0^{(x)} l_0^{(y)} W_{q_x, q_y} (q_x = N_x - 1).
\]
Role of symmetries

On a torus, the system has $N/2+1$ loop symmetry generators from which all other loop symmetries can be obtained. We can specify a particular sector of the Hamiltonian by specifying the eigenvalues of the $N/2-1$ plaquette symmetries and 2 homologically nontrivial symmetries.

\[ Y_{(0,N_y-1)} = -l_0^{(y)}, \quad X_{(N_x-1,0)} = -l_0^{(x)} \]

\[ X_{(2,3)} = \prod_{q_y=0}^{2} W_{(2,q_y)} \]

\[ X_{(N_x-1,2)} = -l_0^{(x)} \prod_{q_y=0}^{1} W_{(N_x-1,q_y)} \]

\[ Y_{(2,N_y-1)} = -l_0^{(y)} \prod_{q_x=0}^{1} \prod_{q_y=0}^{N_y-1} W_{(q_x,q_y)} \]
Fermionization on torus

The general Hamiltonian for an arbitrary vortex configuration

\[ H = \frac{1}{2} \sum_{qq'} \begin{bmatrix} c_{q}^\dagger & c_{q} \end{bmatrix} \begin{bmatrix} \xi_{qq'} & \Delta_{qq'} \\ \Delta_{qq'}^\dagger & -\xi_{qq'}^* \end{bmatrix} \begin{bmatrix} c_{q'} \\ c_{q'}^\dagger \end{bmatrix} \]

presents the Bogoliubov-de Gennes eigenvalue problem

\[ \begin{bmatrix} \xi & \Delta \\ \Delta^\dagger & -\xi^* \end{bmatrix} = \begin{bmatrix} U & V^* \\ V & U^* \end{bmatrix} \begin{bmatrix} E & \Theta \\ \Theta & -E \end{bmatrix} \begin{bmatrix} U & V^* \\ V & U^* \end{bmatrix}^\dagger \]

The system thus reduces to free fermion Hamiltonian

\[ H = \sum_{n=1}^{M} E_n (\gamma_n^\dagger \gamma_n - 1/2) \]

with quasiparticle excitations

\[ \begin{bmatrix} \gamma_1^\dagger, ..., \gamma_M^\dagger, \gamma_1, ..., \gamma_M \end{bmatrix} = \begin{bmatrix} c_{1}^\dagger, ..., c_{M}^\dagger, c_{1}, ..., c_{M} \end{bmatrix} \begin{bmatrix} U & V^* \\ V & U^* \end{bmatrix} \]

and the eigenstates

\[ |gs\rangle_{HC} = \prod_{k} \left( u_k + v_k c_{k}^\dagger c_{-k} \right) |Q_q\rangle, l_x^{(0)}, l_y^{(0)}, \{\theta\} \]
Fermionization on torus: momentum representation

In the momentum representation

\[ H = \sum_{k_x, k_y} E_k (\gamma_k^+ \gamma_k - \frac{1}{2}) \]

The allowed values of momentum \( k_\alpha \) in the various homology sectors on torus are given as

\[ k_\alpha = \theta_\alpha + 2\pi n_\alpha / N_\alpha \]

\[ n_\alpha = 0, 1, \ldots, N_\alpha - 1 \]

where the four topological sectors (in vortex free sector)

\( (l_0^{(x)} l_0^{(y)}) = (\pm 1, \pm 1) \)

correspond to

\[ \theta_\alpha = \frac{l_0^{(\alpha)} + 1}{2} \frac{\pi}{N_\alpha} \]

The configuration

\( (l_0^{(x)} l_0^{(y)}) = (-1, -1) \)

is fully periodic, permitting the momenta \((\pi, \pi)\) exactly.
Non-Abelian phase on torus - vanishing of one BCS state

In the fully symmetric configuration \((l_0^{(x)}l_0^{(y)}) = (-1, -1)\) where momentum \(\pi\) appears exactly, passing the phase transition to the non-Abelian phase leads has the following consequences:

- \(\Delta_{\pi,\pi} = 0\)
- \(\xi_{\pi,\pi} / E_{\pi,\pi} = -1\)

(the sign flips from +1 at transition, \(J_z = J_x + J_y\))

implying that

- \(u_{\pi,\pi} = 0\)
- \(v_{\pi,\pi} = i\)

This cause one of four BCS state on torus

\[
|gs\rangle_{HC} = \prod_k \left( u_k + v_k c_k^+ c_{-k}^+ \right) |\{Q_q\}, l_x^{(0)}, l_y^{(0)}, \{\theta\}\rangle
\]

to vanish as

\[
c_{\pi,\pi}^+ c_{-\pi,-\pi}^+ = (c_{\pi,\pi}^+)^2 = 0
\]

The ground state of the system in the non-Abelian phase on a torus is three-fold degenerate as expected for the Ising theory.
... and beyond
Yao-Kivelson model

\[ H = H_x + H_y + H_b = -J' \sum_{\square-\bullet} \sigma^z \sigma^z - J \sum_{\square-\square} \sigma^y \sigma^x - J \sum_{\bullet-\bullet} \sigma^x \sigma^y \]

\[ |\uparrow\downarrow\rangle = |\uparrow,0\rangle, \quad |\downarrow\uparrow\rangle = |\downarrow,0\rangle \]

\[ |\uparrow\downarrow\rangle = |\uparrow,1\rangle, \quad |\downarrow\uparrow\rangle = |\downarrow,1\rangle \]

\[ J' \gg J \]

\[ H_{\text{eff}} = -J_{\text{eff}}^{(6)} \sum_q \circ_q - J_{\text{eff}}^{(8)} \sum_q (\triangle_q \circ_q \nabla_q) \]

\[ \circ_q = \prod_{\tau_q} \tau_q^y \]
\[ \nabla_q = \prod_{\square} \tau_q^z \]
\[ \triangle_q = \prod_{\bullet} \tau_q^z \]
Yao-Kivelson model

$$H_z = -J' \sum_{\alpha, n} (I - 2b_{q, n}^{\dagger} b_{q,n})$$

$$H_w = -J \sum_{q, n} i\tau_{q,n}(b_{q,n}^{\dagger} - b_{q,n})(b_{q,n+1}^{\dagger} + b_{q,n+1})$$

$$H_b = -J \sum_q \left[ \tau_{q,1}^{x}(b_{q,1}^{\dagger} + b_{q,1})\tau_{q,1,3}^{y}(b_{q,1,3}^{\dagger} + b_{q,1,3}) + \tau_{q,3}^{x}(b_{q,3}^{\dagger} + b_{q,3})\tau_{q,3}^{y}(b_{q,3,2}^{\dagger} + b_{q,3,2}) + \tau_{q,2}^{x}(b_{q,2}^{\dagger} + b_{q,2})\tau_{q,2}^{y}(b_{q,2,3}^{\dagger} + b_{q,2,3}) \right]$$

Plaquette operators

$$\bigcirc_q = (-1)^{b_{q,3}^{\dagger} b_{q,3} + b_{q,2}^{\dagger} b_{q,2} + b_{q,1}^{\dagger} b_{q,1}}$$

$$\triangle_q = \sum_{q} \tau_{q,1}^{x} \tau_{q,1,2}^{y} \tau_{q,2}^{z}$$

$$\nabla_q = \prod_{n=1}^{3} (-1)^{b_{q,n}^{\dagger} b_{q,n}}$$
Yao-Kivelson model

\[ H_z = J' \sum_{q,n} (2c^\dagger_{q,n}c_{q,n} - I) \]

\[ H_w = J \sum_q [(c^\dagger_{q,1} - c_{q,1})(c^\dagger_{q,2} + c_{q,2}) \]
\[ -i \nabla_q ((c^\dagger_{q,1} + c_{q,1})(c^\dagger_{q,3} + c_{q,3}) \]
\[ + (c^\dagger_{q,1} - c_{q,1})(c^\dagger_{q,3} + c_{q,3})]. \]

\[ H_b = J' \sum_q [(c^\dagger_{q,3} - c_{q,3})(c^\dagger_{q,1} + c_{q,1}) \]
\[ -i \square_q (c^\dagger_{q,1} - c_{q,1})(c^\dagger_{q,3} - c_{q,3}) \]
\[ + \square_q (c^\dagger_{q,2} - c_{q,2})(c^\dagger_{q,1} + c_{q,1})]. \]

\[ c^\dagger_{q,n} = b^\dagger_{q,n} S_{q,n}' \]
\[ c_{q,n} = b_{q,n} S_{q,n}' \]
\[ \{c^\dagger_{q,m}, c^\dagger_{q',m}\} = \delta_{qq'} \delta_{n,m} \]
\[ \{c^\dagger_{q,n}, c^\dagger_{q',m}\} = 0, \{c_{q,n}, c_{q',m}\} = 0 \]

\[ c_{q,n} = M^{-1/2} \sum_{k,n} c_{k,n} e^{ik \cdot q} \]

\[ H = \frac{1}{2} \sum_{knk'm} \left[ \begin{array}{cccc} c^\dagger_k c_n & c^\dagger_k c_{k'n} & \xi_{knk'm} & \Delta_{knk'm} \\ c_k c_n & c_k c_{k'n} & \Delta_{knk'm} & \xi_{knk'm} \end{array} \right] \left[ \begin{array}{c} c_{k'n} \\ c_{k'n} \end{array} \right] \]

\[ \xi_{nm} = \left[ \begin{array}{cccc} 2J' & J(1 - \theta^*_x) & -J(1 - \theta^*_y) \\ J(1 - \theta_x) & 2J' & iJ(1 + \theta_x \theta^*_y) \\ -J(1 - \theta_y) & -iJ(1 + \theta_x \theta^*_y) & 2J' \end{array} \right] \]

\[ \Delta_{nm} = \left[ \begin{array}{cccc} 0 & J(1 + \theta^*_x) & -J(1 + \theta^*_y) \\ J(1 + \theta_x) & 0 & iJ(1 - \theta_x \theta^*_y) \\ -J(1 + \theta_y) & -iJ(1 - \theta_x \theta^*_y) & 0 \end{array} \right] \]

\[ |gs\rangle = \prod_{k,n} (u_{k,n} + v_{k,n} a^\dagger_{k,n} a^\dagger_{-k,n}) \{\circ\}, \{\triangledown\}, \{\blacktriangledown\}, \{\emptyset\} \]
Yao-Kivelson model

\[
H = \frac{1}{2} \sum_{knk'm} \begin{bmatrix} c_{kn}^\dagger & c_{kn} \end{bmatrix} \begin{bmatrix} \xi_{knk'm} & \Delta_{knk'm} \\ \Delta_{knk'm}^\dagger & \xi_{knk'm} \end{bmatrix} \begin{bmatrix} c_{k'm} \\ c_{k'm}^\dagger \end{bmatrix}
\]

\[
\begin{bmatrix} \xi_{knk'm} & \Delta_{knk'm} \\ \Delta_{knk'm}^\dagger & \xi_{knk'm} \end{bmatrix} = \begin{bmatrix} \xi_{nm} \delta_{k,k'} & \Delta_{nm} \delta_{k,-k'} \\ \Delta_{nm}^\dagger \delta_{k,-k'} & \xi_{nm} \delta_{k,k'} \end{bmatrix}
\]

\[
\xi_{nm} = \begin{bmatrix} 2J' & J(1-\theta_x^*) & -J(1-\theta_y^*) \\ J(1-\theta_x) & 2J' & iJ(1+\theta_x \theta_y^*) \\ -J(1-\theta_y) & -iJ(1-\theta_x \theta_y) & 2J' \end{bmatrix}
\]

\[
\Delta_{nm} = \begin{bmatrix} 0 & J(1+\theta_x^*) & -J(1+\theta_y^*) \\ -J(1+\theta_x) & 0 & iJ(1-\theta_x \theta_y^*) \\ J(1+\theta_y) & -iJ(1-\theta_x \theta_y) & 0 \end{bmatrix}
\]

Dispersion relations:

Abelian phase

phase transition

Non-abelian phase
Conclusions

Closed expression for the ground state of the Kitaev honeycomb lattice

$$|g_s\rangle_{HC} = \prod (u_k + v_k c_k^\dagger c_{-k}^\dagger) |g_s\rangle_{TC}$$

Combines two powerful wavefunction descriptions:
- BCS product
- stabilizer formalism

Connection with Hartree-Fock-Bogoliubov theory

Shows relations between the $\mathcal{D}(Z_2)$ abelian phase and the Ising non-Abelian phase

Ground state degeneracy of torus and its change on the phase transition to the non-Abelian phase

Arbitrary vortex configuration on torus (e.g. vortex interactions and energies in large systems)

Generalization to Yao-Kivelson type models

References: