Skyrmions in the Moore-Read state at $\nu = 5/2$

Gunnar Möller, **Arkadiusz Wójs**, **Nigel R. Cooper**
Cavendish Laboratory, University of Cambridge

Steven H. Simon
Peierls Centre for Theoretical Physics, Oxford University

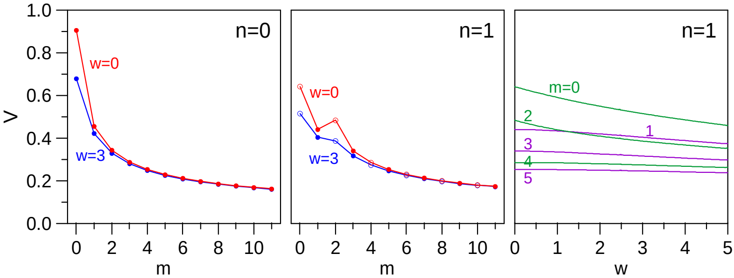
Maynooth Workshop
Quantum Information & Condensed Matter Physics

September 17, 2009

Motivation

Why revisit the role of spin at $\nu = 5/2$?

- Finite width known to be important at $\nu = 5/2$, however, was not considered in previous work.
- Pseudopotentials in finite width $w > 0$ ease reversal of spins:



[m: relative angular momentum; w: sample width]

Overview

Introduction & Motivation

- Spin polarization: status of experiment and theory

Energetics for states at $\nu = 5/2$ on the sphere

- Search for unpolarized groundstates with S=0
- Characterization of the series $N_\phi = 2N - 2$ & $N_\phi = 2N - 4$

Trial wavefunctions for skyrmion states

- review: skyrmion wavefunctions at $\nu = 1$
- generalization to the Moore-Read / weakly paired states

Discussion of partial spin polarization

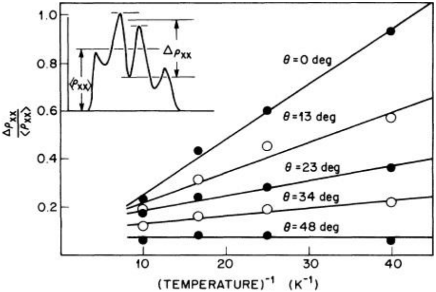
- Skyrmions vs localized quasiparticles

- Conclusions

Introduction

Spin polarization of $\nu = 5/2$ in the early days

- Sensitivity of 5/2 state to tilted field at first suspected to result from partial spin polarization
- Haldane-Rezayi introduce spin singlet (HR) state



[Eisenstein *et al.*, PRL 61, 997 (1988)]

- numerics convincingly support spin-polarized groundstate wavefunction Morf (1998), and
- explain sensitivity to tilted field by proximity to phase transition Haldane, Rezayi (2000)

Introduction

Spin polarization of the even denominator quantum Hall state at $\nu = 5/2$

experimental status

- direct probes of spin cannot establish polarization [current work: Gervais, Pinczuk]
- quasiparticle charge $e/4$, consistent with full spin polarization, but inconclusive on its own

theoretical results

- Numerous theory papers and numerical works support a spin polarized groundstate, adiabatically connected to the Moore-Read state
- estimate of gap significantly larger than experimental value

- What about spinful excitations?

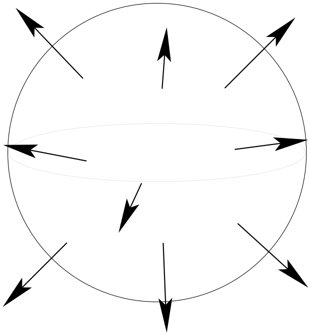
[many papers...]

Numerical studies on the sphere

Our tool: exact diagonalization on the sphere

- Convenient geometry without boundaries
- Shift σ relating integer number of flux N_ϕ and number of particles N naturally separates Hilbert-spaces of competing states

$$N_\phi = \nu^{-1} N - \sigma$$



To study states with given spin, diagonalize Hamiltonian

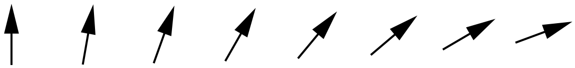
$$\mathcal{H} = \mathcal{H}_{\text{int}} + \alpha \hat{\mathbf{S}}^2$$

in subspace of $S_z = S_{\text{target}}$.

Are skyrmions possible? – spin-wave theory

- At integer QH states and in the LLL, skyrmions are lowest spinful excitations.

$\Rightarrow$ Simple energetic estimate from spin-wave dispersion



a (piece of) spin-wave

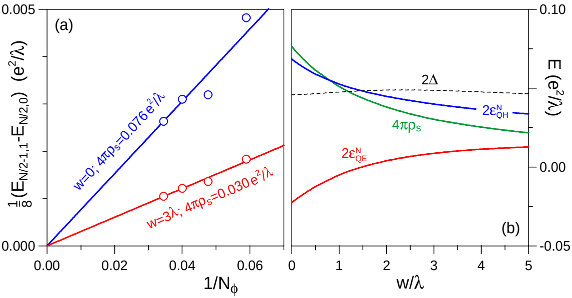
Dispersion of spinwaves is quadratic, involving spin-stiffness ρ_s:

$$E_L = \frac{8\pi}{N}\rho_s L(L+1) \qquad \text{(on the sphere)}$$

- Use to estimate skyrmion energy $E_{\text{sk}} = 4\pi\rho_s$
- Explicitly extract ρ_s from long wavelength spectrum for single reversed spin

Spin-waves – results

Comparing: skyrmion from spin-stiffness vs quasihole excitations



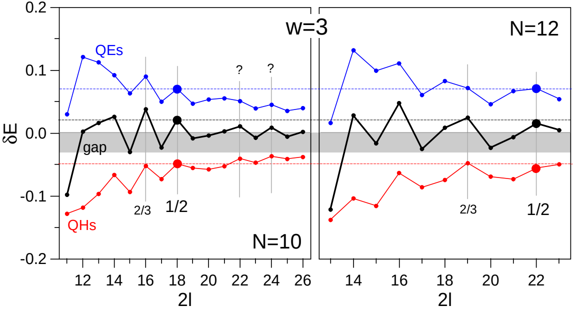
[left: regression for ρ_s from $E_{L=1}$, right: energetic comparison]

- Skyrmions compete favourably with quasiholes in finite width
- Quasielectrons are always preferred
- Attention: need to compare to neutral quasiparticle energies!

[for skyrmions in finite width: Cooper '97]

Numerical search for unpolarized states - I
Scan of N_ϕ at fixed N

Search incompressible states restricted to subspace with $\langle S^2 \rangle = 0$



[Data from ED for the Coulomb Hamiltonian in a layer of width $w = 3\ell_0$]

- gap $\Delta(N_\phi) = E_{N_\phi+1} + E_{N_\phi-1} - 2E_{N_\phi}$ peaks for $N_\phi = 3/2N + 1$ and $N_\phi = 2N - 2$.

Numerical search for unpolarized states - II

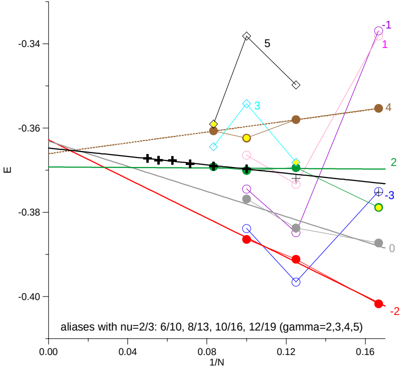
Finite-size scaling of groundstate energies

- Single out potential states with coherent series of groundstate energies in $S = 0$ sector at different N_ϕ

- only groundstate energies for systems with *even* shift σ align in potential series
- important to eliminate aliases with $\nu = 2/3$ state at $\sigma = -1$

- consistent scaling of energy for $\sigma = -2, 0, 2, 4$

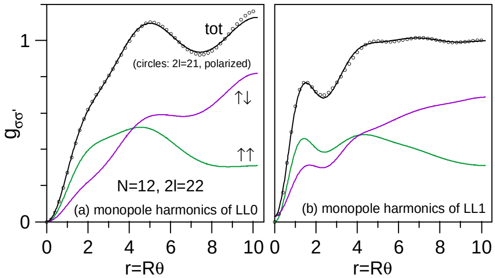
$\Rightarrow$ (Anti-)skyrmions of (anti-)pfaffian?



Groundstate energies for $w = 0$, energies in $e^2/\epsilon\ell_0'$ with rescaled $\ell_0' = \sqrt{N_\phi/\nu N}\,\ell_0$

Characterizing the state at $N_\phi = 2N - 2$ - I
Correlation functions

Features typically associated with a homogeneous quantum liquid not fulfilled [state discarded by Morf ('98) partly on these grounds]



[left: correlations $g_{\uparrow\uparrow}$, $g_{\downarrow\downarrow}$ and $g_{\text{tot}} = g_{\uparrow\uparrow} + g_{\downarrow\downarrow}$ for guiding center coordinates; right: same for electrons]

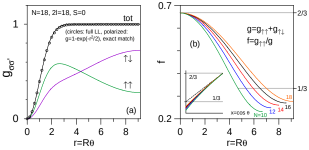
- The $g_{\uparrow\uparrow}(r)$ has a dip at large r, while $g_{\uparrow\downarrow}$ becomes large
- Total correlations g_{tot} closely match those of the polarized 5/2 state at $\sigma = 3$ (length units rescaled for difference in σ)

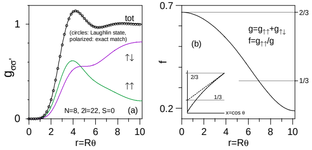
Characterizing the state at $N_\phi = 2N - 2$ - II

Correlation functions, again

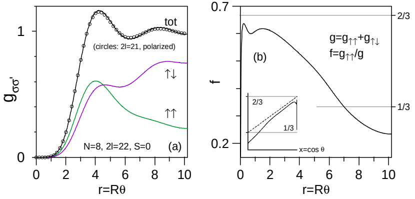
- Compare correlations to known skyrmion states over $\nu = 1$ and $\nu = 1/3$, also showing $f = g_{\uparrow\uparrow}/g_{\text{tot}}$

$\nu = 1$





$\nu = 1/3$



$\nu = 5/2$

- Agreement of g_{tot} with polarized state and similarity with known skyrmion states

Trial wavefunction for skyrmion states
Skyrmion wavefunctions at $\nu = 1$

- Recapitulate known facts about skyrmions from the literature

For every fermionic LLL wavefunction, a Jastrow factor assuring total antisymmetry can be factored out. Therefore, at $\nu = 1$

$$\Psi_{\text{Skyrme}}[z, \chi] = \prod_{i<j} (z_i - z_j) \times \Psi_B[z, \chi]$$
$$= \Psi_{\nu=1} \Psi_B[z, \chi],$$

where Ψ_B is a many-body state of bosons filling orbitals of an effective flux $N_\phi^{\text{boson}} = N_\phi - N_\phi^{\nu=1}$.

- Can construct full spin spectrum, but in particular, there is a unique state Ψ_B with $L = S = 0$ for $N_\phi^{\text{boson}} = 1$.

MacDonald, Fertig and Brey, Phys. Rev. Lett. **76**, 2153 (1996)

Trial wavefunction for skyrmion states

Explicit form of bosonic state for Ψ_B

For $N_\phi^{\text{boson}} = 1$, space with two bosonic orbitals, and two spin-states spanned by normalized vectors $|n_{-\frac{1}{2}\uparrow}, n_{-\frac{1}{2}\downarrow}, n_{\frac{1}{2}\uparrow}, n_{\frac{1}{2}\downarrow}\rangle$. Constraint $L_z = S_z = 0$ entails that

$$|\Psi_B\rangle = \sum_{i=0}^{N/2} c_i \, |i, N/2 - i, N/2 - i, i\rangle$$

Requiring $\hat{\mathbf{S}}^2|\Psi_B\rangle = (\hat{S}_+ + \hat{S}_-)|\Psi_B\rangle = 0, \Rightarrow c_i = [\frac{N}{2} + 1]^{-\frac{1}{2}}(-1)^i$.

General states with different angular momentum / spin quantum numbers can be easily generated by diagonalising $\hat{\mathbf{L}}^2 + \hat{\mathbf{S}}^2$ in this (very small) bosonic Hilbert-space.

Likewise, it is easy to express Ψ_B in position space:
$\Psi_B(\{z_i\}) = \sum_i c_i \,\text{per}\left[\{\Phi_{-\frac{1}{2}}(z_k^\uparrow)\}_{k=1}^{i} \; \{\Phi_{\frac{1}{2}}(z_k^\uparrow)\}_{k=i+1}^{\frac{N}{2}}\right] \times \text{per}\,[(\downarrow)]$

Trial wavefunction for skyrmion states

Generalization to general polarized states

- Generalize to different filling factors using the same Ansatz, but starting from general polarised states $|\Psi_{\text{pol}}\rangle$

$$\Psi_{\text{Skyrmion}}(\{z_i\}, L_z, S_z) = \Psi_{\text{pol}}(\{z_i\}) \times \Psi_B(\{z_i\}, L_z, S_z)$$

- Unlike for the $\nu = 1$ case, the wavefunction is not required mathematically to be separable into these specific factors $\Rightarrow$ test for $\nu = 1/3$ as a reference case

Evaluate overlap $\mathcal{O} = |\int d(z_1, \ldots, z_N)\Psi^*_{\text{skyrmion}}\Psi_{\text{exact}}|^2$ by Monte-Carlo sampling in position space.

N	N_ϕ	$d(\mathcal{H}_{pol})$	$d(\mathcal{H}_{full})$	$\mathcal{O}$
6	16	338	16k	0.95(1)
8	22	8512	1.76M	0.95(2)

Trial wavefunction for skyrmion states

Skyrmions at $\nu = 5/2$

- Using our Ansatz, we can now write

$$\begin{aligned}\Psi^{\nu=5/2}_{\text{Skyrmion}}(\{z_i\}, L_z, S_z) =& \Psi_{\text{MR}}(\{z_i\}) \times \Psi_B(\{z_i\}, L_z, S_z) \\ =& \prod_{i<j}(z_i - z_j)^2 \text{Pf}\left[\frac{1}{z_i - z_j}\right] \\ & \times \Psi_B(\{z_i\}, L_z, S_z)\end{aligned}$$

- But...

Digression: weakly paired states

The Moore-Read state: one of many representatives in the weakly paired phase

- Moore-Read:
  $\Psi_{\mathrm{MR}} = \mathrm{Pf}\left[\frac{1}{z_i - z_j}\right] \prod_{i<j}(z_i - z_j)^2$
- want explicit expression for general paired state in same universality class!
  (see Read & Green, PRB 2000)
- start from BCS state: $|\mathrm{BCS}\rangle = \prod'_{\mathbf{k}}(u_k + v_k c^\dagger_{\mathbf{k}} c^\dagger_{-\mathbf{k}})|0\rangle$
  [variational parameters $u_k, v_k \to g_k = v_k/u_k$]
- in position space: $\langle\{\mathbf{r}_i\}|\mathrm{BCS}\rangle = \mathrm{Pf}\left[\sum_{\mathbf{k}} g_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{r}_l - \mathbf{r}_m)}\right]$

- Composite-fermionize BCS: $[\tilde{\phi}(z_i) = J_i^{-1}\mathcal{P}_{\mathrm{LLL}} J_i \phi(z_i)]$

$$\Psi^{\text{CF-BCS}} = \mathrm{Pf}\left[\sum_{\mathbf{k}} g_{\mathbf{k}}\, \tilde{\phi}_{\mathbf{k}}(z_i)\, \tilde{\phi}_{-\mathbf{k}}(z_j)\right] \prod_{i<j}(z_i - z_j)^2.$$

G. Möller and S. H. Simon, Phys. Rev. B **77**, 075319 (2008).

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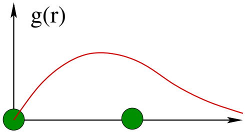

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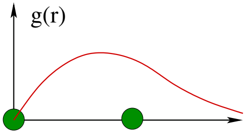

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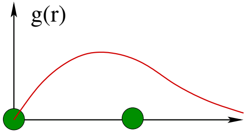
Digression: weakly paired states

Apply concept of general pair wavefunctions for skyrmion states

- Make use of variational degrees of freedom in the pair wavefunction of the polarized weakly paired wavefunction



Moore-Read skyrmion state with generalized pair wavefunction:

$\Rightarrow \Psi^{\nu=\frac{5}{2}}_{Skyrme}[g_k] = \mathrm{Pf}\left[\sum_{\mathbf{k}} g_{\mathbf{k}}\, \tilde\phi_{\mathbf{k}}(z_i)\, \tilde\phi_{-\mathbf{k}}(z_j)\right] \prod_{i<j}(z_i - z_j)^2 \Psi_B^{L=S=0}$,

with the projected CF orbitals $\tilde\phi(z_i) = J_i^{-1}\mathcal{P}_{\mathrm{LLL}} J_i \phi(z_i)$,

and on the sphere, $\tilde\phi_{\mathbf{k}} \to \tilde Y^{-\frac{1}{2}}_{l,m}$ are the CF monopole harmonics in negative flux.

G. Möller and S. H. Simon, Phys. Rev. B **77**, 075319 (2008).
G. Möller and S. H. Simon, Phys. Rev. B **72**, 045344 (2005).

Trial wavefunction for skyrmion states
Results for overlaps

Overlaps for skyrmion wavefunctions at $\nu = 5/2$ are found to be moderately large

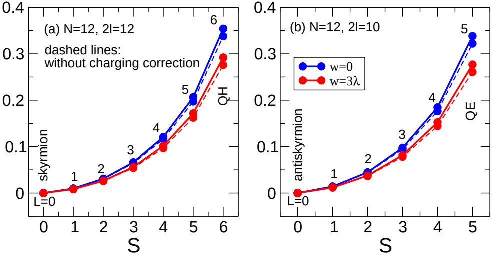
N	$d(\mathcal{H}_{pol})$	$d(\mathcal{H}_{full})$	$\mathcal{O}_{MR}$	$\mathcal{O}_{\text{CF-BCS}}$	$\mathcal{O}_{MR}$	$\mathcal{O}_{\text{CF-BCS}}$
			$w = 0$		$w = 3\ell_0$	
8	151	67k	0.788(9)	0.802(9)	0.81(2)	0.84(3)
10	1514	3.47M	0.51(3)	0.54(3)	0.71(1)	0.72(1)

- Overlaps smaller than for $\nu = 1/3$, but still non-trivial agreement
- Small overlap mostly related to discrepancy of paired trial state and exact Coulomb groundstate → see 'Model Hamiltonians'

Skyrmions at partial spin polarization - I

Generic behaviour for skyrmion state

Having identified the spin-singlet state at $N_\phi = N_\phi^{pol} + 1$, analyze sequence of states with successively higher spin: generic case



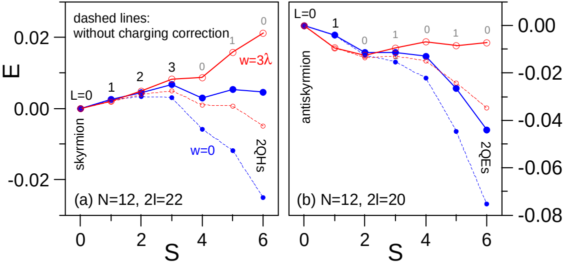
[$\nu = 1$: energy of skyrmion/quasiparticle states versus spin S]

- as polarization increases, a charging correction is required: $\delta E(S) = [S/S_{\max}]^3\, \delta E_{qp}$; $\nu = \frac{5}{2}$: $\delta E_{qp} = \frac{3}{32\sqrt{N}} \frac{e^2}{\epsilon \ell_0}$ (Morf 2002)
- roughly quadratic dispersion; the localized qp has the highest correlation energy (correction negligible at $\nu = 1$)

Skyrmions at partial spin polarization - II

Behaviour for the skyrmion states over $\nu = 5/2$

Spin dependent energy at $\nu = 5/2$



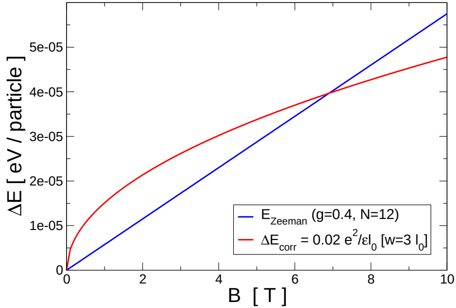
[$\nu = 5/2$: energy of skyrmion/quasiparticle states versus spin S]

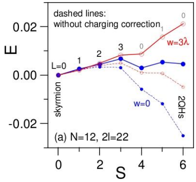
- Kink separating skyrmion-like quadratic dispersion at small S and drop-off towards fully polarized state
- $e/2$ skyrmion formed by binding two $e/4$ quasi-particles, *unlike* $\nu = 1$ or $\nu = 3$ where $q_{\text{skyrmion}} = q_{qp}$ ($\rightarrow$ low L)
  $N = 10$: A. Feiguin *et al.*, Phys. Rev. B **79**, 115322 (2009)

Skyrmions at partial spin polarization - III

Behaviour for the skyrmion states over $\nu = 5/2$

With appropriate charging correction, Skyrmion has *lower* correlation energy than pair of qh's, especially in finite width





[quasihole vs skyrmion energy: $\Delta E = E_{qh} - E_{skyrmion}$]

- Skyrmion might be favourable up to fields $B \sim 6.5T$
- caveat: finite size effects for large skyrmions

Skyrmions at partial spin polarization - IV

Mechanisms to nucleate skyrmions

- at low field / Zeeman coupling, skyrmions are likely the lowest energy excitations

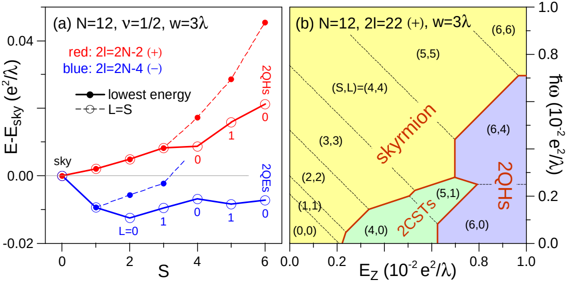
Mechanisms to nucleate skyrmions

- non-zero density of quasiparticles: tuning magnetic field away from center of Hall plateau induces quasiparticles $\longrightarrow$ if qp's are close enough they may be susceptible to bind
- might be better to work at low end of Hall plateau as quasielectrons have less pronounced tendency to bind into skyrmions
- disorder: if two pinning sites are at short separation, mutual binding and introducing a spin-texture may be the energetically most favourable way to accommodate pinned quasiparticles

Skyrmions at partial spin polarization - V

Phase diagram for skyrmions vs quasiholes

- localized $e/2$ fermions may be preferred over $2 \times e/4$ CST by confining disorder potential

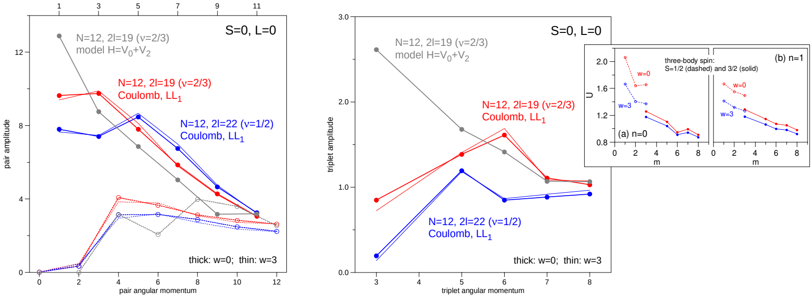


[$\nu = 5/2$: energy of skyrmion/quasiparticle states versus spin S]

Model Hamiltonians - I

A different angle on trial states: towards exact model Hamiltonians

Analyse correlations on the basis of the amplitudes for pairs / triplets with given relative angular momentum



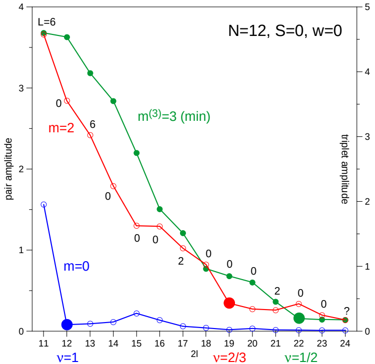
[left: pair-amplitudes for relative angular momentum m; right: triplet amplitudes for $S = 3/2$]

- Pair amplitudes suppressed in V_0 and V_2 channel
- Triplet amplitude suppressed in $\mathcal{V}_{3,3}^{S=3/2}$ channel for $\nu = 5/2$

Model Hamiltonians - II

Evolution of pair / triplet amplitudes with N_ϕ

Plot select pair amplitudes of the respective groundstate for V_0, V_2 and triplet amplitude $\mathcal{V}_{3,3}^{S=3/2}$ as a function of N_ϕ



[Amplitudes $\langle \hat{X} \rangle_\Psi$ calculated for the groundstate of Coulomb interactions in 2nd LL]

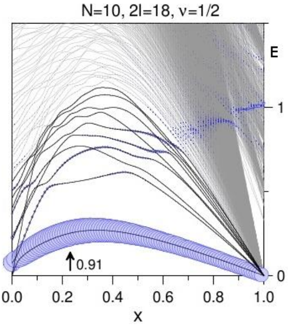
- Pair amplitudes decrease with N_ϕ until Hilbert space is large enough that the GS (nearly) avoids pairs.
- A low background value of the pair amplitude remains for the Coulomb Hamiltonian
- Relation to exact model Hamiltonians evident

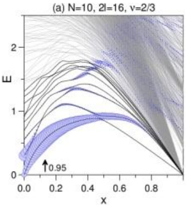
Model Hamiltonians - III

Approximate model Hamiltonians for unpolarized $\nu = 2/3$, $\nu = 1/2$ states

Need to include both pair and triplet amplitudes:

$$\mathcal{H}_{\mathrm{model}}(x) = (1-x)[V_0 + V_2] + x\mathcal{V}_{3,3}^{S=3/2}$$





- $\nu = 5/2$: groundstate of pure 3-body interaction is highly degenerate $\rightarrow$ admixture of $V_0 + V_2$ to split deg.
- yields high overlap with GS of Coulomb interaction ($\rightarrow$ MR)

Conclusions

- We find series of states with $N_\phi = 2N - 2$ and $N_\phi = 2N - 4$ on the sphere; these are the unique candidates with L=S=0.
- We identify these series as (anti-)skyrmions of Moore-Read, and show how to construct explicit trial states (good overlap)
- At $\nu = 5/2$ the skyrmion has twice the charge of qp's
- Appearance of skyrmion can be interpreted as binding of qp's; binding becomes favourable at finite width $w \sim 3\ell_0$; and might occur spontaneously at moderate fields / qp density

- The physics of $\nu = 5/2$ is that of a spin polarized quantum liquid. The groundstate is in the non-abelian weakly paired phase, but its quasielectrons/-holes compete with skyrmions to be the lowest lying excitations
- Skyrmions could be observed at $\nu = 5/2$ in samples with low Zeeman energy, and be a mechanism to deplete spin pol.

Charging correction for localized quasiparticle

- Localized charge causes net energy with respect to homogeneous backgroud
- multiple quasiparticles have mutual repulsion

$$E_{\text{tot}} = E_{MR} + 2\epsilon_{qp-\text{corr}} + E_{\text{charging}} + 2V_{qp-\text{bg}} + V_{qp-qp}$$

where the different terms signify:

- E_{MR} – energy of liquid w/o qp's
- $E_{\text{charging}} = (e/2)^2/2R$ – charging energy of uniformly distributed bg charge $-e/2$
- $V_{qp-\text{bg}} = -(e/4)(e/2)/R$ – interaction of one qp with the homogeneous background
- $V_{qp-qp} = (e/4)^2/2R$ repulsion between two qp's of charge $e/4$ separated by diameter

$\Rightarrow$ convert all to same units of ℓ_0

R. Morf *et al.*, Phys. Rev. B **66**, 075408 (2002)