Quantum Phases of a Supersymmetric Model of Lattice Fermions

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Collaborators and references

UvA, Amsterdam:

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UVa, Charlottesville:

P. Fendley, J. Halverson

P. Fendley, K. Schoutens, J. de Boer, PRL (2003)

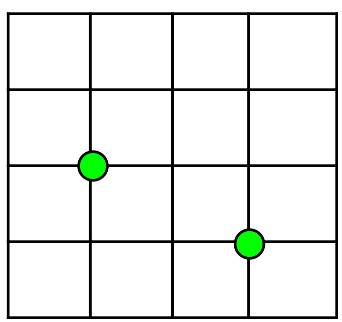
P. Fendley, K. Schoutens, PRL (2005)

L. Huijse, J. Halverson, P. Fendley, K. Schoutens, PRL (2008)

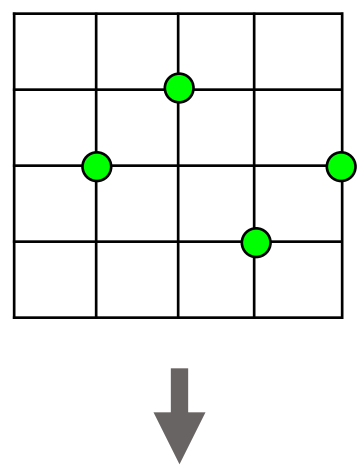
L. Huijse, K. Schoutens, arXiv:0903.0784

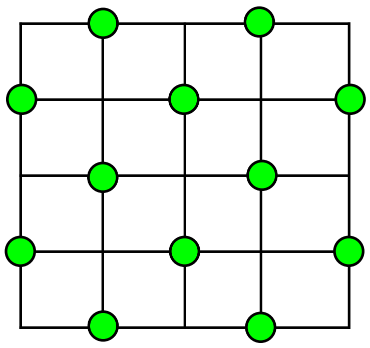
Motivation

challenge: understand quantum phases of strongly repelling lattice fermions at intermediate densities



Fermi liquid



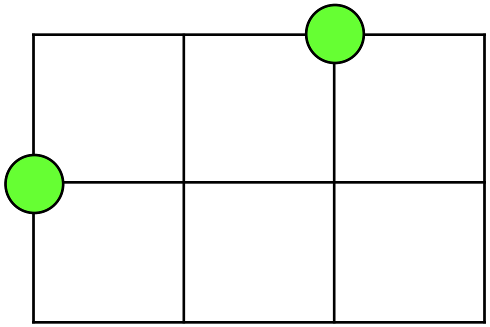


Mott insulator

???

Supersymmetric model for lattice fermions

name of the game:



- lattice models for spin-less fermions tuned to be supersymmetric

key features:

- susy implies delicate balance between kinetic and potential terms, leading to interesting ground state structure
- analytic control due to such tools as the Witten index and cohomology techniques

Supersymmetric model for lattice fermions

characteristics:

- quantum criticality in 1D
  (N=2 superconformal FT)
- superfrustration in 2D
  (extensive ground state entropy)
- supertopological phases in 2D

Outline

- **Supersymmetric quantum mechanics**
- The model
- 1D: Quantum criticality
- 2D: Superfrustration
- 2D: Supertopological phases

Supersymmetric QM: algebraic structure

susy charges Q^+, $Q^-=(Q^+)^+$ and fermion number N_f:

$$(Q^+)^2 = 0, \quad (Q^-)^2 = 0, \quad [N_f, Q^\pm] = \pm Q^\pm$$

Hamiltonian defined as

$$H = \{Q^+, Q^-\}$$

satisfies

$$[H, Q^+] = [H, Q^-] = 0\,, \quad [H, N_f] = 0$$

Spectrum of supersymmetric QM

- $E \geq 0$ for all states

- $E > 0$ states are paired into doublets of the susy algebra

$$\{|\psi>, Q^{+}|\psi>\}, \quad Q^{-}|\psi>=0$$

- $E = 0$ iff a state is a singlet under the susy algebra

$$Q^{+}|\psi>=Q^{-}|\psi>=0$$

- if $E = 0$ ground state exist, supersymmetry is unbroken.

Witten index

$$W = \mathrm{Tr}(-1)^{N_f}$$

- $E>0$ doublets $\{|\psi\rangle, Q^+|\psi\rangle\}$
  with $N_f = f$, $N_f = f+1$ cancel in W
- only $E=0$ groundstates contribute

→ $|W|$ is lower bound on # of ground states

[Witten 1982]

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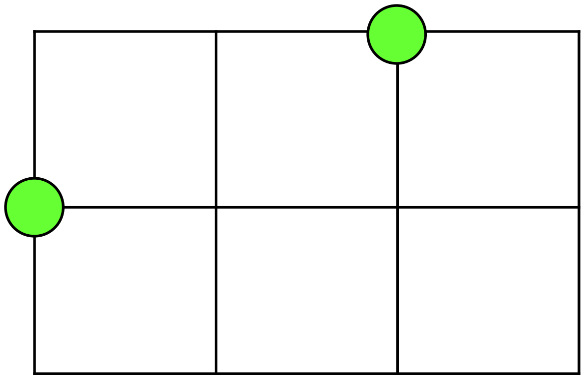
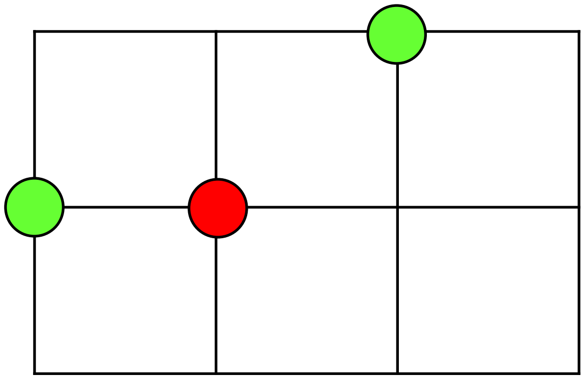
Susy lattice model

configurations:
lattice fermions with nearest neighbor exclusion



Susy lattice model

configurations:
lattice fermions with nearest neighbor exclusion



Susy lattice model

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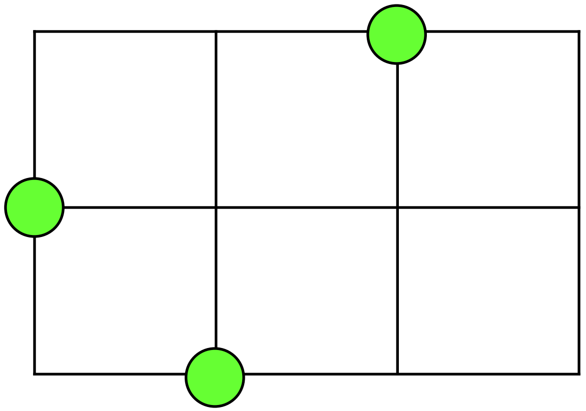

Susy lattice model

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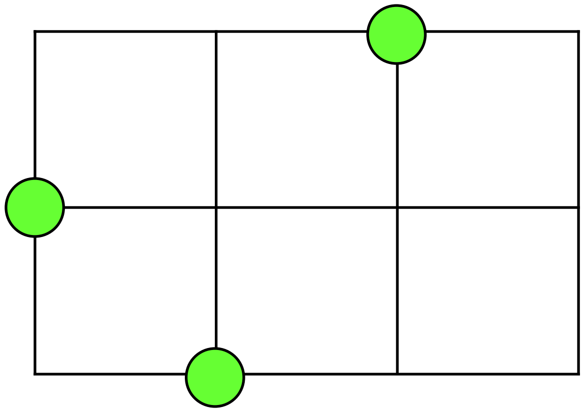


nilpotent supercharges, respecting exclusion rule:

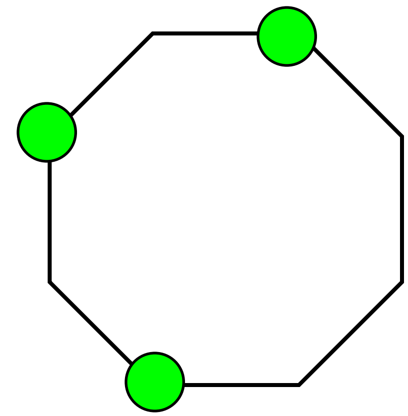
$$Q^+ = \sum_i c_i^+ \prod_\delta (1 - n_{i+\delta}), \quad Q^- = (Q^+)^+ \qquad n_i = c_i^+ c_i$$

Hamiltonian: kinetic (hopping) plus potential terms

$$H = \{Q^+, Q^-\} = H_{kin} + H_{pot}$$

[Fendley - Schoutens - de Boer 2003]

Susy model in 1*D*



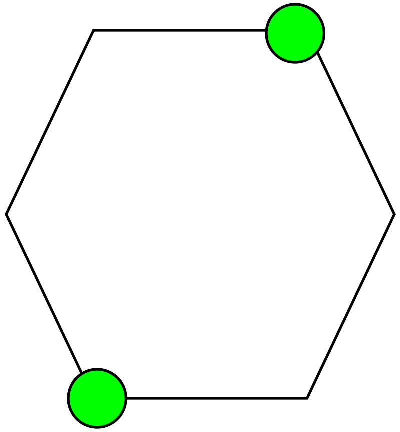
supercharges

$$Q^{+} = \sum_{i} (1 - n_{i-1}) c_{i}^{+} (1 - n_{i+1}), \quad Q^{-} = (Q^{+})^{+}$$

Hamiltonian:

$$H = \sum_{i} [(1 - n_{i-1}) c_{i}^{+} c_{i+1} (1 - n_{i+2}) + \text{h.c.}] + \sum_{i} n_{i-1} n_{i+1} - 2 N_{f} + L$$

L=6 model: Witten index



$$W = \mathrm{Tr}(-1)^{N_f}$$

$N_f = 0$: 1 state

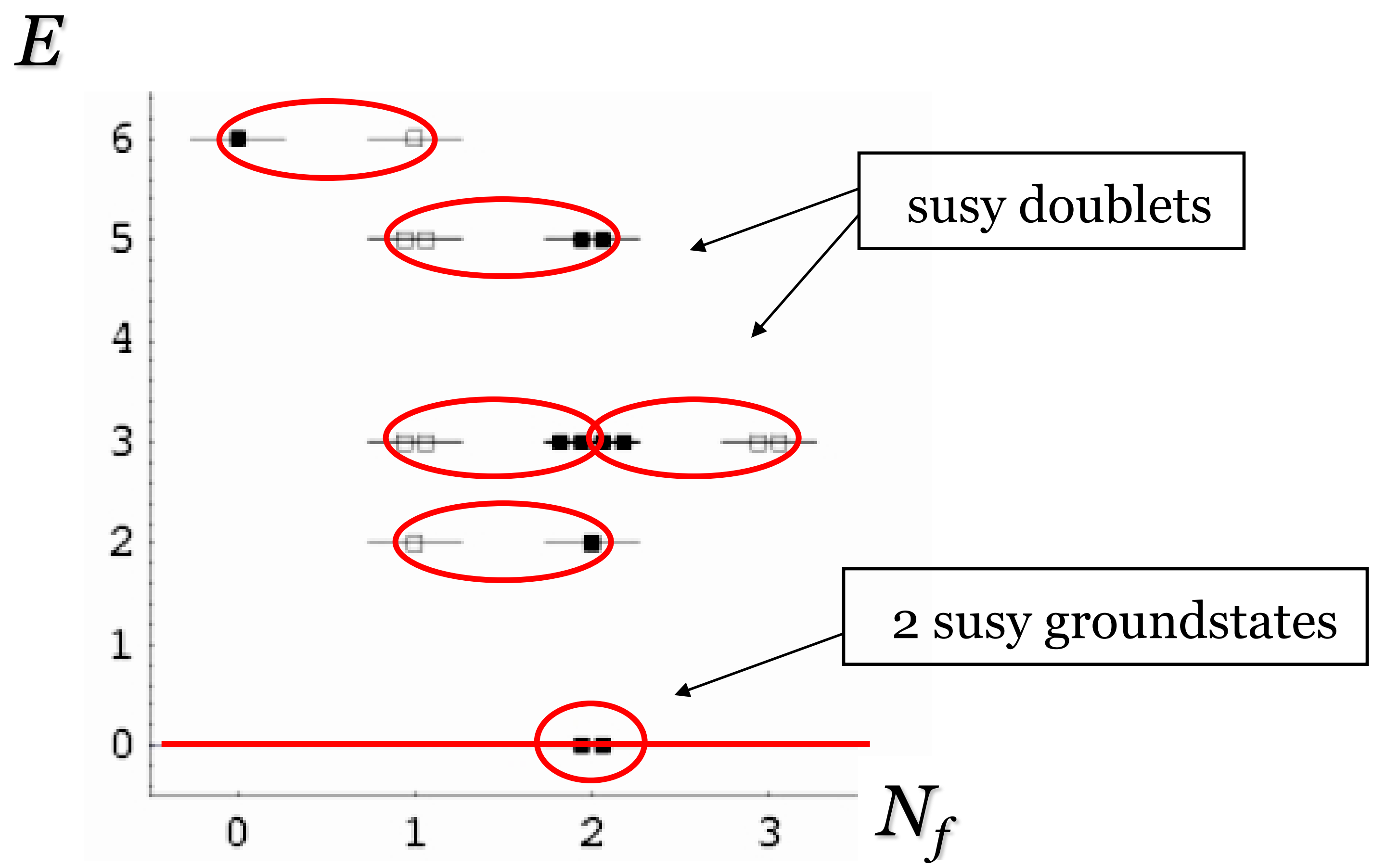
$N_f = 1$: 6 states

$N_f = 2$: 9 states

$N_f = 3$: 2 states

$$\Rightarrow W = 1 - 6 + 9 - 2 = 2$$

Spectrum for $L=6$ sites



Cohomology technique

Lemma

Susy ground states are in 1-1 correspondence with the cohomology

$$H_{Q,N_f} = \mathrm{Ker}[Q^+]_{N_f} / \mathrm{Im}[Q^+]_{N_f-1}$$

of Q^+ in the complex

$$\ldots \xrightarrow{Q^+} H_{N_f} \xrightarrow{Q^+} H_{N_f+1} \xrightarrow{Q^+} \ldots$$

Cohomology technique

Spectral sequence technique for evaluating the cohomology:

- decompose: $Q^+ = Q^+{}_A + Q^+{}_B$,
- first evaluate the cohomology H_B of $Q^+{}_B$,
- next evaluate the cohomology $H_A(H_B)$ of $Q^+{}_A$

A tic-tac-toe lemma relates $H_A(H_B)$ to the full cohomology H_Q . In general, $H_Q \subseteq H_A(H_B)$.

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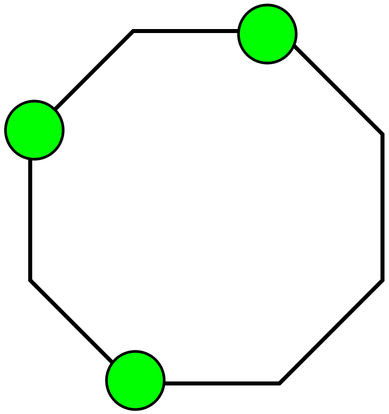
Quantum critical behavior 1D



- periodic chain:
  2 gs for L multiple of 3, else 1 gs
- ground states at filling: $f = N_f / L = \frac{1}{3}$
- exactly solvable via Bethe Ansatz
- continuum limit: $\mathcal{N}=2$ SCFT with central charge $c=1$

N=2 SCFT description for the chain

- finite size spectrum built from vertex operators

$$V_{m,n}\,, \quad (-1)^{m+2n} = -1, \quad h_{L,R} = \frac{3}{8}(m \pm \frac{2}{3}n)^2$$

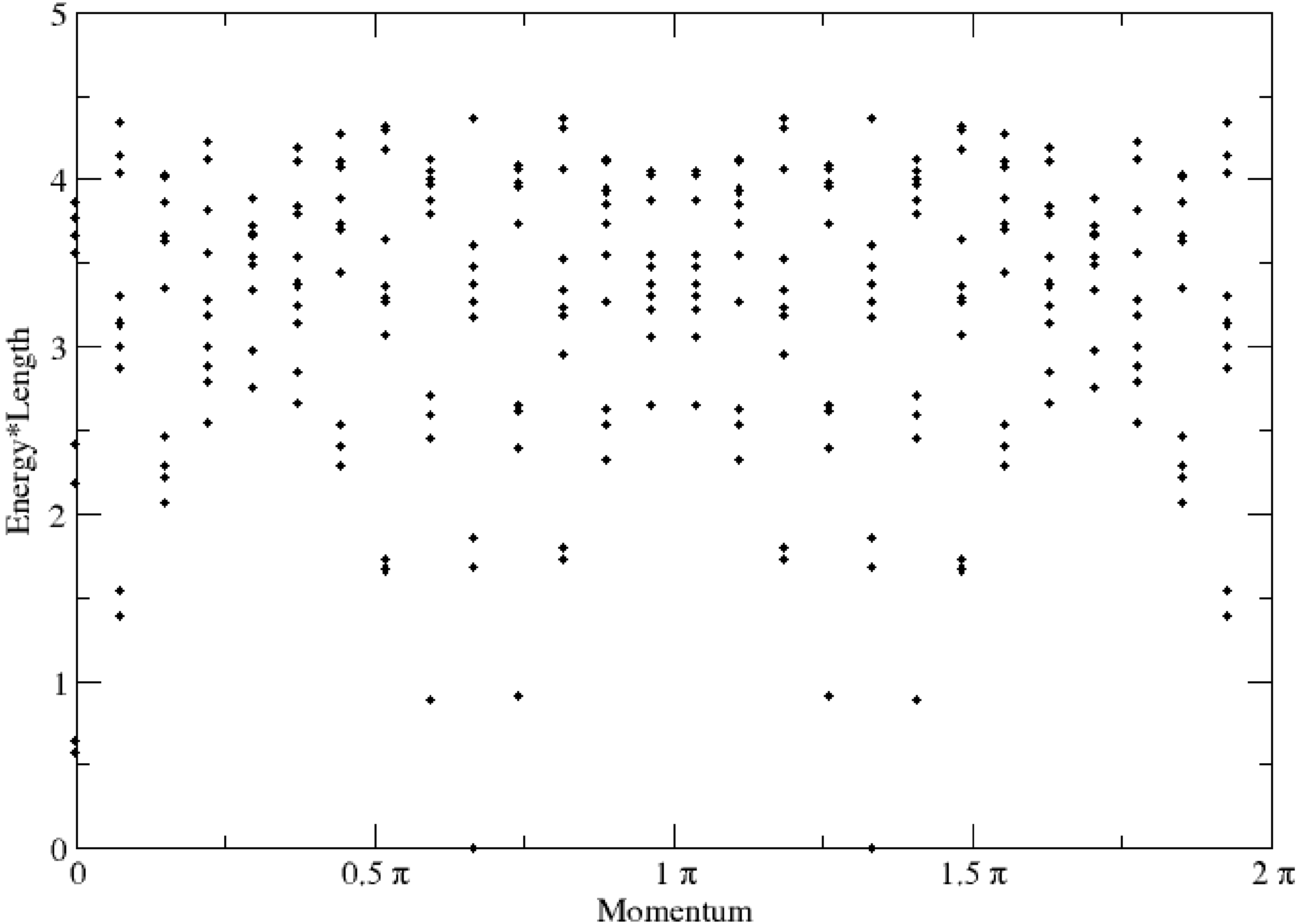
and Virasoro generators $L_{-k,L}, L_{-k,R}$

- lattice model parameters *E, P* and N_f related to conformal dimensions $h_{L,R}$ and *U(1)* charges $q_{L,R}$.

In particular

$$E \propto h_L + h_R - \frac{c}{12}$$

Spectrum for 1D chain, L=27, N_f=9



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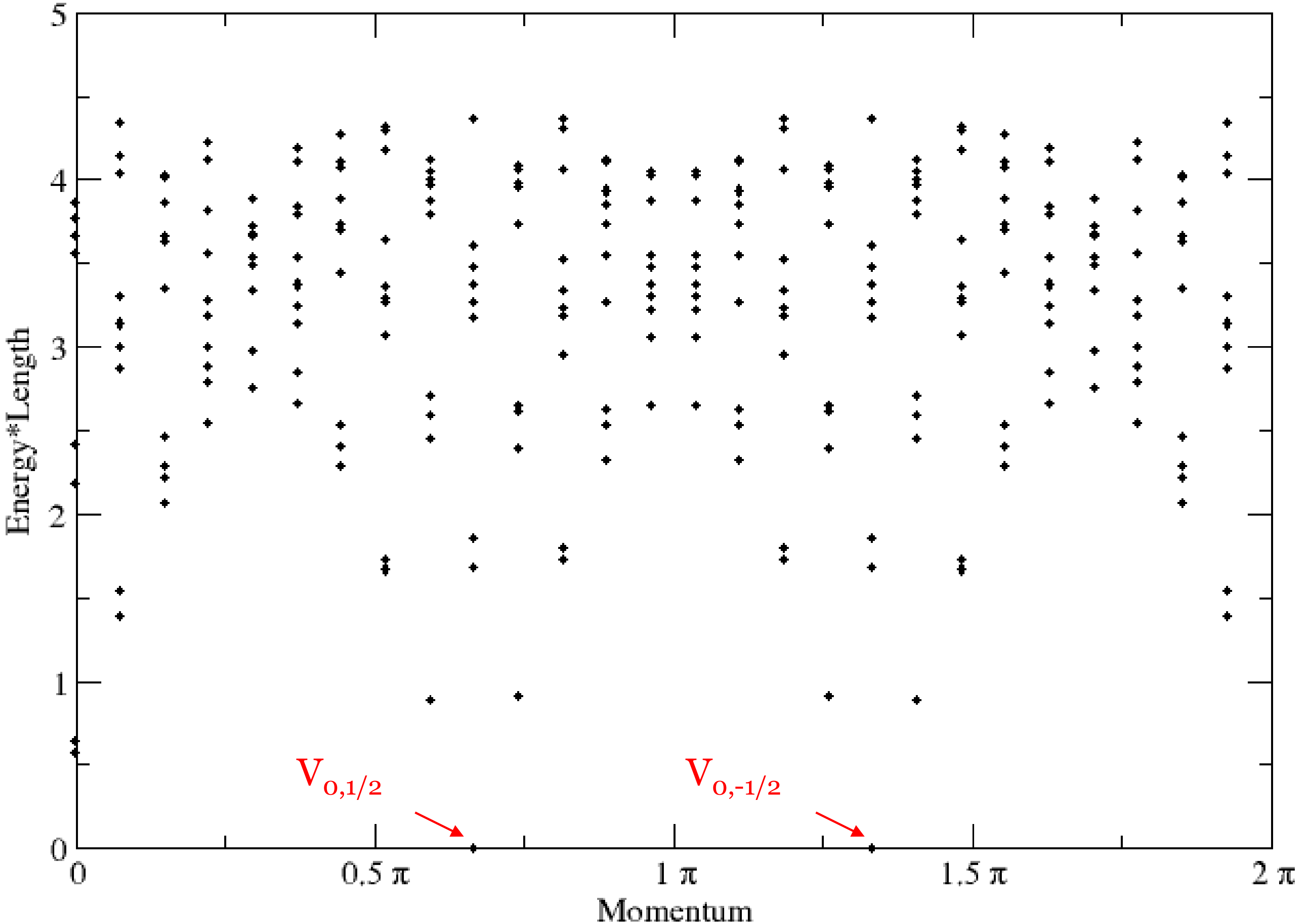

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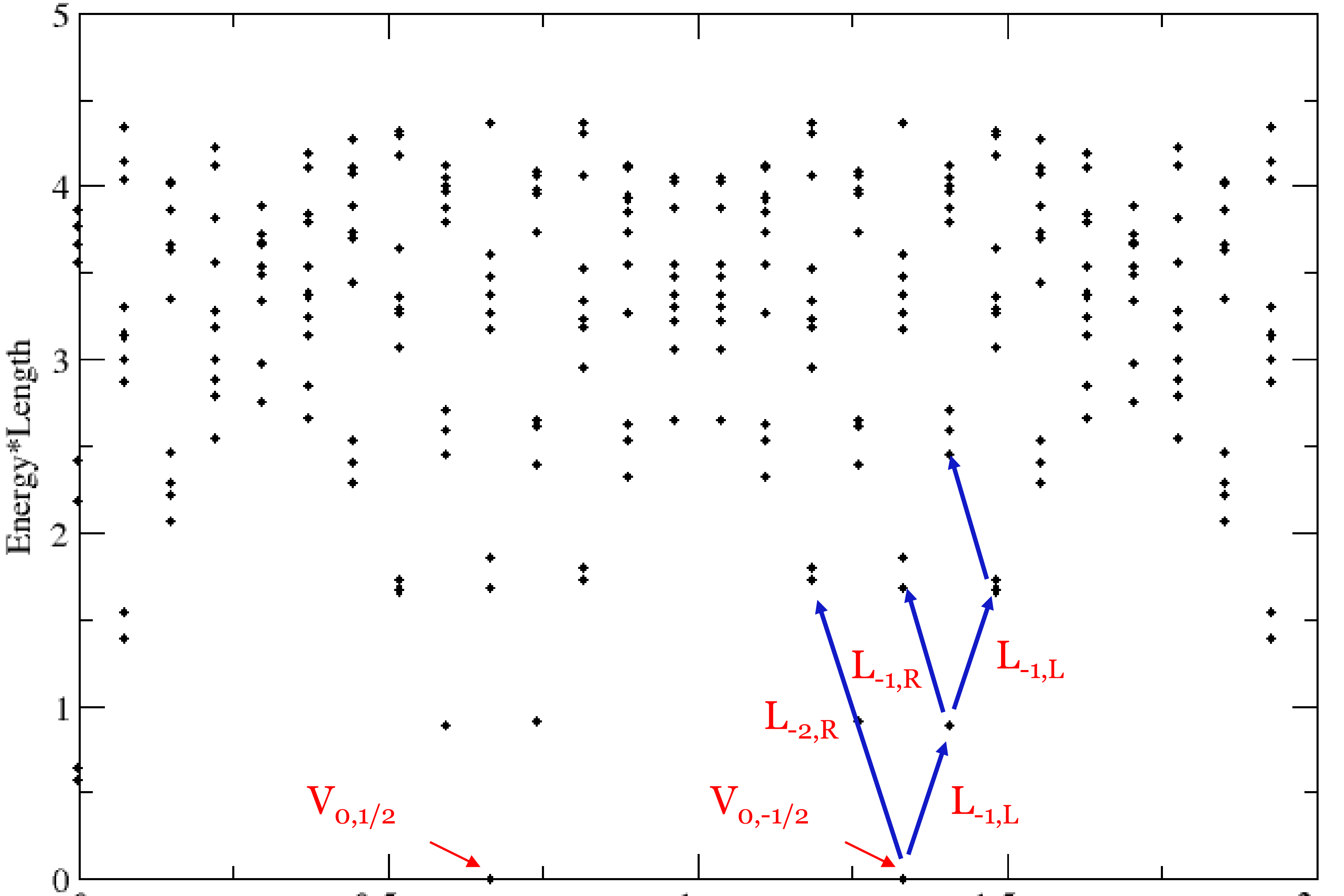

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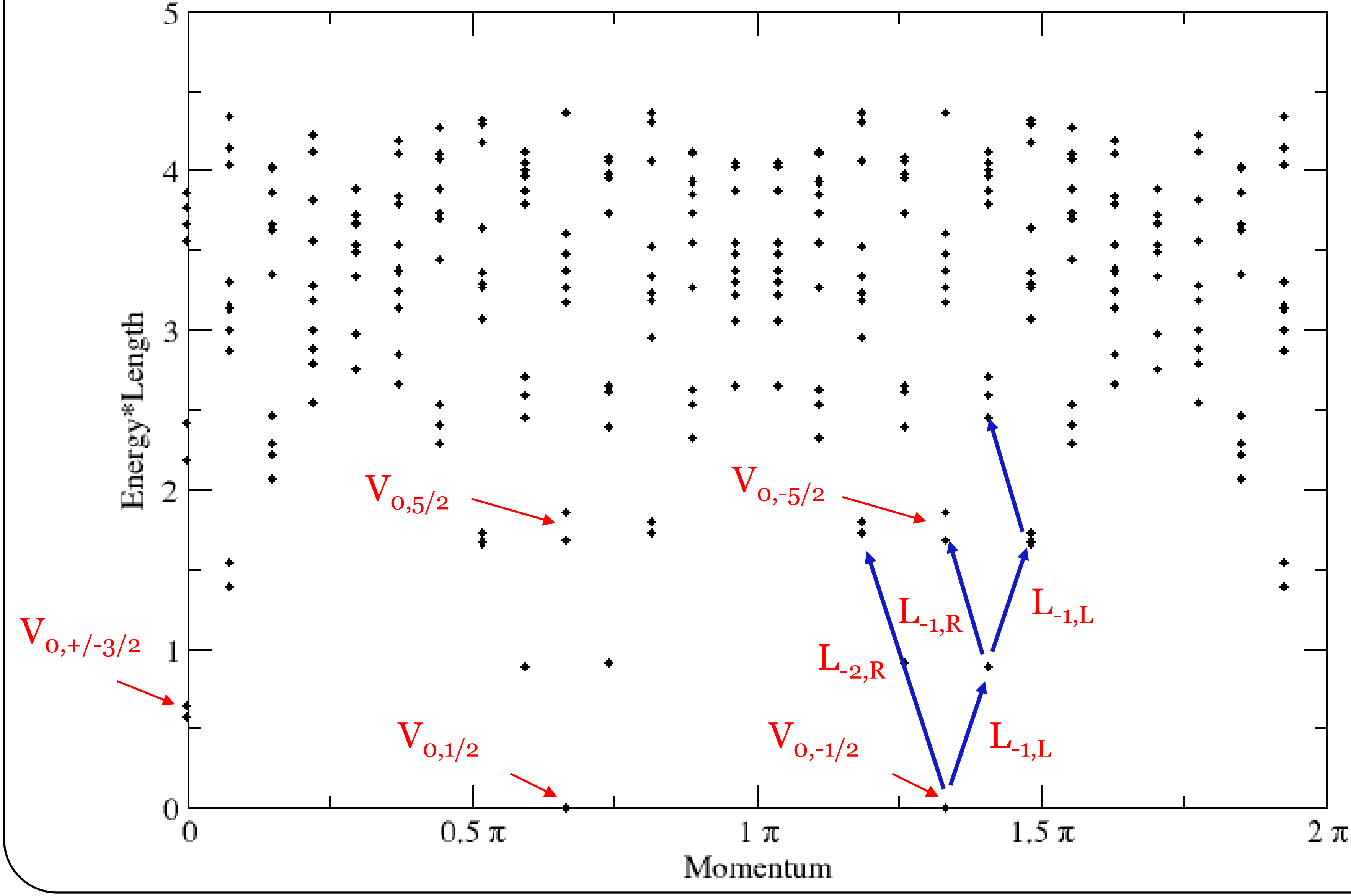

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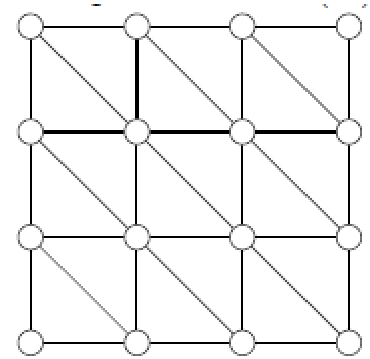
Triangular lattice: Witten index



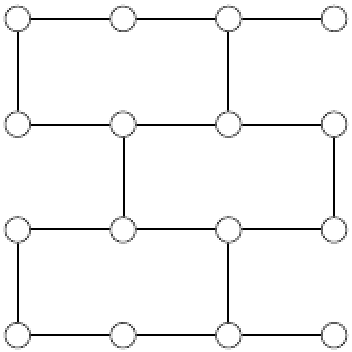
$N \times M$ sites with periodic BC

	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	1	-3	-5	1	11	9	-13	-31	-5	57
3	1	-5	-2	7	1	-14	1	31	-2	-65
4	1	1	7	-23	11	25	-69	193	-29	-279
5	1	11	1	11	36	-49	211	-349	811	-1064
6	1	9	-14	25	-49	-102	-13	-415	1462	-4911
7	1	-13	1	-69	211	-13	-797	3403	-7055	5237
8	1	-31	31	193	-349	-415	3403	881	-28517	50849
9	1	-5	-2	-29	881	1462	-7055	-28517	31399	313315
10	1	57	-65	-279	-1064	-4911	5237	50849	313315	950592
11	1	67	1	859	1651	12607	32418	159083	499060	2011307
12	1	-47	130	-1295	-589	-26006	-152697	-535895	-2573258	-3973827
13	1	-181	1	-77	-1949	67523	330331	-595373	-10989458	-49705161
14	1	-87	-257	3641	12611	-139935	-235717	5651377	4765189	-232675057
15	1	275	-2	-8053	-32664	272486	-1184714	-1867189	134858383	-702709340

$\Rightarrow$ \`superfrustration'

[van Eerten 2005]

Hexagonal lattice: Witten index

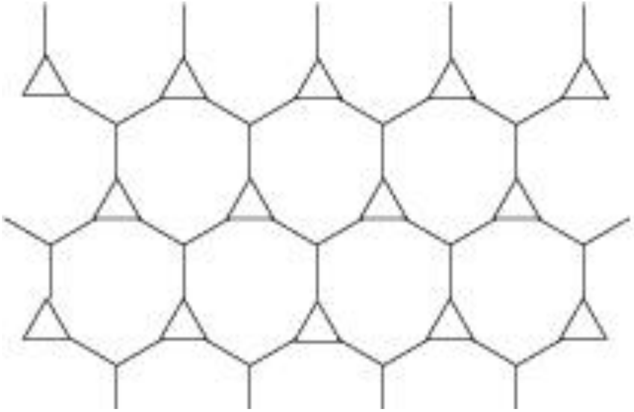


$N \times M$ sites with periodic BC

	2	4	6	8	10	12	14	16	18
2	-1	-1	2	-1	-1	2	-1	-1	2
4	3	7	18	47	123	322	843	2207	5778
6	-1	-1	32	-73	44	356	-1387	2087	2435
8	3	7	18	55	123	322	843	2215	5778
10	-1	-1	152	-321	-171	7412	-26496	10079	393767
12	3	7	156	1511	6648	29224	150069	1039991	6208815
14	-1	-1	338	727	-5671	1850	183560	-279497	-4542907
16	3	7	1362	12183	31803	379810	5970107	55449303	327070578

[van Eerten 2005]

Martini lattice

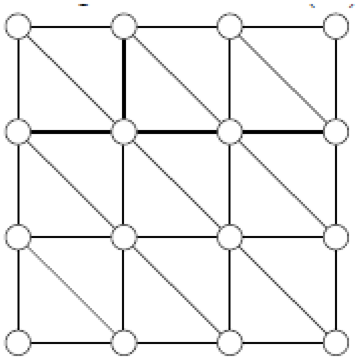


- extensive number of susy ground states, all at filling ¼ (one fermion per triangle)
- susy gs 1-1 with dimer coverings of hexagonal lattice
- exact result for ground state entropy

$$\frac{S_{\mathrm{gs}}}{N} = \frac{1}{\pi}\int_0^{\pi/3} d\theta \, \ln[2\cos\theta] = 0.16153\ldots$$

[Fendley - Schoutens 2005]

Triangular lattice: ground states



Two results

- ground states exist in range of filling fractions

$$\frac{1}{7} \leq \frac{N_f}{MN} \leq \frac{1}{5}$$

[Jonsson 2005]

- upper bound to the number of gs on $M \times N$ sites

$$\frac{S_{\text{gs}}}{MN} \leq \frac{1}{2} \log \frac{1+\sqrt{5}}{2} \approx 0.24$$

[Engström 2007]

Open problems

- ground state entropy in thermodynamic limit?
- nature of these ground states?

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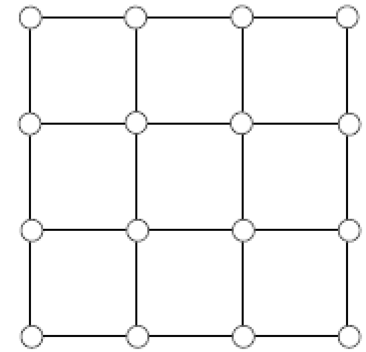
- **2D: Supertopological phases**

Square lattice: Witten index



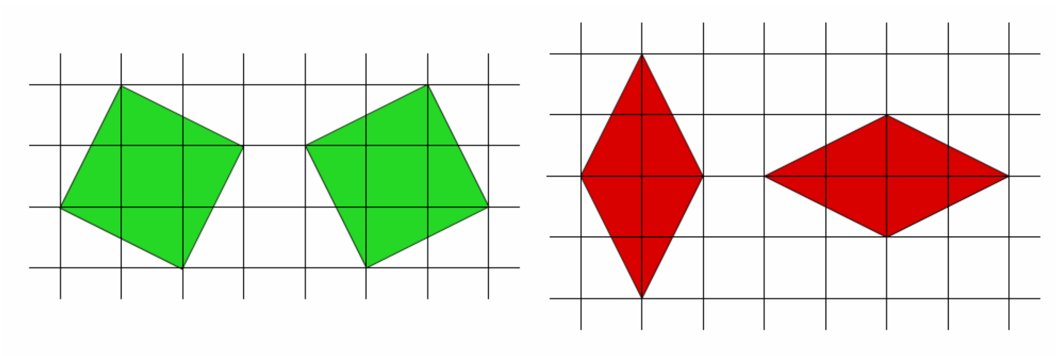
$N \times M$ sites with periodic BC

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	−1	1	3	1	−1	1	3	1	−1	1	3	1	−1	1	3	1	−1	1	3
3	1	1	4	1	1	4	1	1	4	1	1	4	1	1	4	1	1	4	1	1
4	1	3	1	7	1	3	1	7	1	3	1	7	1	3	1	7	1	3	1	7
5	1	1	1	1	−9	1	1	1	1	11	1	1	1	1	−9	1	1	1	1	11
6	1	−1	4	3	1	14	1	3	4	−1	1	18	1	−1	4	3	1	14	1	3
7	1	1	1	1	1	1	1	1	1	1	1	1	1	−27	1	1	1	1	1	1
8	1	3	1	7	1	3	1	7	1	43	1	7	1	3	1	7	1	3	1	47
9	1	1	4	1	1	4	1	1	40	1	1	4	1	1	4	1	1	76	1	1
10	1	−1	1	3	11	−1	1	43	1	9	1	3	1	69	11	43	1	−1	1	13
11	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
12	1	3	4	7	1	18	1	7	4	3	1	166	1	3	4	7	1	126	1	7
13	1	1	1	1	1	1	1	1	1	1	1	1	−51	1	1	1	1	1	1	1
14	1	−1	1	3	1	−1	−27	3	1	69	1	3	1	55	1	451	1	−1	1	73
15	1	1	4	1	−9	4	1	1	4	11	1	4	1	1	174	1	1	4	1	11

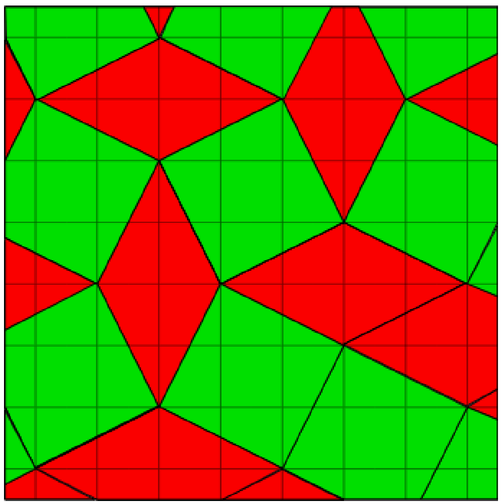
[Fendley - Schoutens - van Eerten 2005]

Square lattice: Witten index

Witten index related to rhombus tilings of the lattice



periodicities $\vec{u}, \vec{v}$



Theorem [Jonsson 2005]

$$W_{\vec{u},\vec{v}} = t_{even} - t_{odd} - (-1)^{d_-} \theta_{d_-} \theta_{d_+}$$

with $d_\pm = \gcd(u_1 \pm u_2, v_1 \pm v_2)$, $\theta_{3p} = 2$, $\theta_{3p\pm1} = -1$

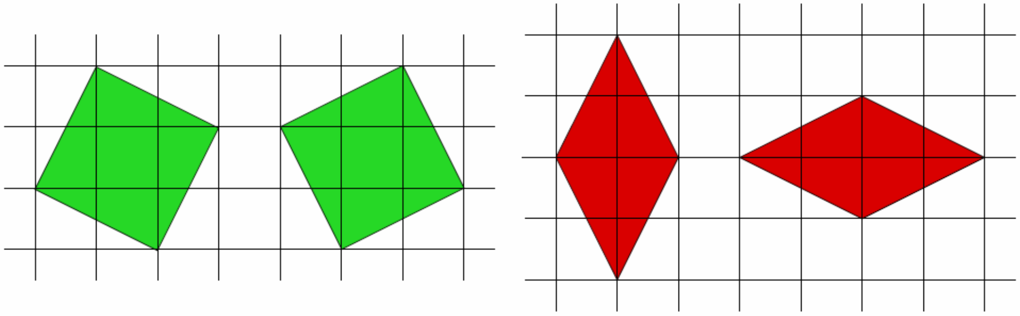
Square lattice: ground states

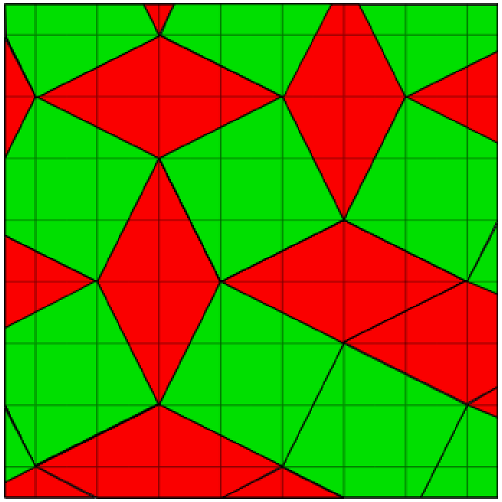
periodicities $\vec{u}, \vec{v}$

$$v_1 + v_2 = 3p$$

$$\vec{u} = (m, -m)$$

number of gs related to rhombus tilings of the lattice, with $N_f = N_t$





Theorem [Fendley, LH - Schoutens 2009]

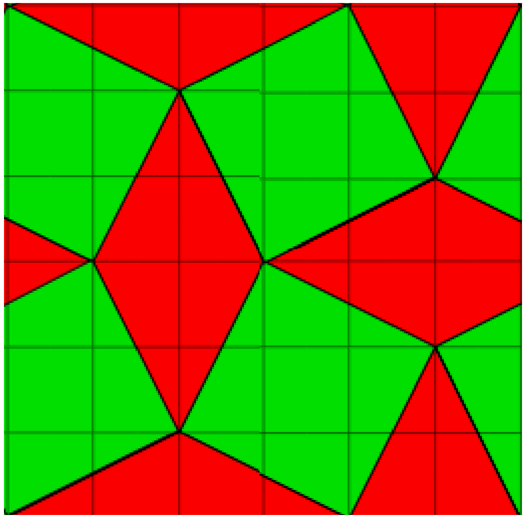
$$\#\ \mathrm{GS} = t_{even} + t_{odd} - (-1)^{(\theta_m+1)p}\theta_{d_-}\theta_{d_+}$$

with $d_{\pm} = \gcd(u_1 \pm u_2, v_1 \pm v_2)$, $\theta_{3p} = 2$, $\theta_{3p\pm1} = -1$

Square lattice: ground states

Example: square lattice 6x6

$$\vec{u} = (6,0), \quad \vec{v} = (0,6)$$



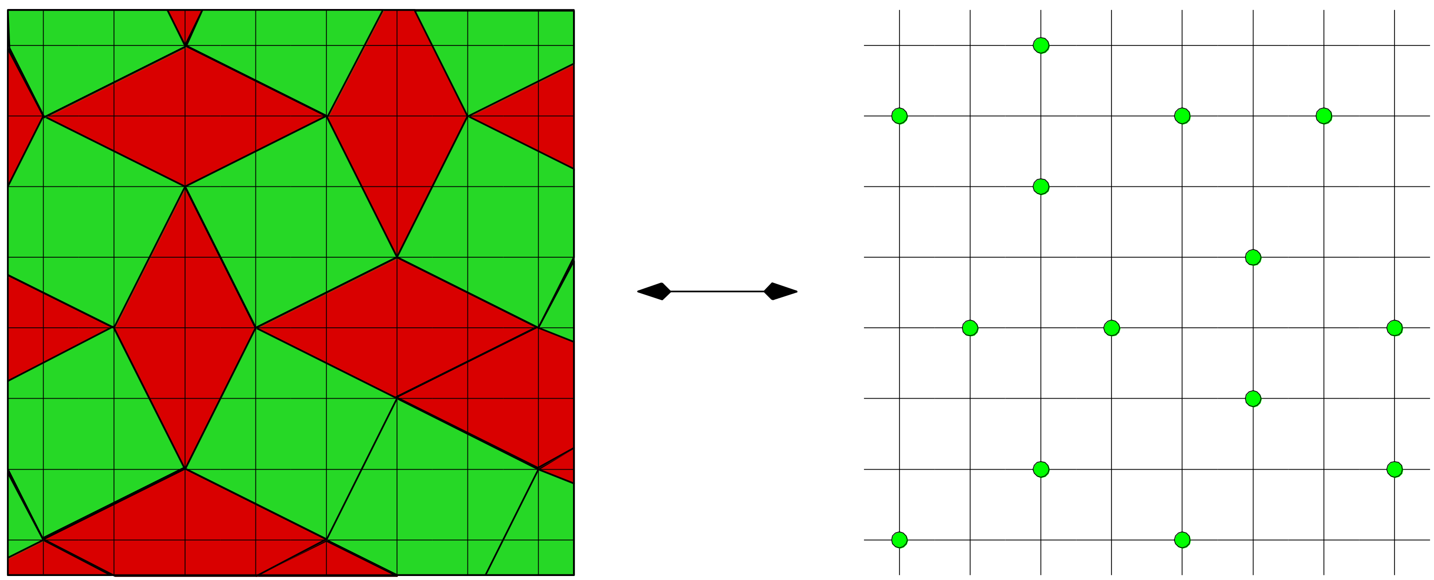
- 18 tilings with N_t=*8*
- correction term equals -4

$\Rightarrow$ 14 groundstates with N_f=*8*, filling 2/9

Square lattice: ground states

- # gs grows exponentially with the linear size of the system
- zero energy ground states found at intermediate filling:

$$\frac{N_f}{L} \in [1/5, 1/4] \cap \mathbb{Q}$$



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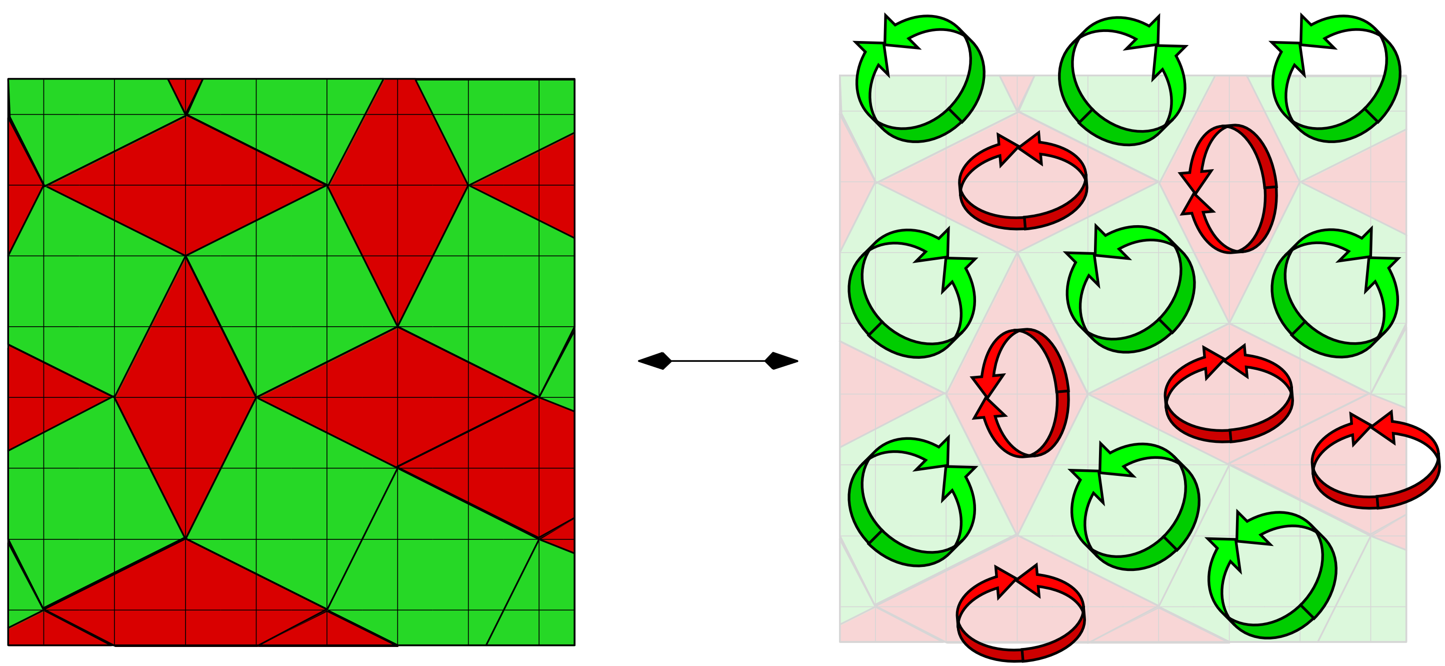
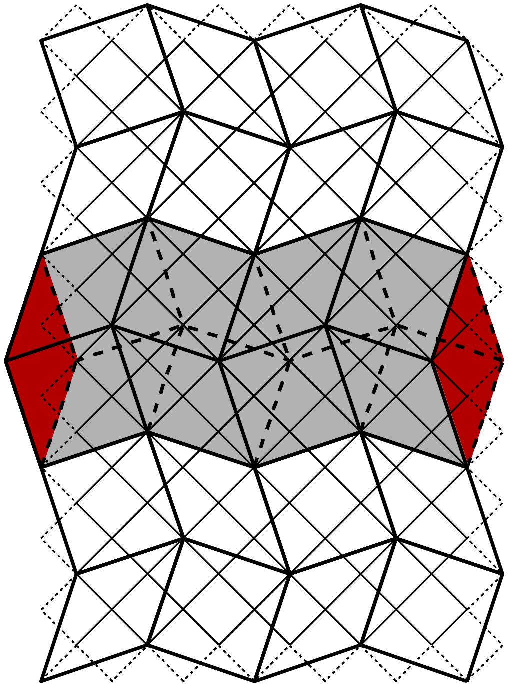


Square lattice: edge states

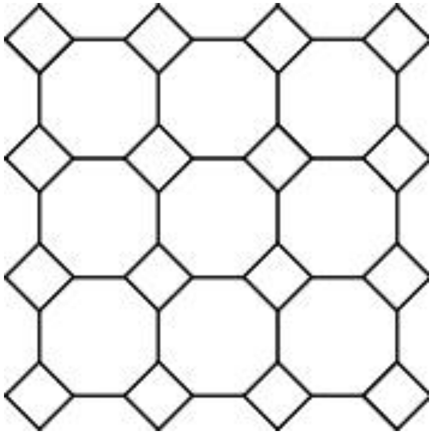
- for `diagonal' open boundary conditions there is a unique gs; expect that `vanished' torus gs's form band of edge modes

- explicit evidence for critical modes from ED studies of various ladder geometries



[LH - Halverson - Fendley - Schoutens 2008]

Octagon-square lattice

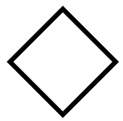


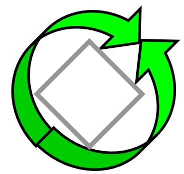
- $N \times M$ plaquettes with open bc : unique gs with one fermion per plaquette: `filled Landau level'
- $N \times M$ plaquettes with closed bc: 2^M+2^N-1 gs
- gapless defects that interact through `Dirac strings'
- …

$\Rightarrow$ `supertopological phase'

Single plaquette

plaquette
(1 gs)

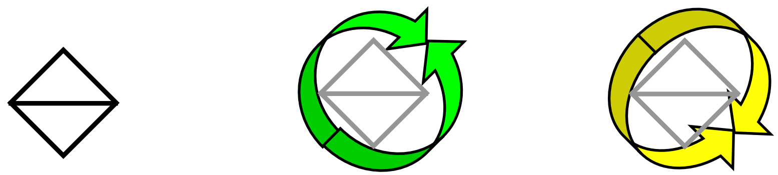




Single plaquette

plaquette
(1 gs)

H-defect
(2 gs)

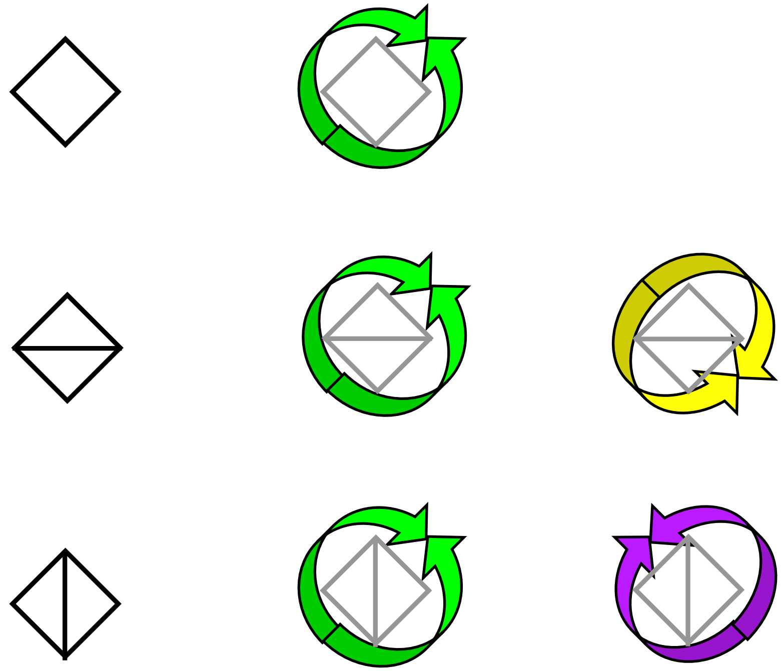


Single plaquette

plaquette
(1 gs)

H-defect
(2 gs)

V-defect
(2 gs)



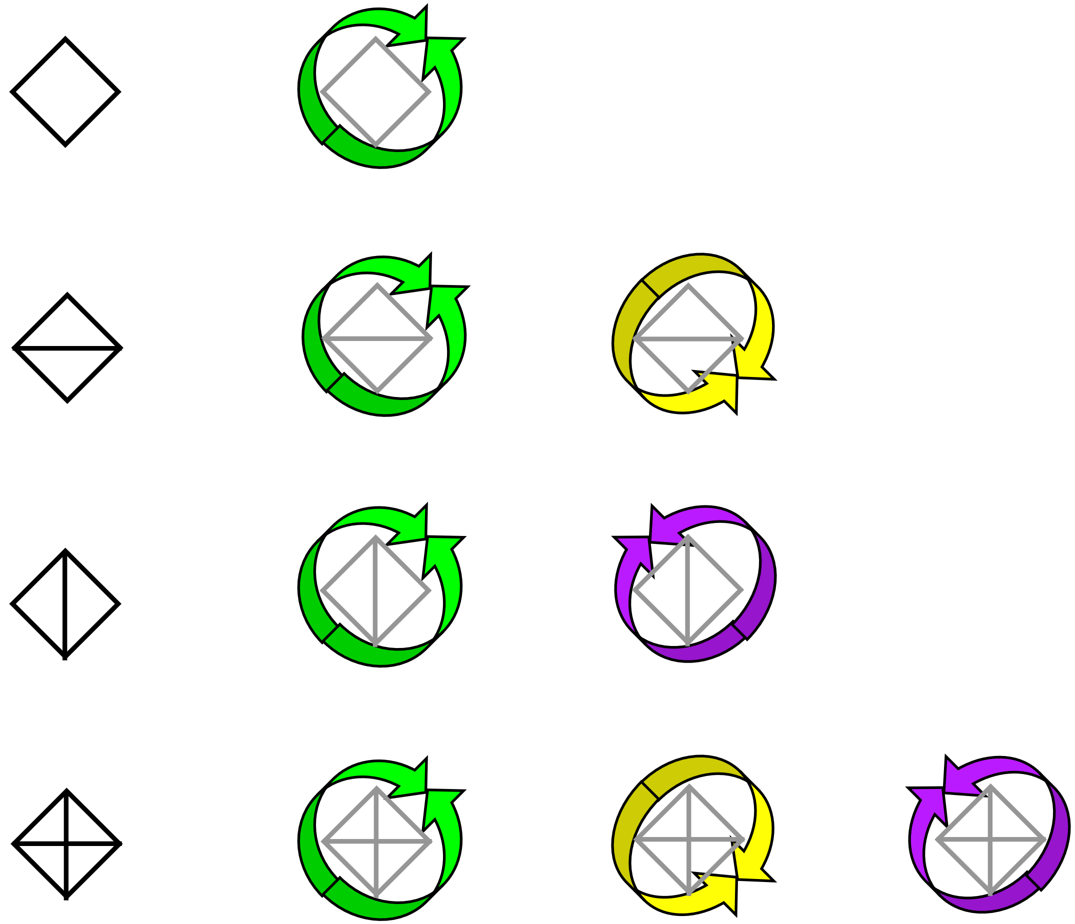
Single plaquette

plaquette
(1 gs)

H-defect
(2 gs)

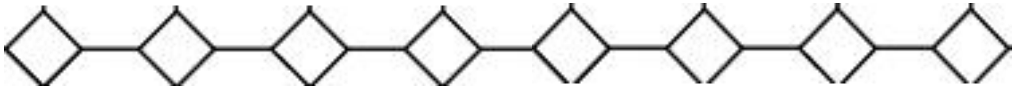
V-defect
(2 gs)

HV-defect
(3 gs)



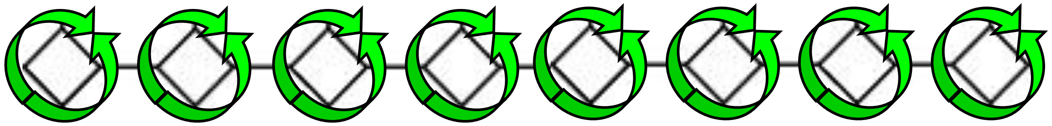
1D plaquette chain (open)

open bc



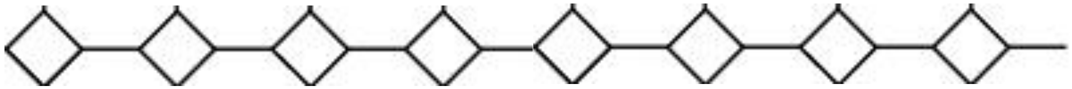
1D plaquette chain (open)

open bc
(1 gs)



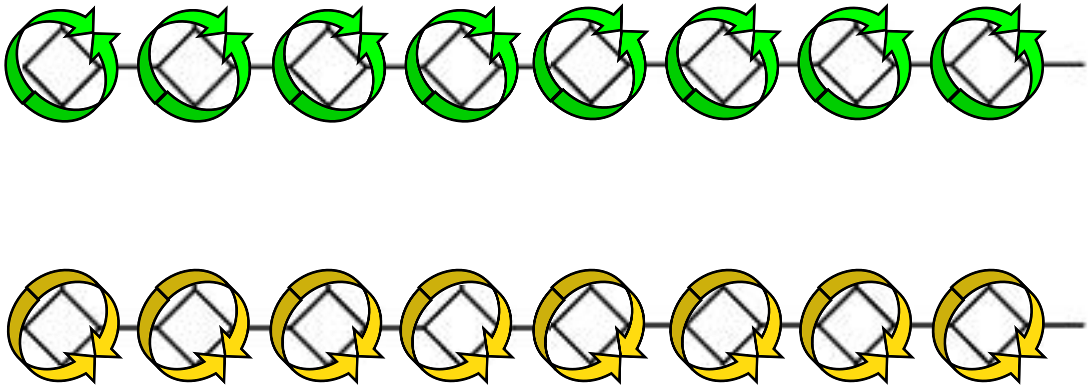
1D plaquette chain (closed)

closed bc



1D plaquette chain (closed)

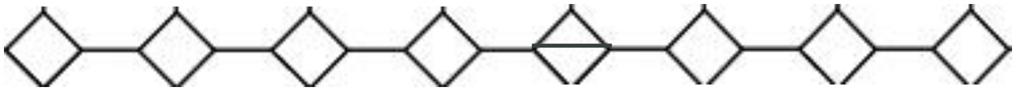
closed bc
(2 gs)



[Maps to staggered 1D chain]

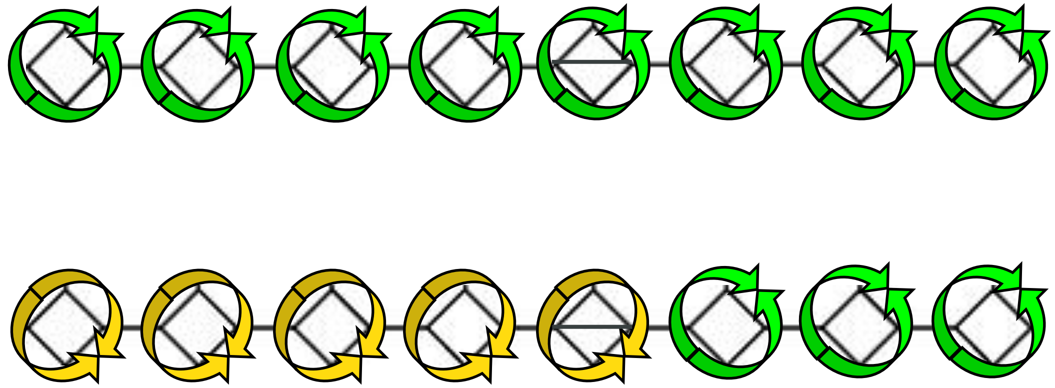
1D plaquette chain (H-defect)

H-defect



1D plaquette chain (H-defect)

H-defect
(2 gs)

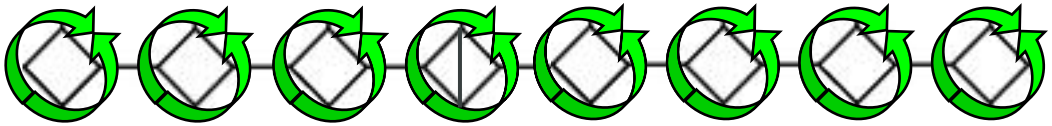


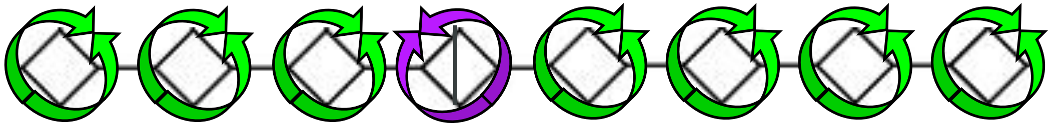
1D plaquette chain (V-defect)

V-defect

1D plaquette chain (V-defect)

V-defect
(2 gs)

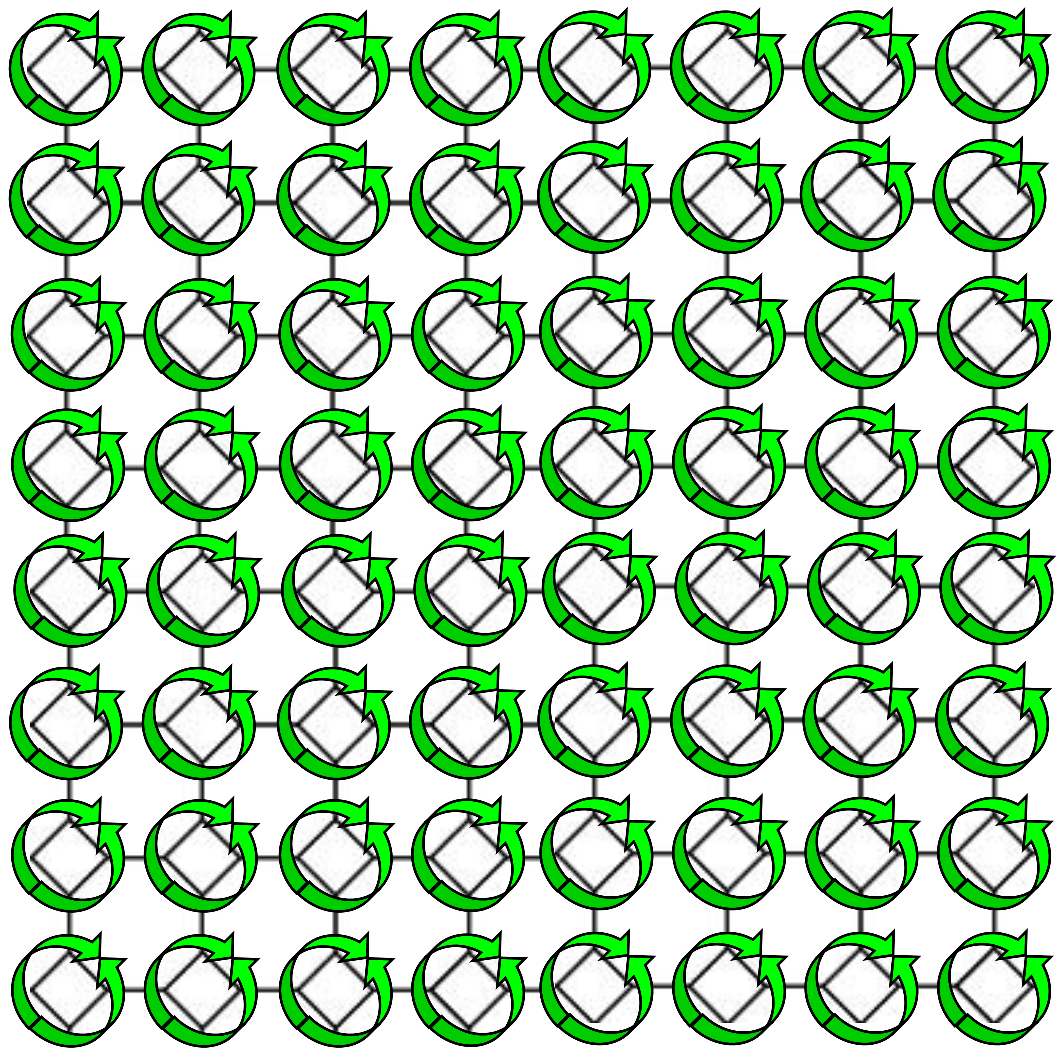




2D lattice (open)

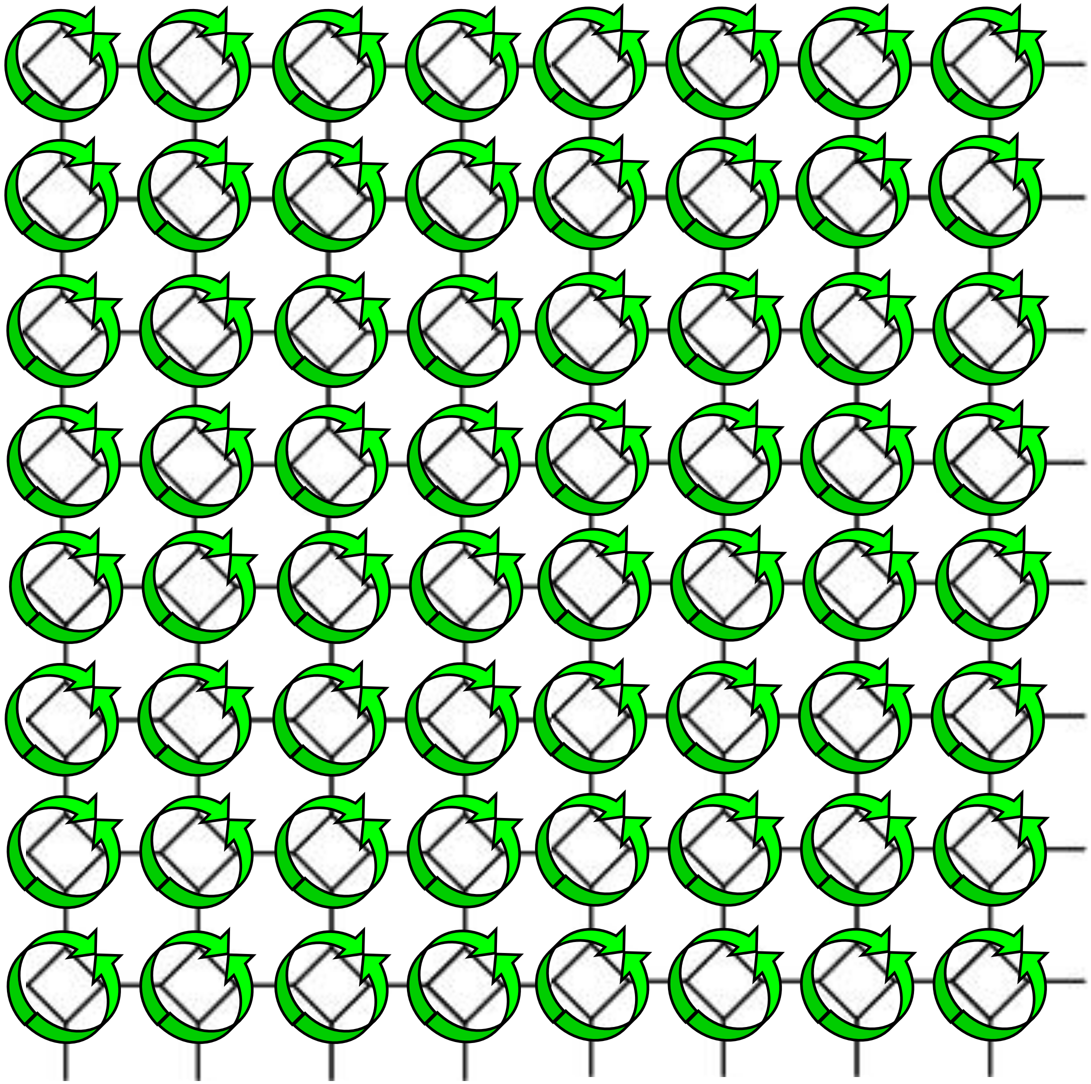
open bc
(1 gs)

“filled
Landau
level”



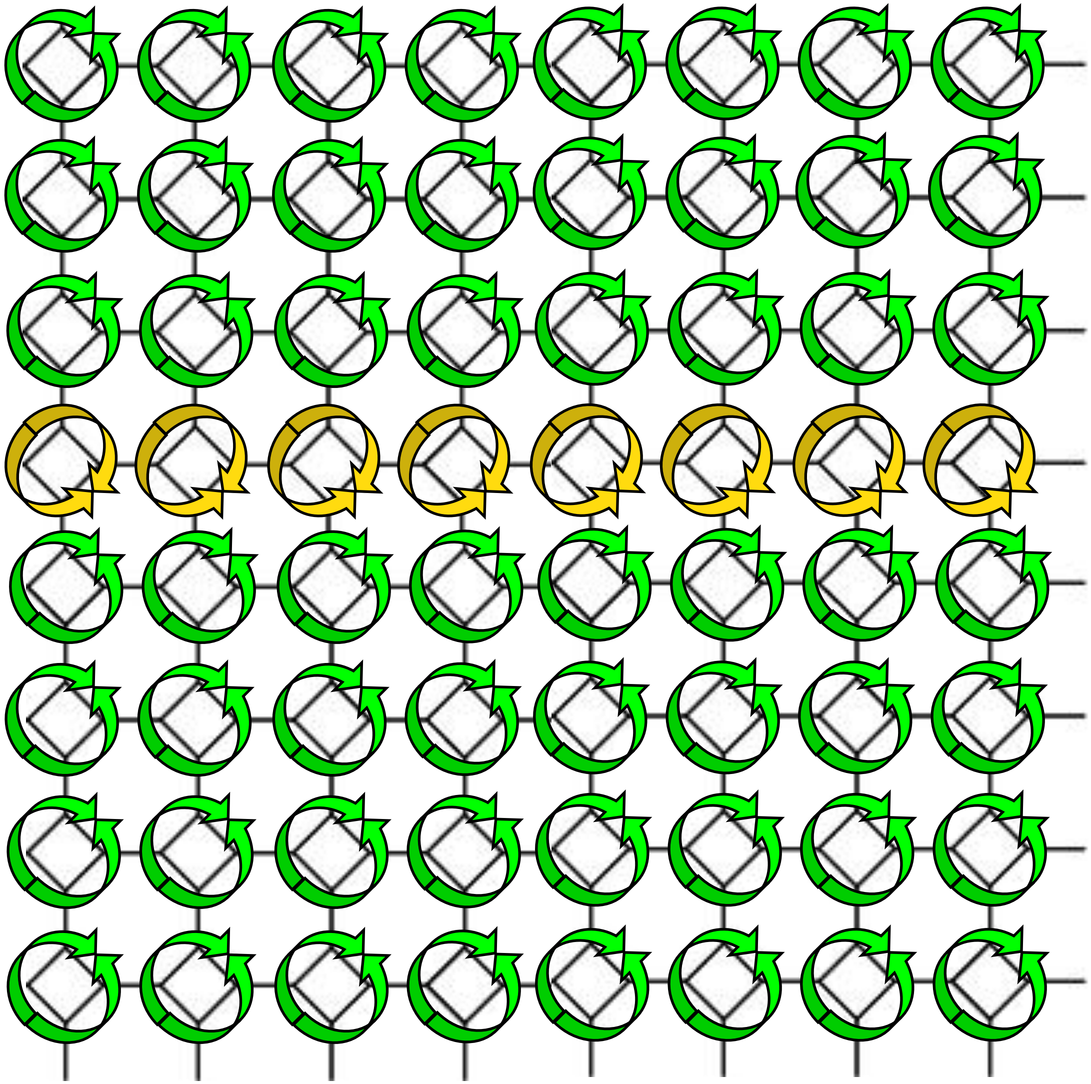
2D lattice (closed)

closed bc
(2^M+2^N-1 gs)



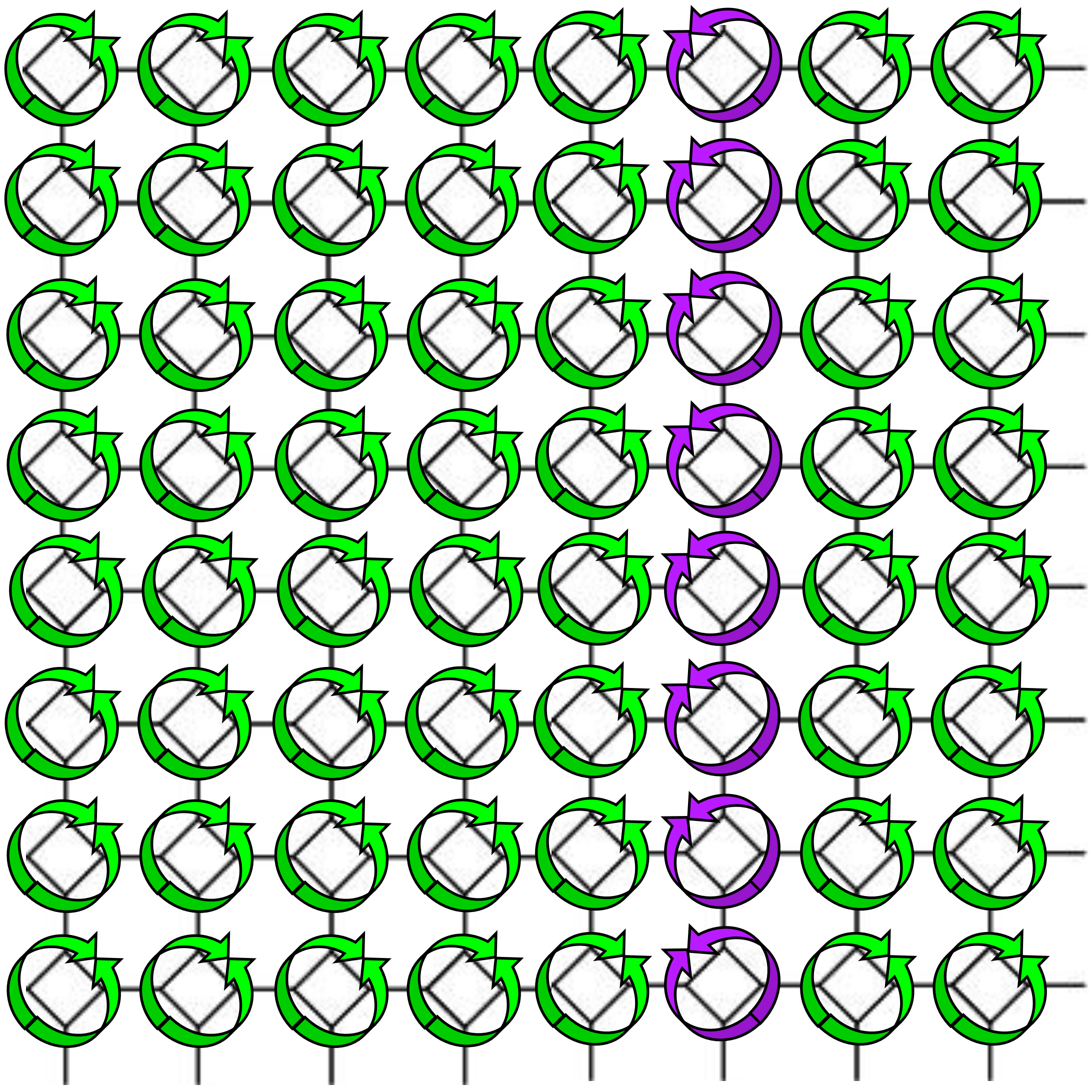
2D lattice (closed)

closed bc
(2^M+2^N-1 gs)



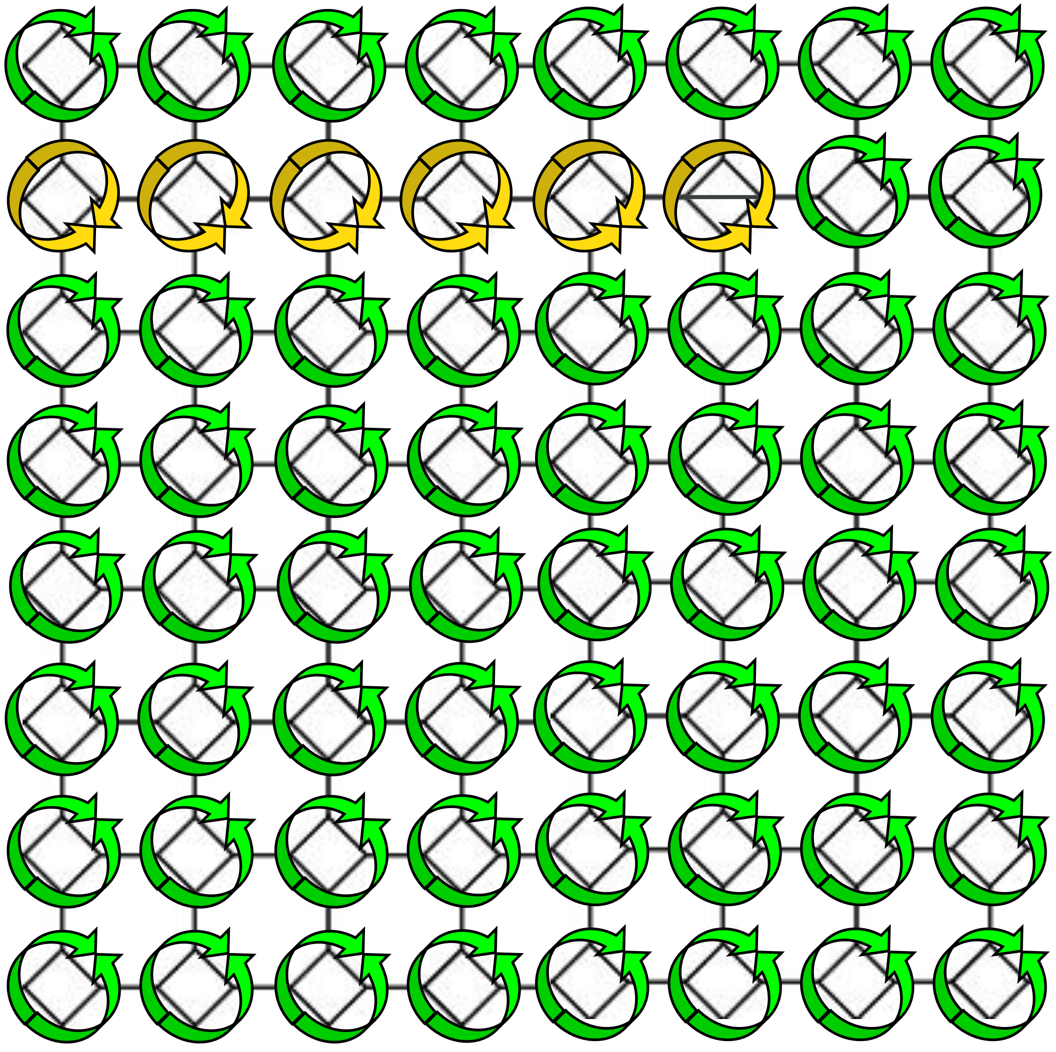
2D lattice (closed)

closed bc
(2^M+2^N-1 gs)



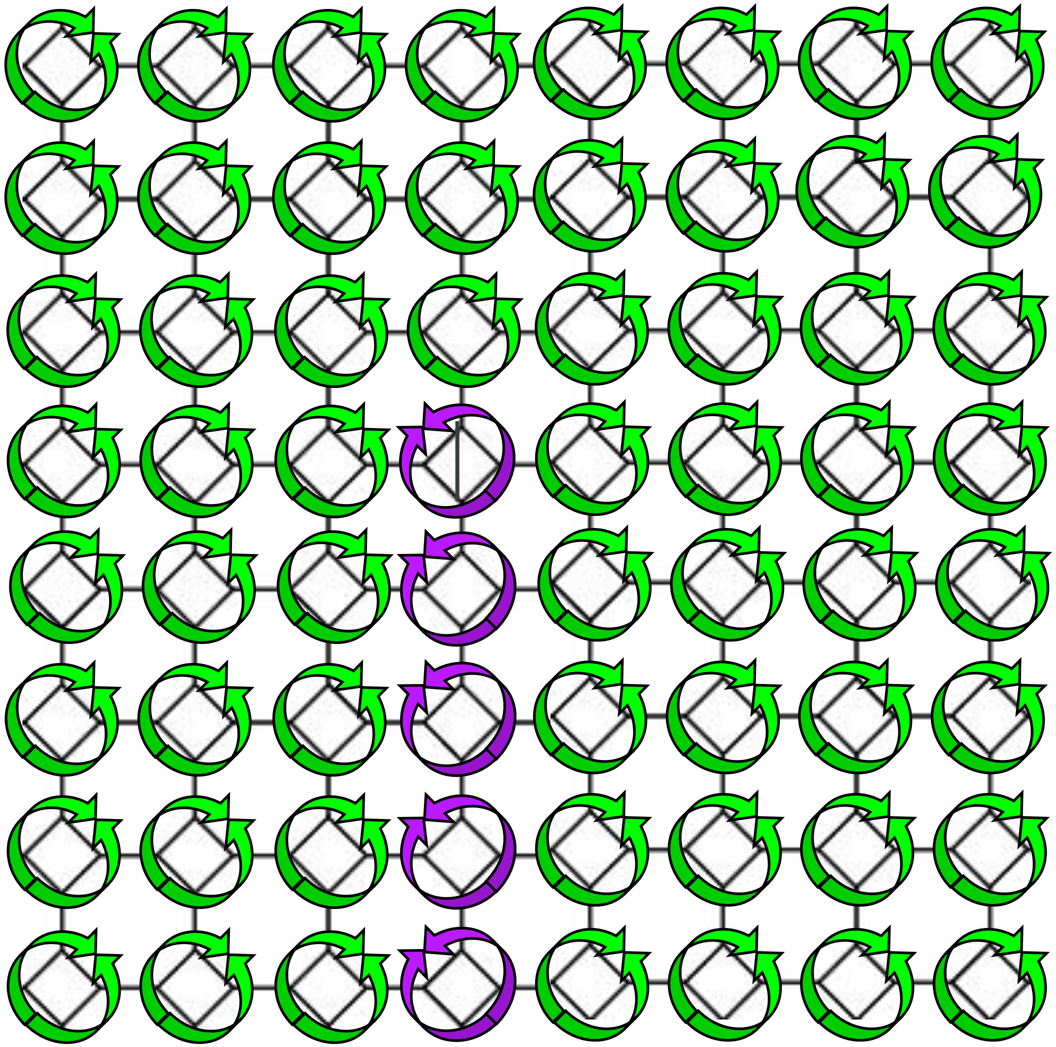
2D lattice (H-defect)

H-defect
(2 gs)



2D lattice (V-defect)

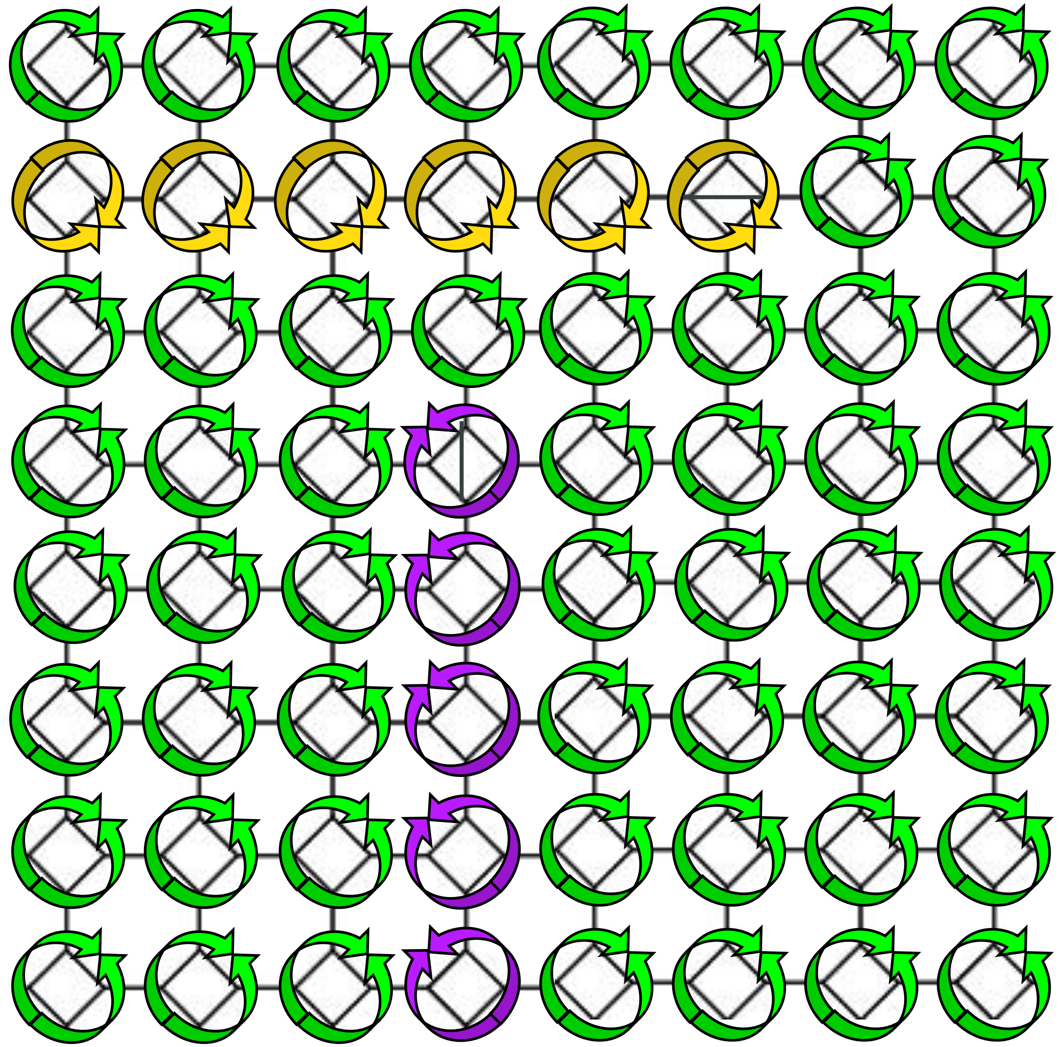
V-defect
(2 gs)



2D lattice (2 defects)

H-defect
plus
V-defect
(4 gs)

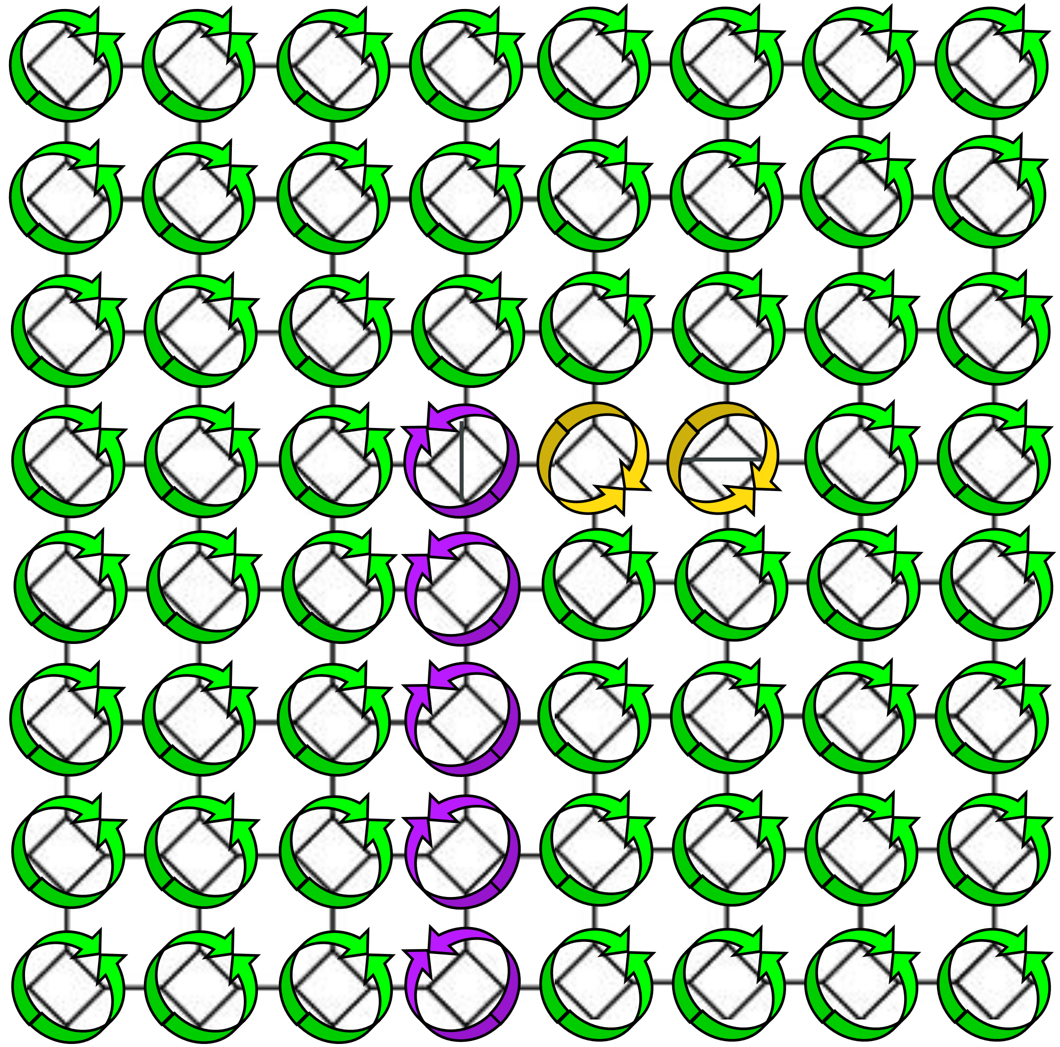
(I)



2D lattice (2 defects)

H-defect
plus
V-defect
(4 gs)

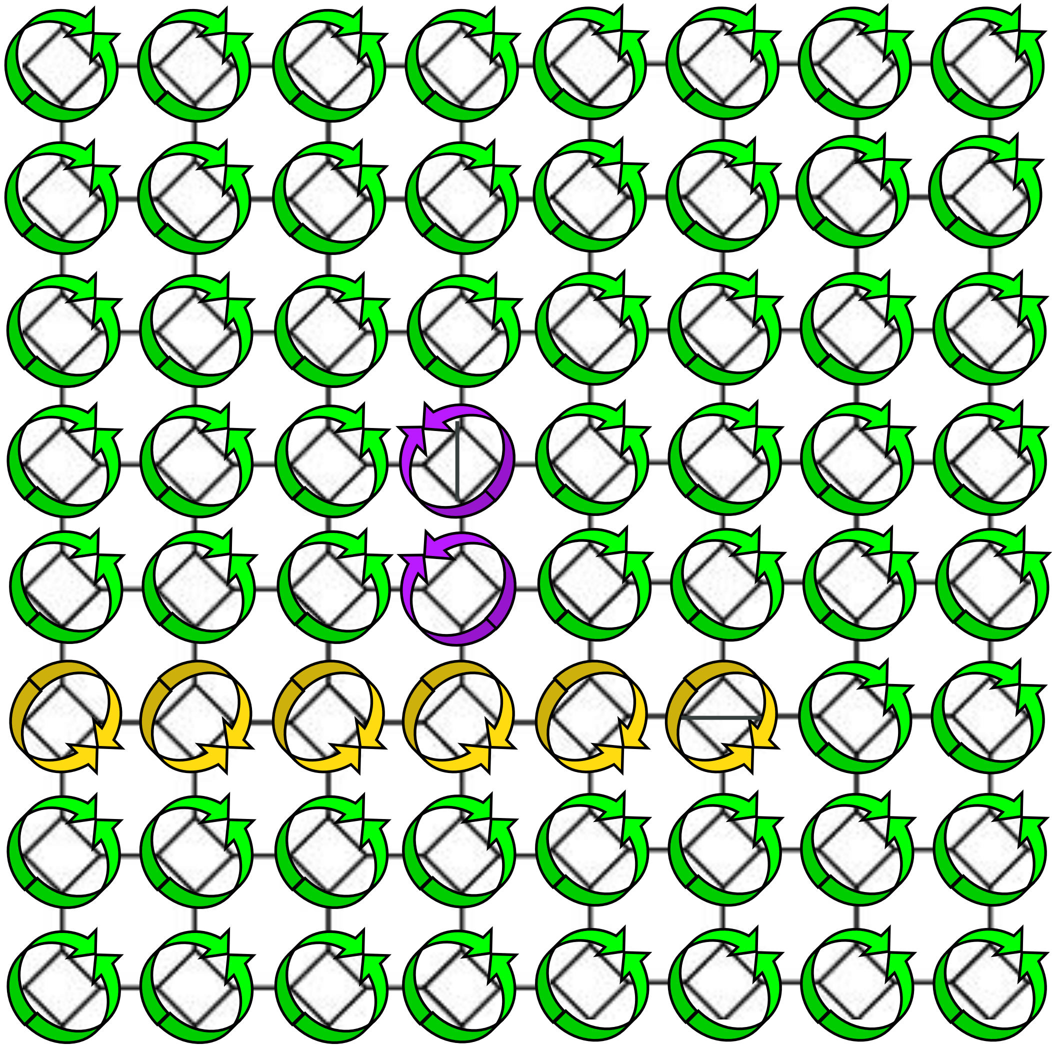
(II)



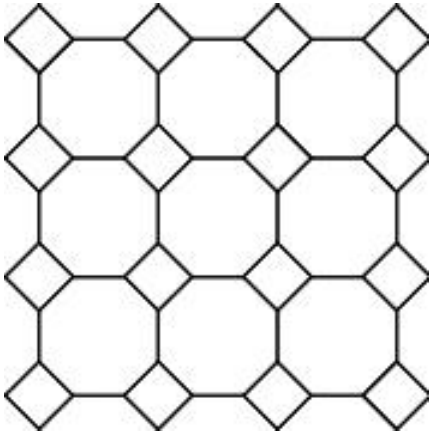
2D lattice (2 defects)

H-defect
plus
V-defect
(4 gs)

(III)



Supertopological phase?

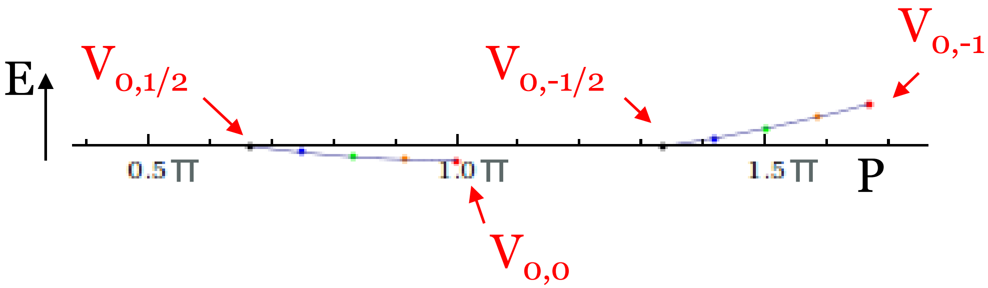


need to understand

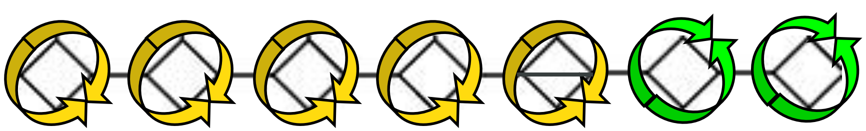
- gap above torus gs?
- edge modes for open system?
- topological interactions and braiding
  of H, V and HV defects?
- ...

Supersymmetric model for lattice fermions

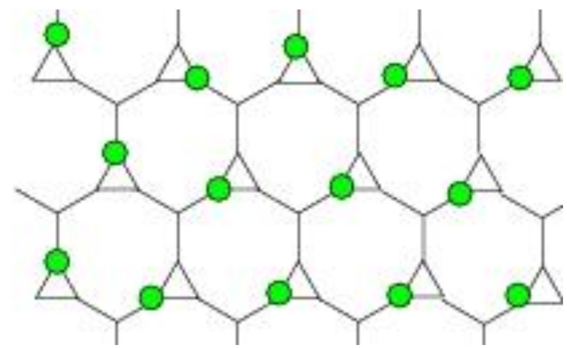
1D: superconformal criticality



2D: supertopological phases



2D: superfrustration



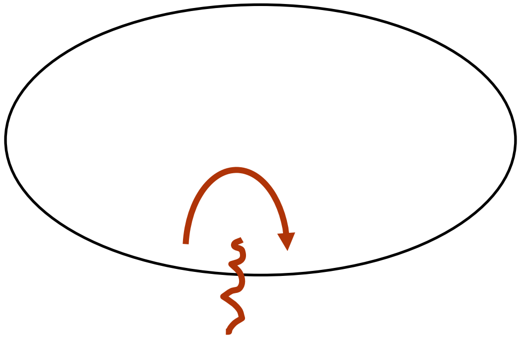
$$\frac{S_{gs}}{N} = \frac{1}{\pi}\int_0^{\pi/3} d\theta \, \ln[2\cos\theta] = 0.16153\ldots$$

Thank you

Boundary twist: spectral flow

wave function picks up a phase exp($2\pi i\alpha$)
as a particle hops over a "boundary"

twist: α: $0 \leftrightarrow 1/2$
"pbc $\leftrightarrow$ apbc" = "R $\leftrightarrow$ NS sector"



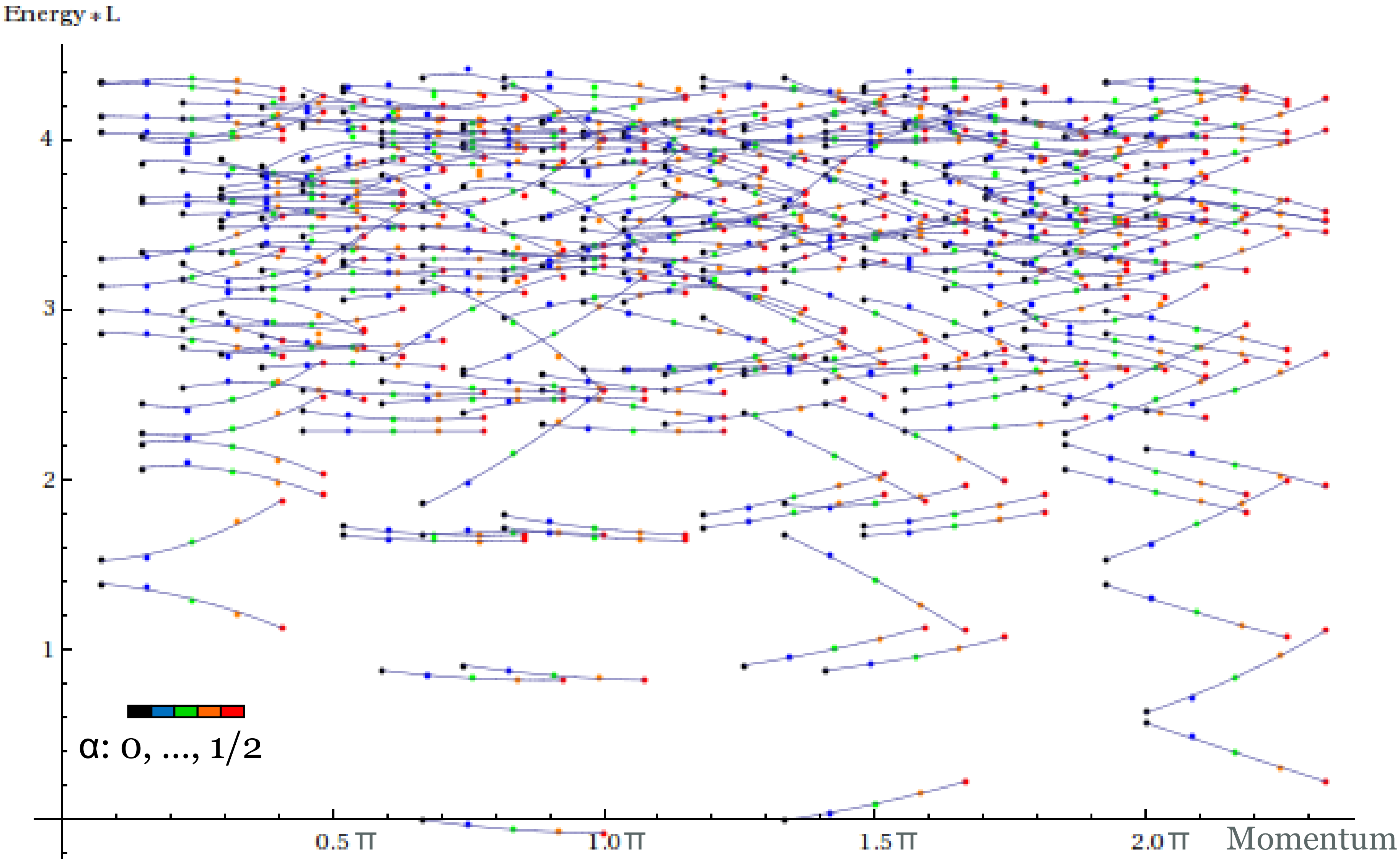
in SCFT: twist operator: $V_{0,\alpha}$
⟶ energy is parabolic function of twist parameter

$$E_\alpha = E_0 - \alpha Q_0 + \alpha^2 c/3$$

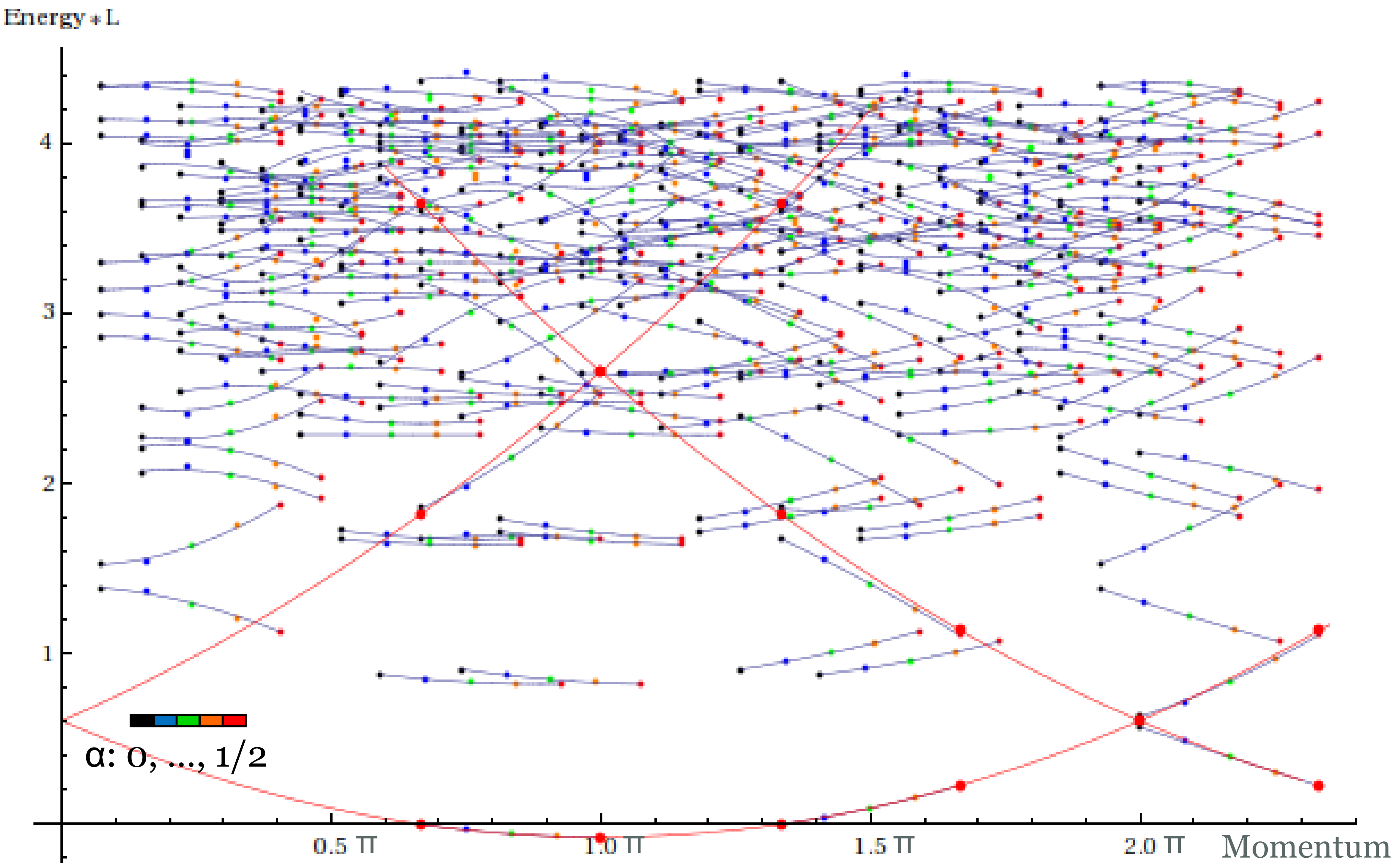
$$\mathrm{R}: \quad \alpha = 0$$

$$\mathrm{NS}: \quad \alpha = 1/2$$

Spectral flow for 1D chain, L=27 , N_f=9



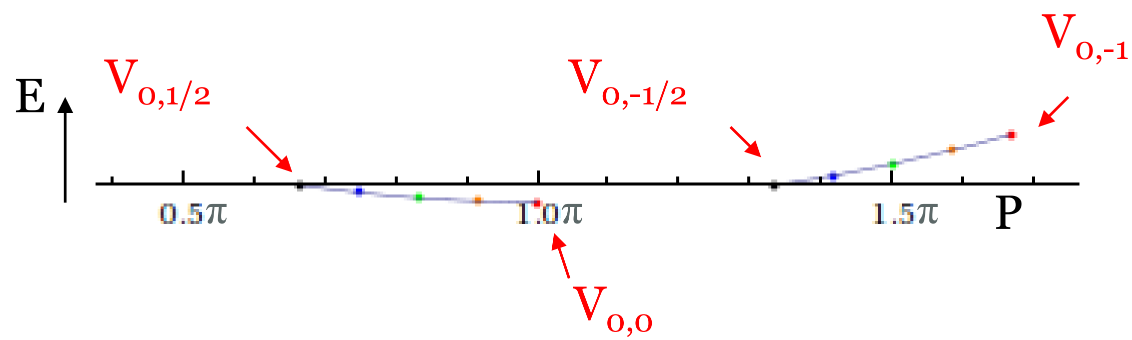
Spectral flow for 1D chain, L=27 , N_f=9



What can we learn from spectral flow?

- 3 fit parameters
- 4 unknowns:

  E, Q_0, c and v_F
- → ratios
- for 1D chain we extract:



numerics

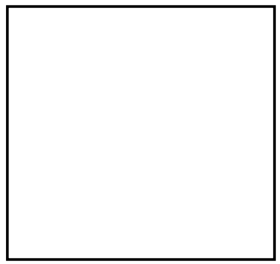
sector	E/c	Q_0/c	c*v_F
R	0	-0.334	3.92
NS	-0.083	0	3.92
R	0	0.342	3.89
NS	0.254	0.675	3.89

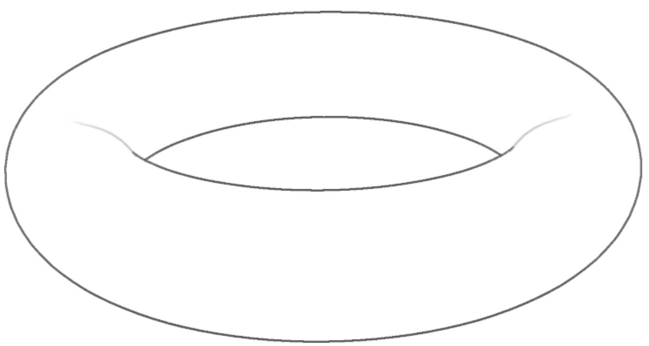
SCFT

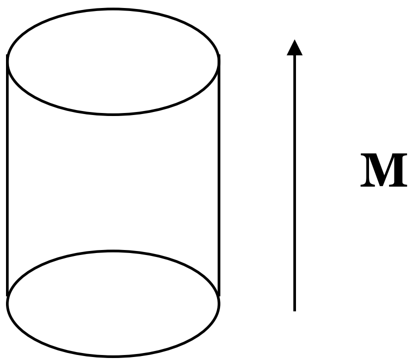
state	E	Q_0
$V_{0,1/2}$	0	-1/3
$V_{0,0}$	-1/12	0
$V_{0,-1/2}$	0	1/3
$V_{0,-1}$	1/4	2/3

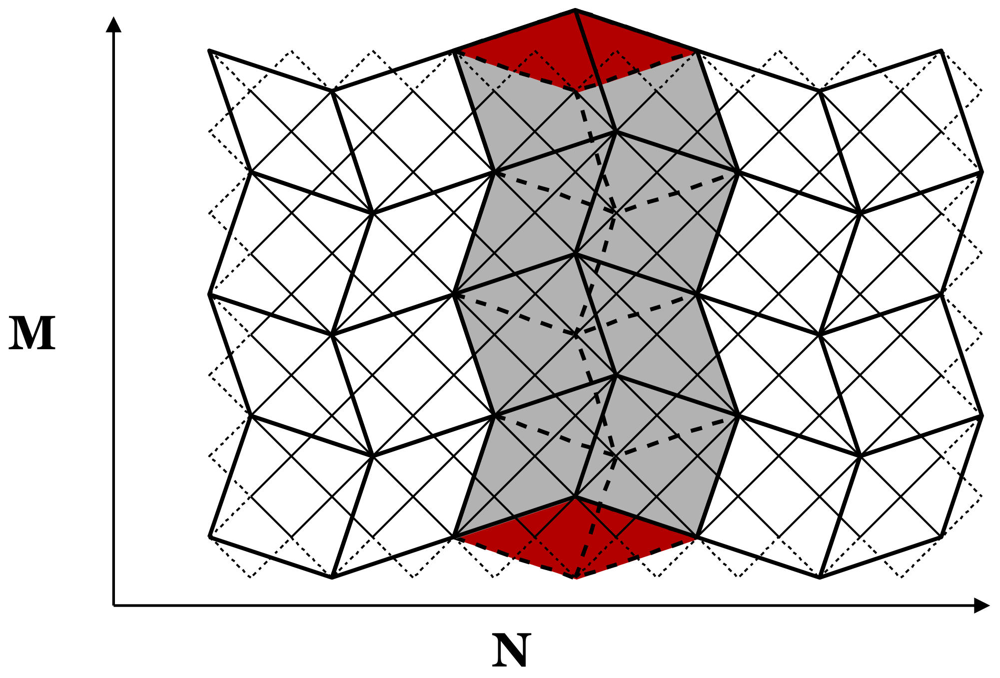
Edge modes (heuristic argument)

- plane: #gs = 1
- cylinder: #gs ~ 2^{M}
- torus : #gs ~ 2^{M+N}



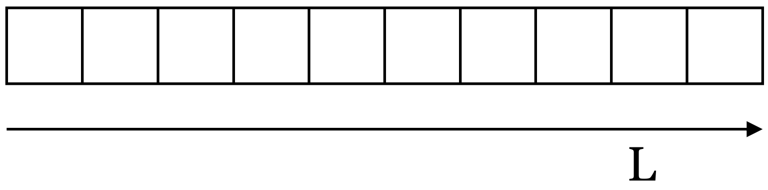


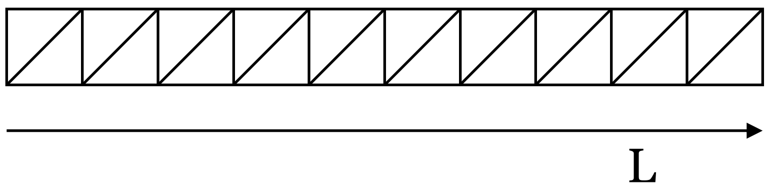


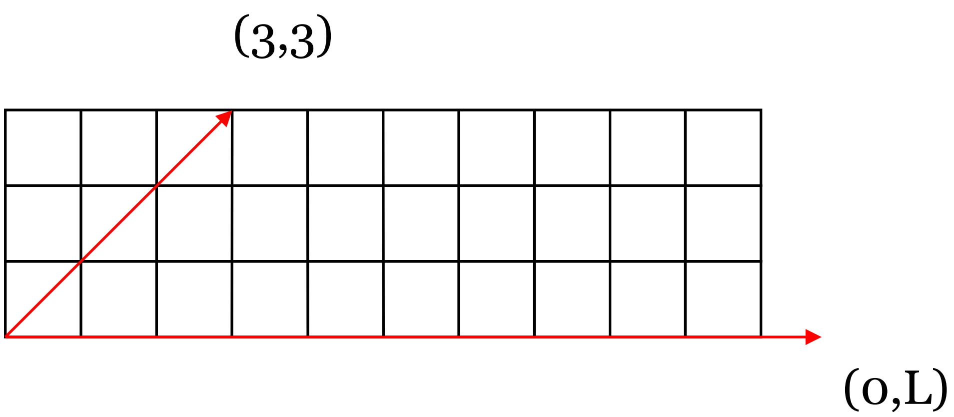


Spectral flow for the square lattice

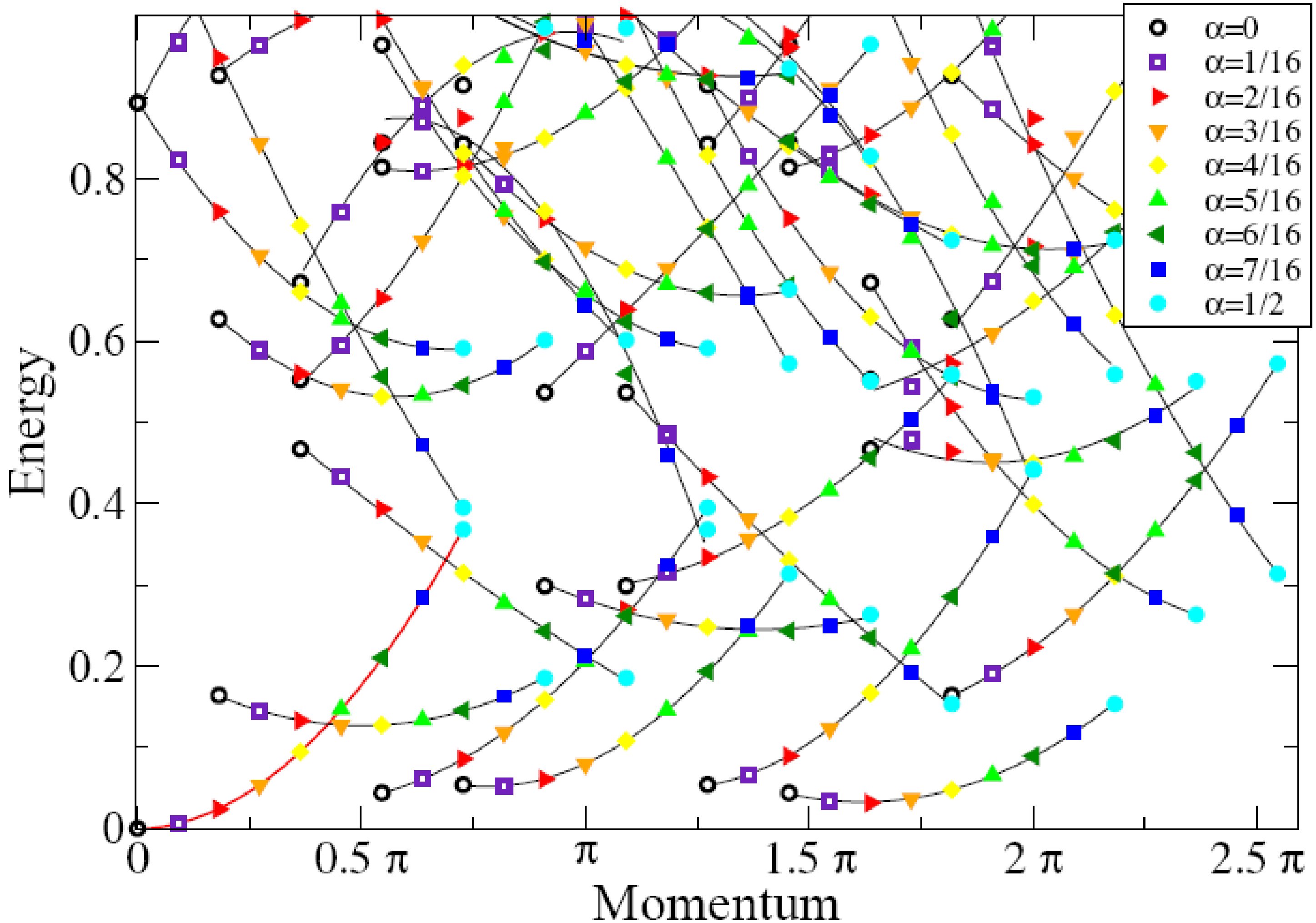
- square ladder
  (2,0)x(0,L)
- zigzag ladder
  (2,1)x(0,L)
  GS for $\nu \in [1/5, 1/4]$
- (3,3)x(0,L)
  fermions can hop past each other







Spectral flow results (3,3)x(0,11), N_f=8



Spectral flow results

$(L,0) \times (3,3)$

N	f	E/c	Q/c
18	4	-0.0851	0.004
36	8	-0.0841	-0.002
15	4	0.0898	0.349
21	4	0.0850	0.337
24	5	0.0850	0.337
30	7	0.0853	0.338
33	**8**	**0.0855**	**0.338**

$(L,0) \times (1,2)$

N	f	E/c	Q/c
9	2	-0.0858	-0.005
18	4	-0.0842	-0.002
27	6	-0.0839	-0.001
17	4	0.0844	0.336
26	6	0.0840	0.335
35	8	0.0839	0.335
14	3	0.2666	0.701
23	5	0.2458	0.657
32	7	0.2432	0.652

$(L,0) \times (0,2)$

N	f	E/c	Q/c
16	4	-0.0897	-0.014
24	6	-0.0889	-0.012
32	8	-0.0885	-0.011
12	3	0.0911	0.350
20	5	0.0900	0.348
28	7	0.0894	0.347
14	4	0.0855	0.338
22	6	0.0849	0.337
30	8	0.0847	0.336

Spectral flow results

$(L,0)\times(3,3)$

N	f	E/c	Q/c
18	4	-0.0851	0.004
36	8	-0.0841	-0.002
15	4	0.0898	0.349
21	4	0.0850	0.337
24	5	0.0850	0.337
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$(L,0)\times(0,2)$

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14	4	0.0855	0.338
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30	8	0.0847	0.336

minimal models in SCFT: $c = \dfrac{3k}{k+2}$

$$E/c = \frac{4l-k}{12k} \text{ and } Q_0/c = \frac{2l}{3k}$$

$$l=0:\ (-1/12,0),\ l=k/2:\ (1/12,1/3),\ l=k:\ (1/4,2/3)$$