Jack wavefunctions and $\mathcal{W}$ theories

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CFT, Jacks and trial wavefunctions in the fractional quantum Hall effect

- In the lowest Landau level, wavefunctions are analytic
- Model wavefunctions can be constructed using Conformal field theory

Parafermions and the Read-Rezayi states

Ground state wavefunctions are polynomials satisfying specific clustering properties: they vanish as a cluster of $k+1$ particles come together

- $\Rightarrow$ Jack polynomials with generalized clustering properties: they vanish with power r as a cluster of $k+1$ particles come together

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Connection with CFT: these Jacks are described as correlators of certain CFTs called $\mathcal{W}$ theories

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Relating Jack wavefunctions and CFT correlation functions

Jack polynomials

- $J_\lambda^\alpha(z_1, \cdots, z_N)$
- eigenvector of the Calogero-Sutherland Hamiltonian

Correlation functions

- $\langle \Psi(z_1)\Psi(z_2)\ldots\Psi(z_N)\rangle$
- Ψ has degenerate descendants $\Rightarrow$ correlation functions satisfy a PDE

Link between these objects

They both satisfy the same Partial Differential Equation !

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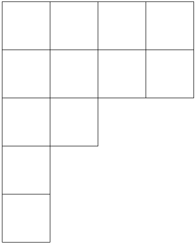
Symmetric Polynomials

Monomial basis $\{m_\lambda\}$

The monomial function m_λ is a symmetric polynomial in n variables $\{z_i, i = 1, \ldots, n\}$:

$$m_\lambda(\{z_i\}) = \mathcal{S}(\prod_{i=1}^{n} z_i^{\lambda_i})$$



Partitions $\lambda = (\lambda_1, \ldots, \lambda_N)$

- λ_i are positive integers
- $\lambda_i \geq \lambda_{i+1}$

For $\lambda = (4, 4, 2, 1, 1)$:

$$m_\lambda = \mathcal{S}\left(z_1^4 z_2^4 z_3^2 z_4 z_5\right)$$

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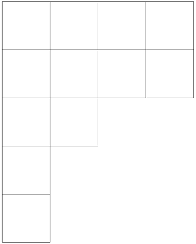

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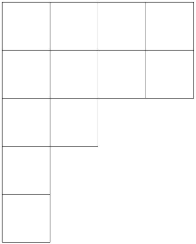

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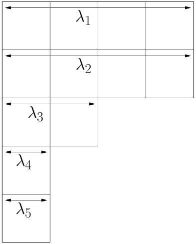

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- symmetric and homogeneous polynomials of N variables
- indexed by partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$
- depend rationally on a parameter α : the expansion over the m_λ basis takes the form

$$J^{\alpha}_{\lambda} = m_\lambda + \sum_{\mu<\lambda} u_{\lambda\mu}(\alpha) m_\mu .$$

The Jacks J^{α}_{λ} are eigenfunctions of the Calogero-Sutherland Hamiltonian :

$$\mathcal{H}^{\mathrm{CS}}(\alpha) = \sum_{i=1}^{N} (z_i \partial_i)^2 + \frac{1}{\alpha} \sum_{i<j} \frac{z_i + z_j}{z_i - z_j} (z_i \partial_i - z_j \partial_j)$$

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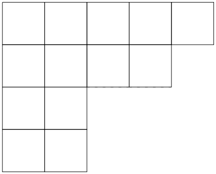

(k, r) admissible partitions

$$\lambda_i - \lambda_{i+k} \geq r$$

Jack Polynomials with (k, r) clustering properties

- for the special value $\alpha = -(k+1)/(r-1)$
- and for a (k, r) admissible partition λ

[Feigin et al (2001)]

- These Jacks are well defined.
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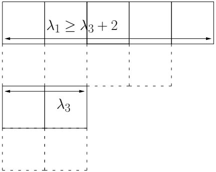

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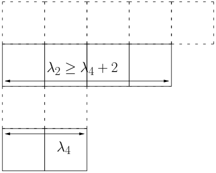

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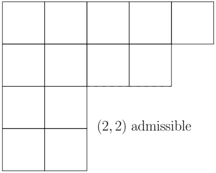

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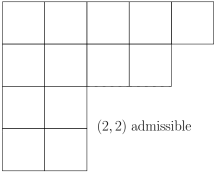

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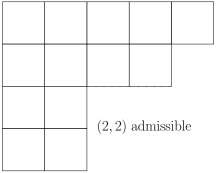

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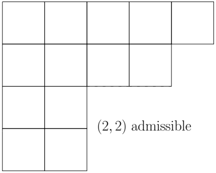

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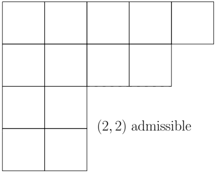

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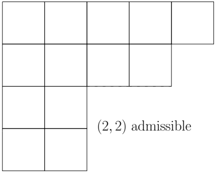

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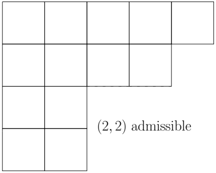

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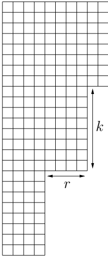

Densest (k, r) admissible partitions

The root partition for the wavefunction with the highest density is given by the occupation numbers

$$\lambda = [k\underbrace{00\ldots0}_{r-1}k\underbrace{00\ldots0}_{r-1}k\ldots]$$

Trial wavefunctions generalizing the Read-Rezayi states

These Jacks have been considered as trial many-body wavefunctions for non-Ablian FQH states [**Bernevig and Haldane (2007)**]

- at (bosonic) filling fraction $\nu = k/r$
- $r = 2$ boils down to the Read-Rezayi $\mathbb{Z}_k$ state
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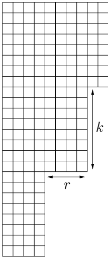

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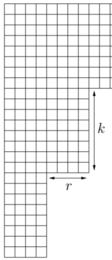

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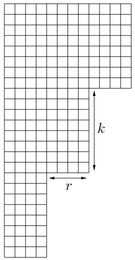

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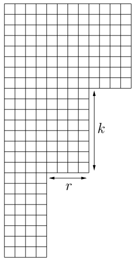

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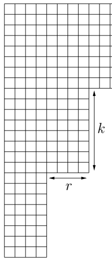

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Conformal field theories as wavefunctions generators

To describe a N particles quantum Hall ground state, a polynomial $P_N(\{z_i\})$ has to be a $\mathrm{SU}(2)$ spin singlet :

$$\begin{aligned} L^- P_N = & \quad \textstyle\sum_i \partial_i P_N(\{z_i\}) & = 0 \\ L^z P_N = & \quad \textstyle\sum_i \left(z_i \partial_i - \frac{N_\phi}{2} \right) P_N(\{z_i\}) & = 0 \\ L^+ P_N = & \quad \textstyle\sum_i \left(-z_i^2 \partial_i + z_i N_\phi \right) P_N(\{z_i\}) & = 0 \end{aligned}$$

All these properties are automatically ensured by global conformal invariance for **single channel** correlators :

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- additional $\mathbb{Z}_k$ symmetry encoded in the fusion rules of a set of chiral operators $\Psi_q(z)$:

$$[\Psi_n] \times [\Psi_m] = [\Psi_{n+m}]$$

consistency (bootstrap) fixes the conformal dimensions :

$$\Delta_n = \frac{r}{2} \frac{n(k-n)}{k}$$

- $r \geq 2$ is an integer :
  - $r = 2$: FZ parafermions **[Fateev, Zamolodchikov (1985)]** $\Rightarrow$ Read-Rezayi states
  - $r = 3$: (for k even) non unitary **[Jacob, Mathieu (2002)]** $\Rightarrow$ Gaffnian
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Parafermionic correlators and clustering properties

Parafermionic correlators

Let's consider a parafermionic CFT $\mathbb{Z}_k^{(r)}$. The following function is a symmetric polynomial

$$P_N^{(k,r)}(\{z_i\}) \;\hat{=}\; \langle \Psi(z_1)\ldots\Psi(z_N)\rangle \prod_{i<j}(z_i - z_j)^{2\Delta_1-\Delta_2}$$
$$= \langle \Psi(z_1)\ldots\Psi(z_N)\rangle \prod_{i<j}(z_i - z_j)^{r/k} \,.$$

and is a $\mathrm{SU}(2)$ singlet.

Clustering properties

More interestingly, this polynomial vanishes as r powers when $k+1$ particles come to the same point !

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WA_{k-1} conformal field theories : some basic properties

Extended conformal symmetry

- These theories have first been introduced in the case $k = 3$ by Fateev and Zamolodchikov (1987) : the so-called $\mathcal{W}_3$ theory
- generalized to any k by Fateev and Lykyanov (1988)
- they are the prototype of CFT with extended symmetries : in addition to the stress-energy tensor $T(z)$, the chiral algebra contains $k-2$ currents $W^{(s)}(z)$ of integer spin $s = 3, \ldots, k-1$.

Minimal models

For a discrete serie of values of the central charge, these CFT are minimal. The central charge of the $\mathrm{WA}_{k-1}(p, p')$ models is:

$$c(p, p') = (k-1)\left(1 - \frac{k(k+1)(p-p')^2}{pp'}\right)$$

p and p' are coprimes, and these models are unitary for $p' = p + 1$.

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WA_1 theories : minimal models of Virasoro algebra

Virasoro algebra

The conformal symmetry is encoded in a single current : the stress-enery tensor $T(z)$. Its mode obey the celebrated Virasoro algebra :

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}$$

Primary fields

Primary fields are anihilated by all positive modes L_n :

$$T(z)\Phi_\Delta(0) = \frac{\Delta}{z^2}\Phi_\Delta(0) + \frac{1}{z}\partial\Phi_\Delta(0) + O(1)$$

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Minimal models of Virasoro algebra $\mathrm{WA}_1(p, p')$

- central charge

$$c = 1 - \frac{6(p - p')^2}{pp'}$$

- finite number of primary fields $\Phi_{(n|n')}$ labeled by the Kac table :

$$1 \leq n \leq p' - 1$$
$$1 \leq n' \leq p - 1$$

- with conformal dimension

$$\Delta_{(n,n')} = \frac{(np - n'p')^2 - (p - p')^2}{4pp'}$$

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$\mathrm{WA}_1(3, 2+r)$ theories and parafermions

$$\begin{tikzcd}[row sep=small] & \Phi_{(n|n'+1)} \\ \Phi_{(1|2)} \times \Phi_{(n|n')} \arrow[ur] \arrow[dr] & \\ & \Phi_{(n|n'-1)} \end{tikzcd}$$

Fermionic field $\Phi_{(1|2)}$

In the theory $\mathrm{WA}_1(3, 2+r)$ the field $\Psi = \Phi_{(1|2)}$ obey the fusion rules :

$$\Psi \times \Psi = \mathbb{I}$$

and its conformal dimension is $\Delta_{(1|2)} = \frac{r}{4}$

$\Rightarrow$ This is a particular realization of a $\mathbb{Z}_2^{(r)}$ parafermionic field theory

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Null vector at level 2 for the field $\Psi = \Phi_{(1|2)}$

The following field

$$\chi_2 = \left(L_{-2} - \frac{3}{r+2} L_{-1}^2 \right) \Psi$$

This degeneracy translates into a PDE for correlators:

$$\partial^2 \langle \Psi(z) \Phi_1(w_1) \Phi_2(w_2) \cdots \rangle = \frac{r+2}{3} \langle L_{-2} \Psi(z) \Phi_1(w_1) \Phi_2(w_2) \cdots \rangle$$

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$$\chi_2 = \left(L_{-2} - \frac{3}{r+2} L_{-1}^2 \right) \Psi$$

This degeneracy translates into a PDE for correlators:

$$\partial^2 \langle \Psi(z) \Phi_1(w_1) \Phi_2(w_2) \cdots \rangle = \frac{r+2}{3} \langle L_{-2} \Psi(z) \Phi_1(w_1) \Phi_2(w_2) \cdots \rangle$$

Virasoro modes have a geometric interpretation

$$\langle (L_{-2}\Phi(z)) \Phi_1(w_1) \Phi_2(w_2) \cdots \rangle = \sum_j \hat{\mathcal{D}}_j \langle \Phi(z) \Phi_1(w_1) \Phi_2(w_2) \cdots \rangle$$

where $\mathcal{D}_j$ are differential operators acting on the j^{th} field:

$$\hat{\mathcal{D}}_j = \frac{1}{(z-w_j)^2} \Delta_j + \frac{1}{(z-w_j)} \partial_{w_j}$$

$\mathrm{WA}_1(3, 2+r)$ theories and PDE

Null vector at level 2

$$\sum_{i=1}^{N} z_i^2 \partial_i^2 \langle \Psi(z_1)\Psi(z_2)\cdots\Psi(z_N)\rangle = \frac{r+2}{3} \sum_{i=1}^{N} z_i^2 L_{-2}^{(i)} \langle \Psi(z_1)\Psi(z_2)\cdots\Psi(z_N)\rangle$$

translates into the following PDE :

$$\mathcal{H}^{\mathrm{WA}_1}(r)\langle \Psi(z_1)\Psi(z_2)\cdots\Psi(z_N)\rangle = 0$$

$\mathcal{H}^{\mathrm{WA}_1}$ is a differential operator of order 2:

$$\sum_i (z_i\partial_i)^2 + \gamma_1(r) \sum_{i\neq j} \frac{z_j^2}{(z_j - z_i)^2} + \gamma_2(r) \sum_{i\neq j} \frac{z_i z_j(\partial_j - \partial_i)}{(z_j - z_i)} + N\gamma_3(r)$$

$$\gamma_1 = -\frac{r(r+2)}{12}, \qquad \gamma_2 = \frac{r+2}{6} \quad \text{et} \quad \gamma_3 = -\frac{r(r-1)}{12}$$

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$\mathrm{WA}_1(3, 2+r)$ theories and Jacks **[Cardy (2004)]**

Jack polynomial

By restauring the charge part, we consider the following polynomial wavefunction :

$$P_N \hat{=} \langle \Psi(z_1) \ldots \Psi(z_N) \rangle \prod_{i<j} (z_i - z_j)^{r/2} .$$

$\Rightarrow$ It is an eigenvalue of the Calogero-Sutherland Hamiltonian for $\alpha = -\frac{2+1}{r-1}$, corresponding to the densest $(2, r)$ admissible partition !

This proves the following relation :

$$\langle \Psi(z_1) \ldots \Psi(z_N) \rangle \prod_{i<j} (z_i - z_j)^{r/2} . = J^{-3/(r-1)}_{[20^{r-1}20^{r-1}\ldots 2]}$$

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WA_2 algebra

The algebra is generated by two currents $T(z)$ and $W(z)$:

$$
\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \\
[L_n, W_m] &= (2n-m)\,W_{n+m} \\
[W_n, W_m] &= \frac{16}{22+5c}(n-m)\Lambda_{n+m} + \frac{c}{360}n(n^2-1)(n^2-4)\delta_{n+m,0} \\
&+ (n-m)\left[\frac{(n+m+2)(n+m+3)}{15} - \frac{(n+2)(m+2)}{6}\right]L_{n+m}
\end{aligned}
$$

Primary fields $\Phi_{\Delta,\omega}$

$$
\begin{aligned}
T(z)\Phi_{\Delta,\omega}(0) &= \frac{\Delta\Phi(0)}{z^2} + \frac{\partial\Phi(0)}{z} + \dots \\
W(z)\Phi_{\Delta,\omega}(0) &= \frac{\omega\Phi(0)}{z^3} + \frac{W_{-1}\Phi(0)}{z^2} + \frac{W_{-2}\Phi(0)}{z} + \dots
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$\mathrm{WA}_2(p, p')$ minimal models

- central charge

$$c = 2\left(1 - \frac{12(p-p')^2}{pp'}\right)$$

- finite number of primary fields $\Phi_{(n_1,n_2|n_1',n_2')}$ labeled by the Kac table :

$$n_1 + n_2 \leq p' - 1$$
$$n_1' + n_2' \leq p - 1$$

- with conformal dimension

$$\Delta_{(n_1,n_2|n_1',n_2')} = \frac{(\vec{n}p - \vec{n}'p')^2 - \vec{\rho}^2(p-p')^2}{2pp'}$$

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$\mathrm{WA}_2(4, 3+r)$ CFT : parafermionic $\mathbb{Z}_3^{(r)}$ theories

$$
\Phi_{(11|21)} \times \Phi_{(n_1,n_2|n_1',n_2')} \longrightarrow \left\{ \begin{array}{l} \Phi_{(n_1,n_2|n_1'+1,n_2')} \\ \Phi_{(n_1,n_2|n_1'-1,n_2'+1)} \\ \Phi_{(n_1,n_2|n_1',n_2'-1)} \end{array} \right.
$$

$\mathrm{WA}_2(4, 3+r)$ CFT : parafermionic $\mathbb{Z}_3^{(r)}$ theories

$$\begin{tikzcd}[column sep=large, row sep=small]
 & \Phi_{(1,1|3,1)} \\
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 & \Phi_{(1,1|2,0)}
\end{tikzcd}$$

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$$\Phi_{(1,1|2,1)} \times \Phi_{(1,1|2,1)} \longrightarrow \Phi_{(1,1|3,1)},\quad \Phi_{(1,1|1,2)},\quad \cancel{\Phi_{(1,1|2,0)}}$$

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\qquad
\boxed{
\begin{array}{c}
\Psi \times \Psi = \Psi^\dagger \\
\\
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Parafermionic fields $\Psi = \Phi_{(1,1|2,1)}$ and $\Psi^\dagger = \Phi_{(1,1|1,2)}$

In the theory $\mathrm{WA}_2(4, 3+r)$ the field $\Psi = \Phi_{(1|2)}$ obey the fusion rules :

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and their conformal dimension is $\Delta_{(1,1|2,1)} = \Delta_{(1,1|1,2)} = \frac{r}{3}$

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Null vectors... but !

Null vectors at level 1 and 2 for the field $\Psi = \Phi_{(1,1|2,1)}$

The parafermionic field admits the following null vectors :

$$\begin{aligned} \left(W_{-1} - \frac{3\omega}{2\Delta} L_{-1}\right)\Psi &= 0 \\ \left(W_{-2} - \frac{12\omega}{\Delta(5\Delta+1)} L_{-1}^2 - \frac{6\omega(\Delta+1)}{\Delta(5\Delta+1)} L_{-2}\right)\Psi &= 0 \end{aligned}$$

But !

No geometrical interpretation of the modes W_n ...

How to get rid of these modes ?

Using the asymptotic behavior of the current $W(z)$

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Asymptotic behavior of $W(z)$

$$W(z) = \sum_n \frac{W_n}{z^{n+3}} \quad \text{and} \quad W(z) \overset{z\to\infty}{\sim} \frac{1}{z^6}$$

Correlation functions of the form $\langle W(z)\Phi_1(z_1)\cdots\Phi_N(z_N)\rangle$ can be expanded into :

$$\langle W(z)\Phi_1(z_1)\cdots\Phi_N(z_N)\rangle = \sum_{j=1}^{N}\left(\frac{\omega_j}{(z-z_j)^3} + \frac{W_{-1}^{(j)}}{(z-z_j)^2} + \frac{W_{-2}^{(j)}}{(z-z_j)}\right)\langle\Phi_1(z_1)\cdots\Phi_N(z_N)\rangle$$

By comparing this expansion and the asymptotic behavior of the current $W(z)$ we get five relations, including :

$$\sum_{j=1}^{N}\left(z_j^2 W_{-2}^{(j)} + 2z_j W_{-1}^{(j)} + \omega_j\right)\langle\Phi_1(z_1)\Phi_2(z_2)\cdots\Phi_N(z_N)\rangle = 0$$

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Partial differential equation

Plugging the null vectors

$$W_{-1}\Psi = \frac{3\omega}{2\Delta} L_{-1}\Psi$$
$$W_{-2}\Psi = \left(\frac{12\omega}{\Delta(5\Delta+1)} L_{-1}^2 + \frac{6\omega(\Delta+1)}{\Delta(5\Delta+1)} L_{-2} \right) \Psi$$

into the equation

$$\sum_{j=1}^{N} \left(z_j^2 \underbrace{W_{-2}^{(j)}} + 2 z_j \underbrace{W_{-1}^{(j)}} + \omega_j \right) \langle \Psi(z_1)\Psi(z_2)\cdots\Psi(z_N)\rangle = 0$$

$\Rightarrow$ We are left with Virasoro modes only !

and we get a partial differential equation for $\langle \Psi(z_1)\Psi(z_2)\cdots\Psi(z_N)\rangle$

Partial differential equation

Plugging the null vectors

$$W_{-1}\Psi = \frac{3\omega}{2\Delta} L_{-1}\Psi$$
$$W_{-2}\Psi = \left(\frac{12\omega}{\Delta(5\Delta+1)} L_{-1}^2 + \frac{6\omega(\Delta+1)}{\Delta(5\Delta+1)} L_{-2} \right) \Psi$$

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$$\mathcal{H}^{\mathrm{WA}_2}(r)\langle\Psi(z_1)\Psi(z_2)\cdots\Psi(z_N)\rangle = 0$$

où $\mathcal{H}^{\mathrm{WA}_2}$ is a differential operator of order 2.

Restauring the charge part, this PDE becomes an eigenvector equation for the Calogero-Sutherland Hamiltonian for $\alpha = -\frac{3+1}{r-1}$, corresponding to the densest $(3, r)$ admissible partition !

This proves the conjecture for $k = 3$:

$$\langle\Psi(z_1)\ldots\Psi(z_N)\rangle \prod_{i<j}(z_i - z_j)^{r/2} . = J^{-4/(r-1)}_{[30^{r-1}30^{r-1}\cdots 3]}$$

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WA_{k-1} theories

WA_{k-1} algebra

The algebra is generated by $k-1$ currents $W^{(s)}(z)$:

$\Rightarrow$ Commutation relations are rather untractable

Huge number of descendants

Level n	Number of fields $p(n)$	Descendants
0	1	Φ
1	$k-1$	$W_{-1}^{(2)}\Phi,\ W_{-1}^{(3)}\Phi, \ldots\ W_{-1}^{(k)}\Phi$
2	$(k-1)(k+2)/2$	$W_{-2}^{(i)}\Phi,\ W_{-1}^{(i)}W_{-1}^{(j)}\Phi$

Generating function :

$$\Phi_k(x) = \left(\frac{1}{\varphi(x)}\right)^{k-1} = \prod_{n=1}^{\infty}\left(\frac{1}{1-x^n}\right)^{k-1} = \sum_{n=0}^{\infty} p(n)x^n$$

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Parafermionic fields in $\mathrm{WA}_{k-1}(k+1, k+r)$ theories

Parafermions

The $\mathrm{WA}_{k-1}(k+1, k+r)$ theories are a special case of $\mathbb{Z}_k^{(r)}$ parafermionic theories, with :

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\Psi_1 &= \Phi_{(1,1,\ldots,1|2,1,\ldots 1)} \\
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Null vectors

In order to derive a Calogero-Sutherland type PDE, it is sufficient to show that these parafermionic field have null vectors of the form:

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\left(W_{-1}^{(3)} + \beta L_{-1}\right) \Psi = 0 \\
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Characters

$$\chi_{(\lambda|\mu)}(x) = \Phi_k(x) \sum_{w \in \hat{W}} \epsilon(w) x^{\Delta_{(w(\lambda)|\mu)}}$$

counts the number of descendants of the primary field $\Phi_{(\lambda|\mu)}$

For the field $\Phi_{(1,1,\dots,1|2,1,\dots 1)}$

The parafermionic field $\Psi = \Phi_{(1,1,\dots,1|2,1,\dots 1)}$ has :

- only has one state at level one: $L_{-1}\Psi$
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Partial differential equation in the general case (k, r)

PDE for parafermionic correlators

$$\mathcal{H}^{\mathrm{WA}_{k-1}}(r)\langle\Psi(z_1)\Psi(z_2)\cdots\Psi(z_N)\rangle = 0$$

where $\mathcal{H}^{\mathrm{WA}_{k-1}}$ is a differential operator of order 2:

$$\sum_j (z_j\partial_j)^2 + \gamma_1 \sum_{i\neq j} \frac{z_j^2}{(z_j - z_i)^2} + \gamma_2 \sum_{i\neq j} \frac{z_i z_j(\partial_j - \partial_i)}{(z_j - z_i)} + N\gamma_3$$

with

$$\gamma_1(k, r) = -\frac{r(rk - r + k^2 - k)}{k^2(k+1)},$$

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Polynomial

The polynomial $P_N^{(k,r)}$ defined as :

$$P_N^{(k,r)} = \langle \Psi(z_1)\Psi(z_2)\cdots\Psi(z_N)\rangle \prod_{i<j}(z_i - z_j)^{\frac{r}{k}}$$

is an eigenvector of the Calogero-Sutherland Hamiltonian, with the eigenvalue corresponding to the parameters :

$$\alpha = -\frac{k+1}{r-1}$$

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Conclusion

- By using the Ward identities associated to the spin 3 curent $W^{(3)}(z)$ and the degeneracy properties of the Ψ_1 and Ψ_{k-1} representations, we showed that their $N-$points correlation functions satisfy a second order differential equation.
- This equation can be transformed into a Calogero Hamiltonian with negative rational coupling $\alpha = -(k+1)/(r-1)$.

$\Rightarrow$ this proves that the $N-$points correlation functions of Ψ can be written in term of a single Jack polynomial.

- This relation between Jacks and $\mathcal{W}$ theories is an interesting result for $\mathcal{W}$ conformal field theories, since computing correlation function of these higher spin symmetry CFTs is usually an hard problem.

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Perspectives

- Wavefunctions with quasiholes are related to the follwing correlators:

$$\langle \sigma(w_1) \cdots \sigma(w_M) \Psi(z_1) \cdots \Psi(z_N) \rangle$$

These correlators also obey a partial differential equation

This could have some interesting applications, even for the Read-Rezayi states !

- Coulomb gas techniques associated with these CFTs :
  $\rightarrow$ integral representation of these Jacks
  $\rightarrow$ integral representation of the conformal blocks for quasihole wavefunctions
  This is interesting to investigate the properties of the quasihole excitations, and to get information beyond their braiding properties and the dimension of the Hilbert space.

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