Strongly Correlated

Ultracold Quantum Gases

Thomas Busch
Physics Department

science foundation ireland
fondúireacht eolaíochta éireann

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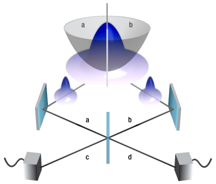
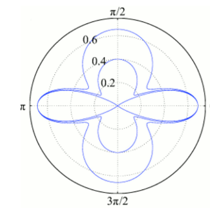
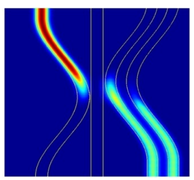
Ultracold Quantum Gases



Quantifying entanglement in strongly correlated quantum gases

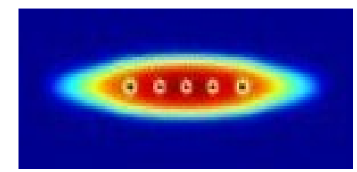
Non-classical light sources in degenerate Fermi gases

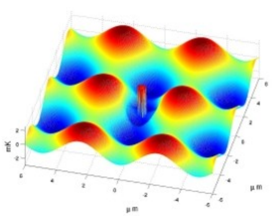




Single particle engineering using adiabatic methods

Long-lived vortex flux qubits in superfluid BECs





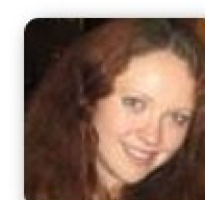
Sub-micron fibres in optical lattices
for global access quantum computing

Dr. Thomas Busch

John Goold

Suzanne McEndoo

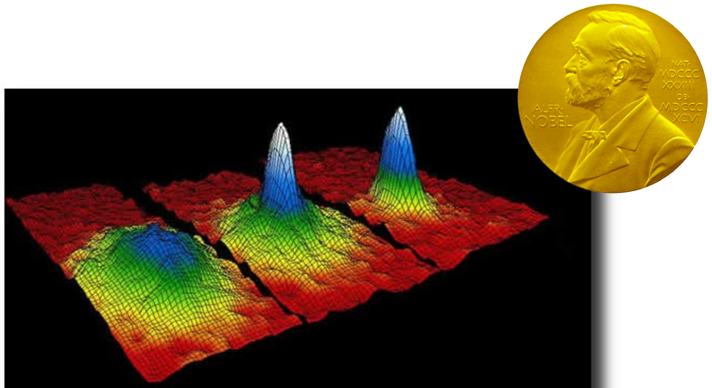
Brian O'Sullivan



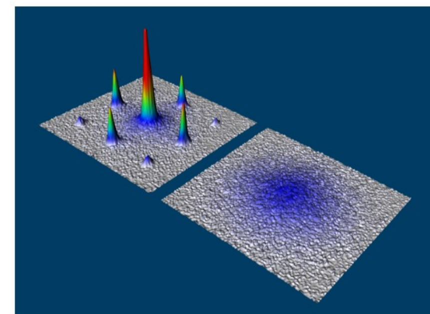
Tara Hennessy

Motivation

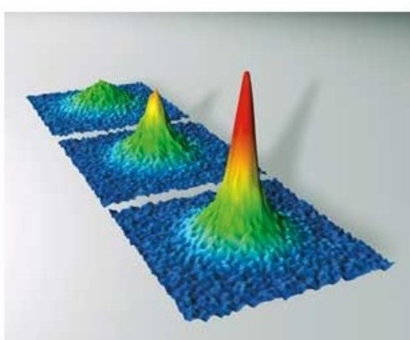
New states of matter:



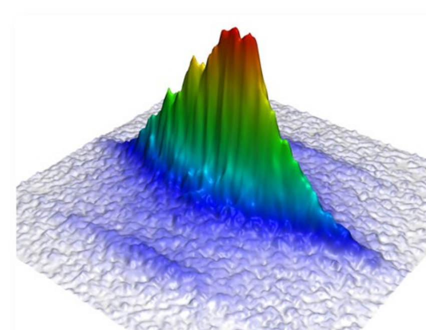
1995 - Bose-Einstein Condensation



2002 - Mott Transitions



2004 – Fermionic Condensates



2004 – Tonks Gas

Outline

1. Introduction into cold atoms

   Brief

2. When Bosons and Fermions become alike:

   Tonks-Giradeau gas

3. Interesting Dynamics:

   Tonks-Girardeau gas in a double well

4. Applications in Quantum Information:

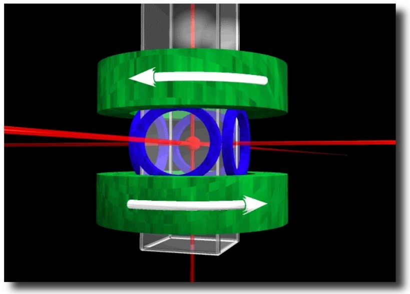
   Entanglement of modes

5. Experimental Systems*:*

   Atom-Ion Gases

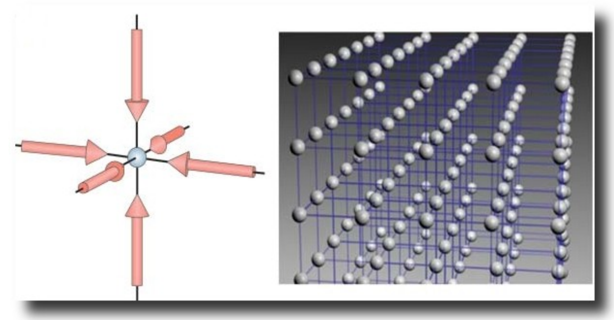
Trapping

Magneto Optical Trap

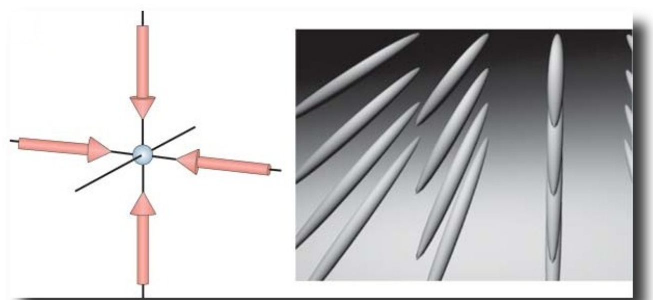


$$a_0 = \sqrt{\frac{\hbar}{m\omega}} \sim \mu m$$

Optical Lattices

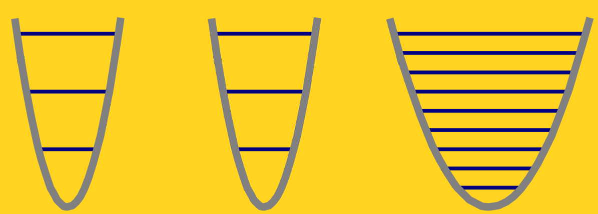


$$a_0 \approx \frac{\lambda}{2} \sim nm$$



effectively lower dimensional system

$$k_B T \ll \hbar\omega_T$$



transverse dynamics can be frozen out!

Quantum Statistics

Bosons (integer spin):

Bose-condensation in three dimensions is very well described by mean field theory using the NLSE.

→ due to the interparticle interaction these systems are *non-linear*

Fermions (half-integer spin):

Two fermions do not have s-wave scattering due to symmetry reasons and at low temperature higher order amplitudes become very small

→ systems can be described as *ideal* gases

One-dimensional Systems (Bosons only)

High Density Limit:

Non-linear Schrödinger Equation can be exactly solved for $V(x) = 0$

$$E\psi(x) = -\frac{\hbar^2}{2m}\nabla^2\psi(x) + V(x)\psi(x) + \underline{g|\psi|^2\psi(x)}$$

→ Dark and bright soliton solutions

Low Density Limit:

Bosonic gas of interacting particles: Tonks gas

$$E\Psi = \sum_{n=1}^{N}\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_n^2} + \frac{1}{2}m\omega^2 x_n^2\right)\Psi + \underline{\sum_{i<j} U(|x_i - x_j|)\Psi}$$

→ Bosons become indistinguishable from fermions

Bose-Fermi Mapping

1. N neutral, bosonic atoms with point-like interactions

$$H_0 = \sum_{j=1}^{N} -\frac{\hbar^2}{2m}\frac{d^2}{dx_j^2} + V(x_1,\ldots,x_N,t) + a\sum_{i<j}^{N}\delta(|x_i - x_j|)$$

2. assume a $\rightarrow \infty$ and replace the interaction term by a constraint

$$\Psi = 0 \qquad \text{if} \qquad |x_i - x_j| = 0 \qquad i \neq j$$

3. equivalent to the Pauli exclusion principle!

➡ *Solve fermionic problem and symmetrise!*

Bose-Fermi Mapping

So, we need:

1. a system where the single particle eigenfunctions are known (and where they are *nice!*)

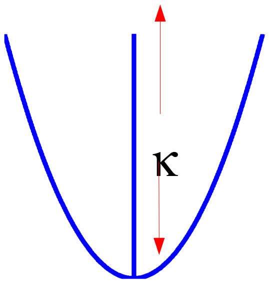
   → free space, box, harmonic oscillator,...

2. a system where the Slater determinant can be calculated (analytically)

   → probably best if eigenfunctions were polynomials

The δ-split Harmonic Oscillator

$$H_0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2 + \kappa\delta(x)$$



→ the *odd* eigenfunctions of the HO are still good eigenfunctions!

→ the *even* ones have to be found

Scaling all quantities: $a_0 = \sqrt{\hbar/2m\omega}$ $\epsilon_0 = \hbar\omega$ for $\kappa = 0$

$$\left(-\frac{d^2}{dx^2} + \frac{1}{4}x^2 + \tilde{\kappa}\delta(x) + \epsilon_n\right)\phi_n(x) = 0$$

For *x > 0* this is Whittakers equation!

The δ-split Harmonic Oscillator

x > 0

$$U(\epsilon_n, x) = \cos\left(\frac{\pi}{4} + \frac{\pi\epsilon_n}{2}\right) Y_1 - \sin\left(\frac{\pi}{4} + \frac{\pi\epsilon_n}{2}\right) Y_2$$

$$Y_1 = \frac{\Gamma\left(\frac{1}{4} - \frac{1}{2}\epsilon_n\right)}{\sqrt{\pi}\, 2^{\frac{1}{4}+\frac{1}{2}\epsilon_n}} e^{\frac{1}{4}x^2} M\left(\frac{1}{4} + \frac{1}{2}\epsilon_n, \frac{1}{2}, \frac{1}{2}x^2\right)$$

$$Y_2 = \frac{\Gamma\left(\frac{3}{4} - \frac{1}{2}\epsilon_n\right)}{\sqrt{\pi}\, 2^{-\frac{1}{4}-\frac{1}{2}\epsilon_n}} e^{-\frac{1}{4}x^2} xM\left(\frac{3}{4} + \frac{1}{2}\epsilon_n, \frac{3}{2}, \frac{1}{2}x^2\right)$$

for any value of κ!

x < 0 since we are looking for the even eigenfunctions

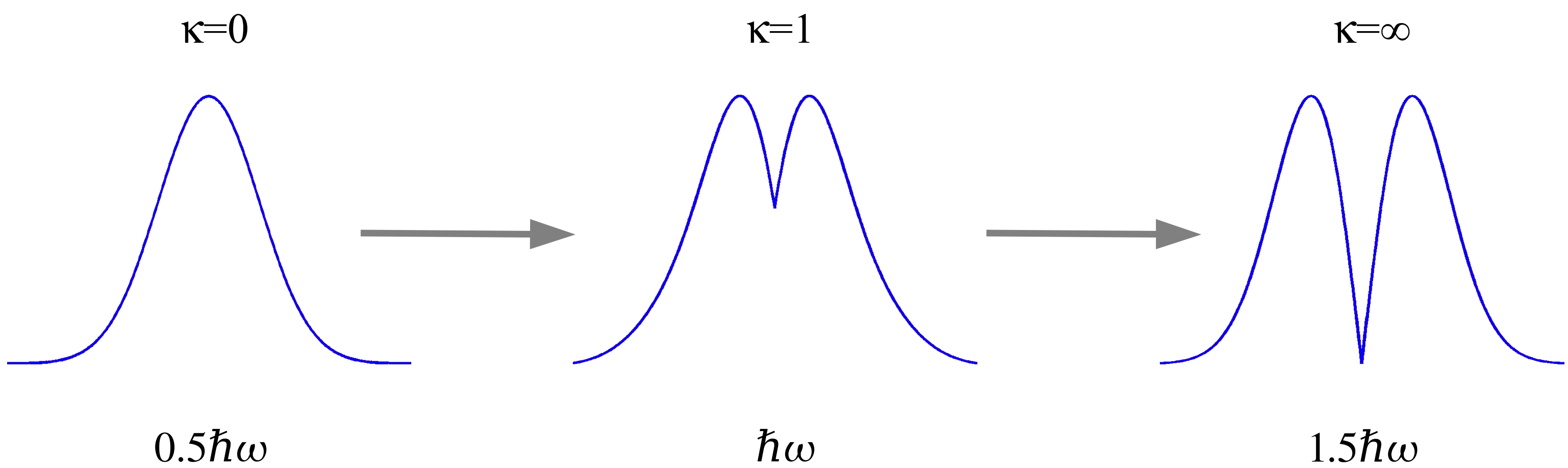
$$\phi_n(x) = CU(\epsilon_n, |x|)$$

x = 0 evaluate the continuity condition:

$$\frac{d}{dx}\phi_n(0^+) - \frac{d}{dx}\phi_n(0^-) = \tilde{\kappa}\phi_n(0)$$

Ground State Eigenfunction

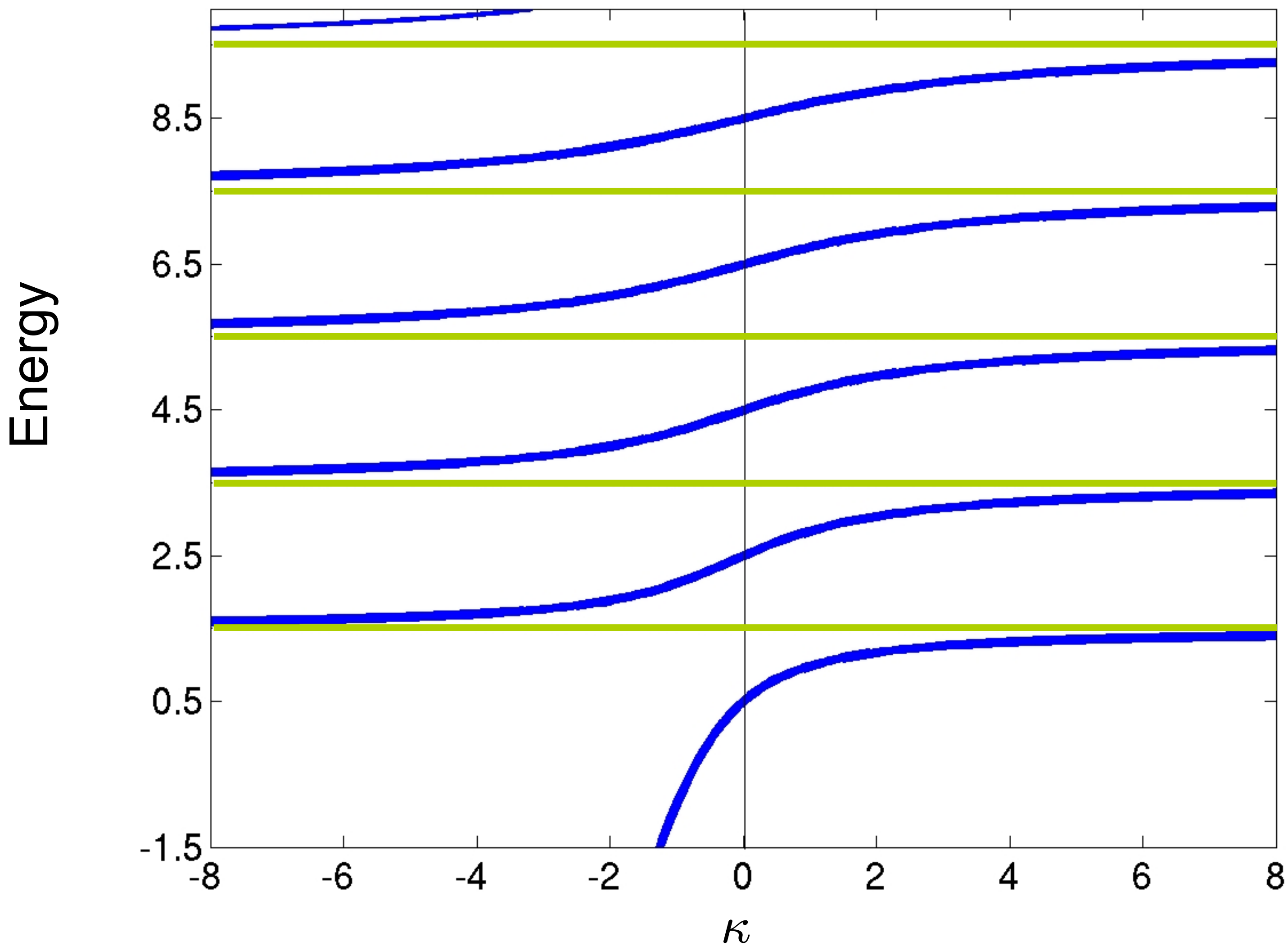
With increasing central potential height the magnitude at the centre of the even eigenfunctions decreases:



- same functional behaviour for all other even states
- for $\kappa = \infty$ even and odd states become degenerate

Eigenvalues

$$\frac{\Gamma\left(\frac{3}{4}+\frac{1}{2}\epsilon_n\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}\epsilon_n\right)} = -\tilde{\kappa}$$



Many Particles in a δ-split trap

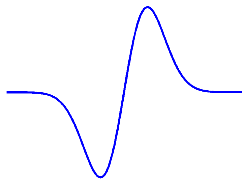
Next: calculate the Slater determinant...

$$\psi_F(x_1,\ldots,x_N) = \frac{1}{\sqrt{N!}} \det_{(n,j)=(0,1)}^{N-1,N} \phi_n(x_j)$$

Example: infinitely high barrier ($\kappa \to \infty$)

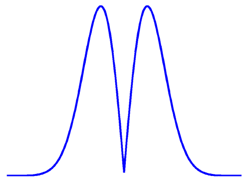
$$\psi_n(x) = C_n e^{-\frac{x^2}{2}} H_n(x)$$

for *n* odd



$$\psi_n(x) = C_{n+1} e^{-\frac{|x|^2}{2}} H_{n+1}(|x|)$$

for *n* even



$$C_n = \left(\sqrt{\pi} a_0 2^n n!\right)^{-\frac{1}{2}}$$

Many Particles in a δ-split trap

Exact many particle wavefunction can be derived:

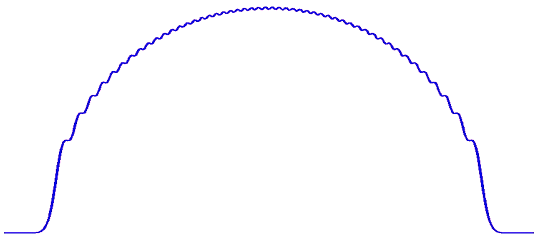
$$\psi_F(x_1,\ldots,x_N) \propto 2^{\frac{N^2}{8}} \left[\prod_{j}^{N/2} x_j\right] \prod_{(j,k)=(1,j+1)}^{(N/2,N/2)} (x_j^2 - x_k^2)$$

Because we know the ground state is real:

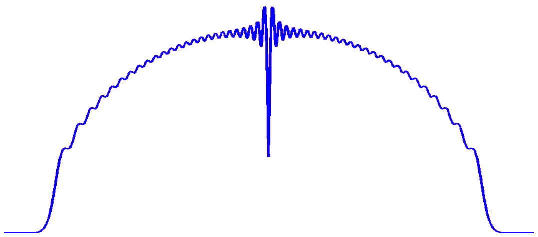
$$\Rightarrow \quad \psi_B(x_1,\ldots,x_N) = |\psi_F(x_1,\ldots,x_n)|$$

$$\Rightarrow \quad \rho_B(x_1,\ldots,x_n) = \rho_F(x_1,\ldots,x_n)$$

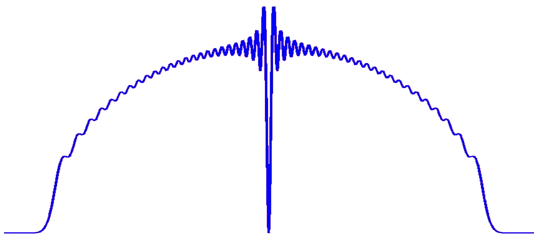
Bosons and fermions become indistiguishable!



$\kappa = 0$



$\kappa = 3$

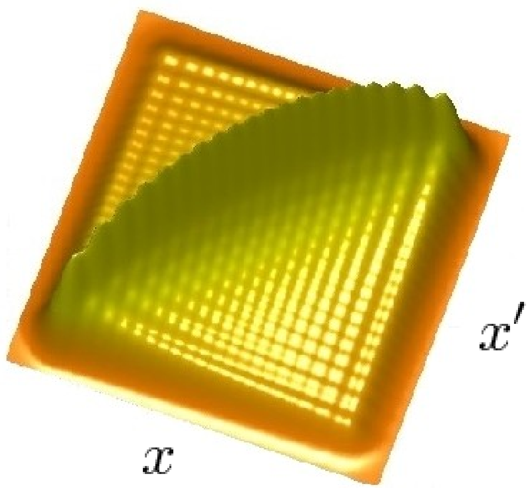


$\kappa = \infty$

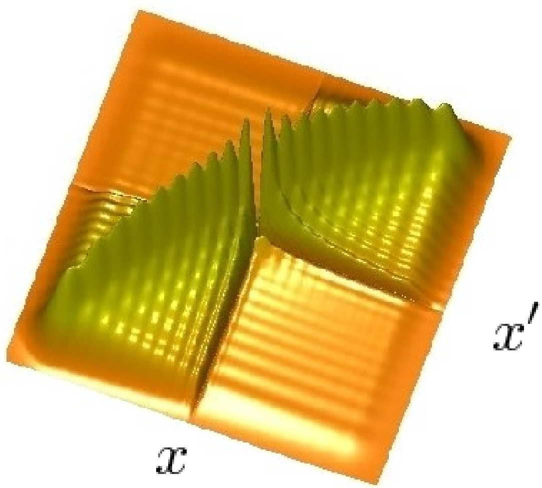
Reduced Single Particle Density Matrix

The self correlations are given by:

$$\rho(x, x') = \int \psi_B(x, x_2, \ldots, x_N) \times \psi_B(x', x_2, \ldots, x_n)\, dx_2 \ldots dx_N$$



no barrier



high barrier

classical result: $\rho(x, x') = \delta(x - x')$

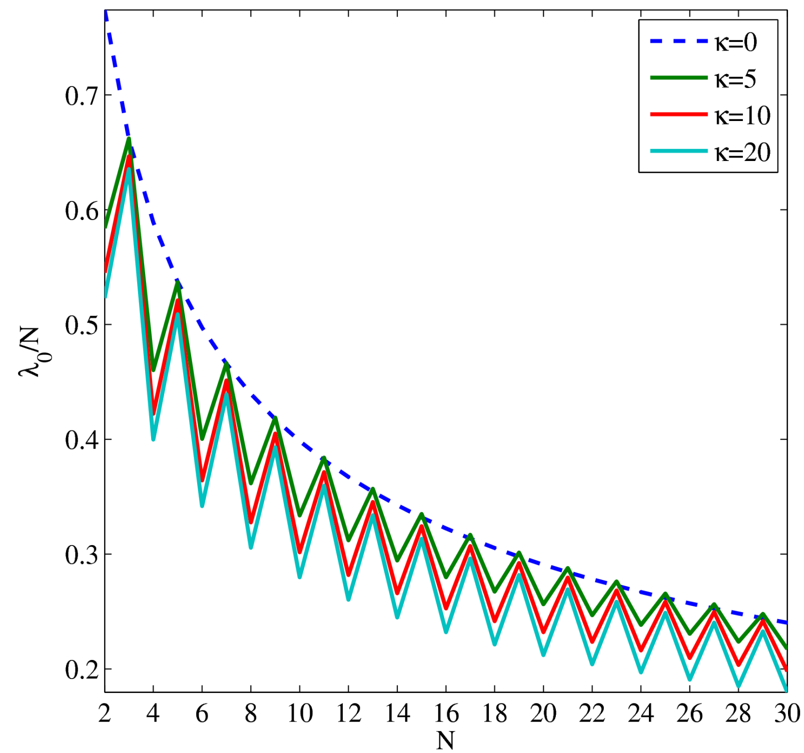
Coherences

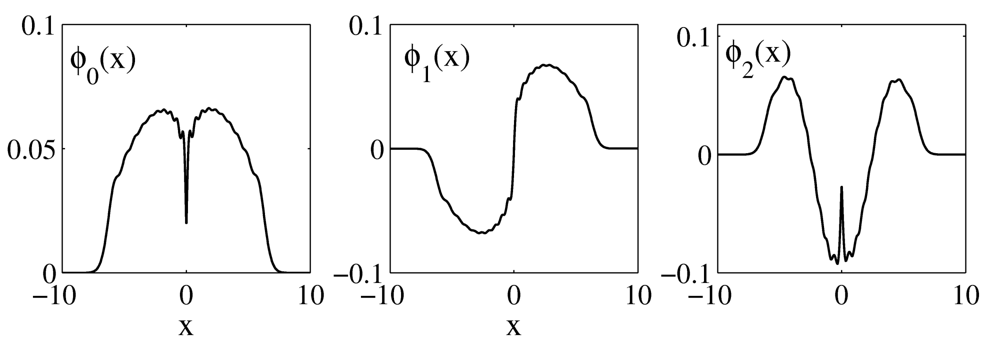
low dimension & strong interaction

Tonks gas is not Bose condensed!

change of basis by diagonalising
reduced single particle density matrix

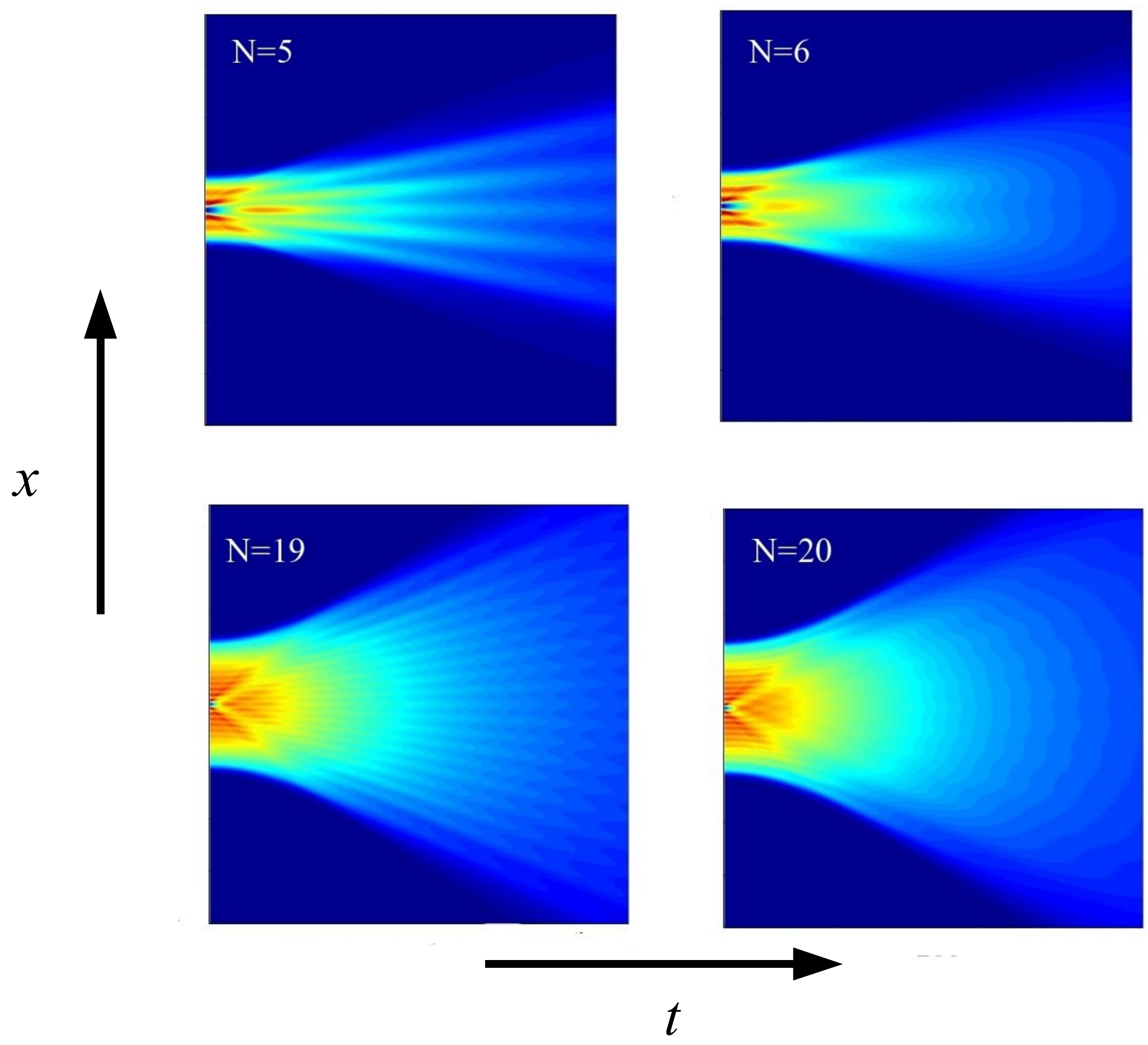
ground state occupation / coherences



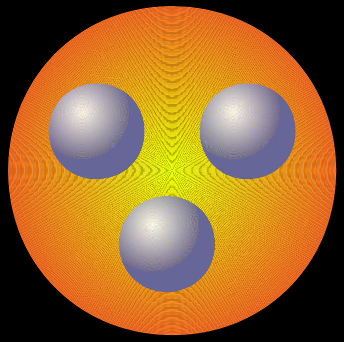


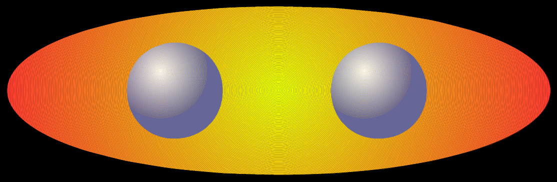
Interferences

Switch all trapping potentials off:



Entanglement in Ultracold Gases





Why is this all interesting?

Cold atoms are a well suited system to do quantum information:

well isolated but also highly
controllable!

⇒ Tonks gas, as an exactly solvable model, lets us calculate many of the properties of interest in quantum information

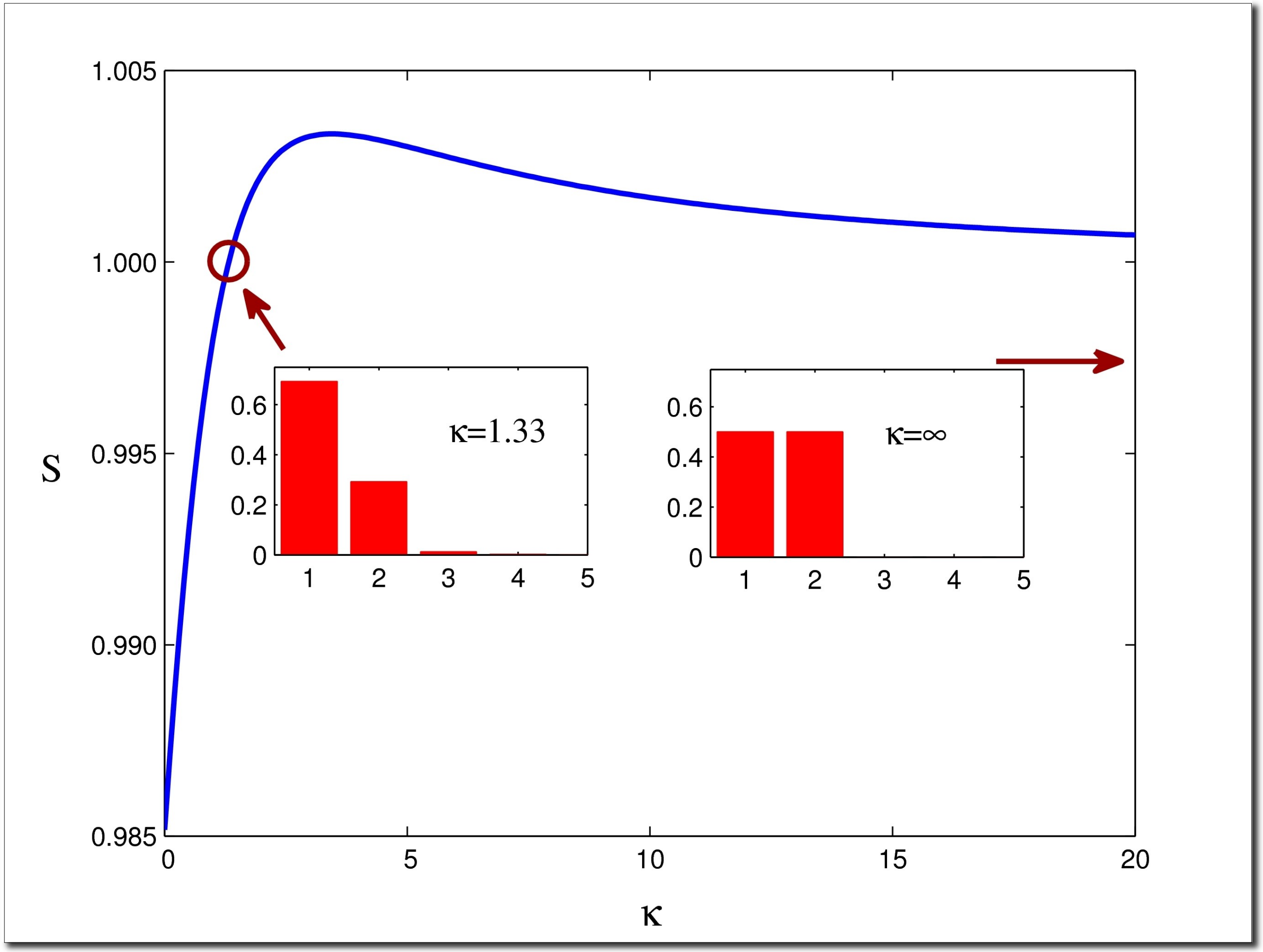
Example: Entanglement

$$S(\rho) = -\mathrm{Tr}(\rho \ln \rho)$$

von Neumann entropy

(only for a two particle system though...)

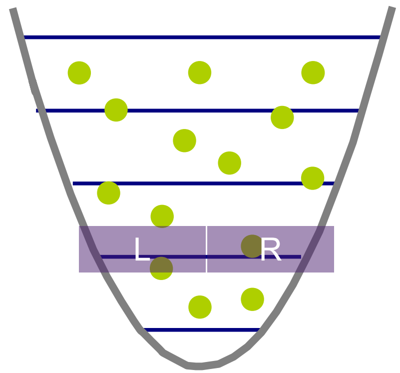
Two Particle Entanglement



Indistinguishability?

How about many particle entanglement?

Idea:



- let two particles interact with the gas in two different regions of the trap
- in second quantisation the regions can be described as modes

$$|\phi_G\rangle \sim |L\rangle + |R\rangle \longrightarrow |\phi_{LR}\rangle \sim |10\rangle + |01\rangle$$

- calculate the entanglement of the state of the two sensors

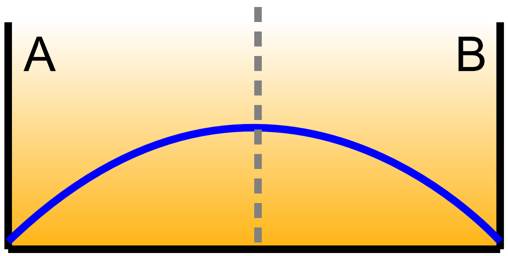
Why is that interesting?

For ideal Bose gas:

increase in entanglement ⟷ BEC transition temperature

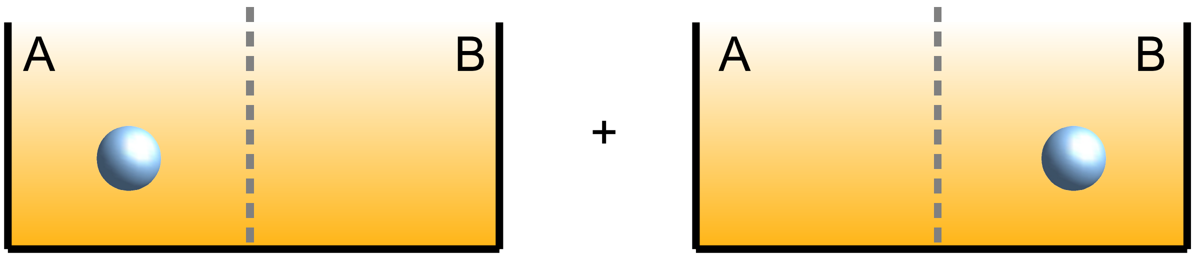
Spatial Mode Entanglement

1st Quantisation



Single particle is in a superposition between left and right

2nd Quantisation

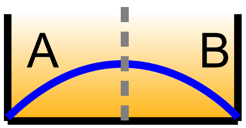


non-local particle number entanglement between modes A and B

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|1\rangle_A|0\rangle_B + |0\rangle_A|1\rangle_B)$$

Spatial Mode Entanglement

<u>Language: non-relativistic quantum field theory</u>



→ construct mode operators

$$\hat{\psi}^\dagger_{A,B} = \int_{A,B} dx\, g(x)\hat{\psi}^\dagger(x)$$

bosonic quantum field operator

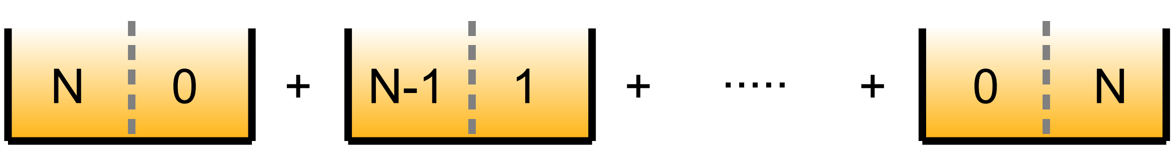
mode function

$$\int |g(x)|^2 = 1 \qquad \left[\hat{\psi}_i, \hat{\psi}_j^\dagger\right] = \delta_{ij}$$

→ number of particles in the gas $N = \mathrm{tr}\left[\hat{\psi}^\dagger_A \hat{\psi}_A\, \rho\right] + \mathrm{tr}\left[\hat{\psi}^\dagger_B \hat{\psi}_B\, \rho\right]$

→ N particle BEC split in the middle is described therefore as

$$|\Psi\rangle = \frac{1}{\sqrt{N!}}\left(\frac{\hat{\psi}^\dagger_A}{\sqrt{2}} + \frac{\hat{\psi}^\dagger_B}{\sqrt{2}}\right)^N |0\rangle = \frac{1}{\sqrt{2^N}}\sum_{n=0}^{N}\frac{\sqrt{N!}}{\sqrt{n!(N-n)!}}|n, N-n\rangle$$

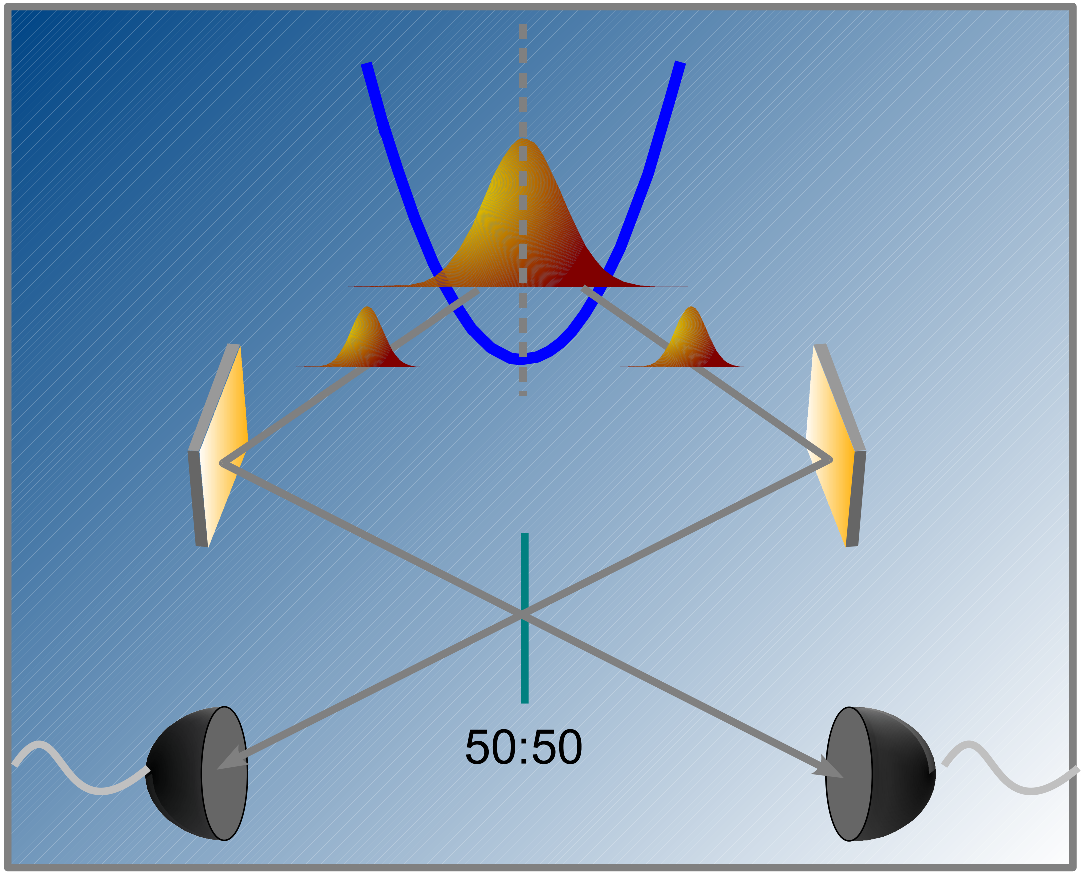


Interference Detection Scheme

joint measurement of the two modes

$$\hat{\psi}_C^\dagger = \frac{1}{\sqrt{2}}\left(\hat{\psi}_A^\dagger + \hat{\psi}_B^\dagger\right)$$

$$N_C = \mathrm{tr}\left[\hat{\psi}_C^\dagger \hat{\psi}_C \, \rho\right]$$



$$\hat{\psi}_D^\dagger = \frac{1}{\sqrt{2}}\left(\hat{\psi}_A^\dagger - \hat{\psi}_B^\dagger\right)$$

$$N_D = \mathrm{tr}\left[\hat{\psi}_D^\dagger \hat{\psi}_D \, \rho\right]$$

assume a fixed total particle number

⇒ pure, separable state cannot show total destructive interference

Ch. Simon, Phys. Rev. A **66**, 052323 (2002)

Interference Detection Scheme

Calculate detector outcomes:

$$N_C = \mathrm{tr}\left[\hat{\psi}_C^\dagger \hat{\psi}_C\ \rho\right] = \frac{1}{2}\Big(\mathrm{tr}[\hat{\psi}_A^\dagger \hat{\psi}_A\ \rho] + \mathrm{tr}[\hat{\psi}_B^\dagger \hat{\psi}_B\ \rho] + 2\mathrm{tr}[\hat{\psi}_A^\dagger \hat{\psi}_B\ \rho]\Big) = \frac{N}{2} + \epsilon_{AB}$$

$$N_D = \mathrm{tr}\left[\hat{\psi}_D^\dagger \hat{\psi}_D\ \rho\right] = \frac{1}{2}\Big(\mathrm{tr}[\hat{\psi}_A^\dagger \hat{\psi}_A\ \rho] + \mathrm{tr}[\hat{\psi}_B^\dagger \hat{\psi}_B\ \rho] - 2\mathrm{tr}[\hat{\psi}_A^\dagger \hat{\psi}_B\ \rho]\Big) = \frac{N}{2} - \epsilon_{AB}$$

$$\epsilon_{AB} = \int_A dx \int_B dx'\ g(x)g(x')\ \rho^{(1)}(x,x')$$

reduced single particle density matrix

fully separable state: $\rho_{\mathrm{sep}} = \sum_i p_i |n_i\rangle\langle_i|_A \otimes |N-n_i\rangle\langle N-n_i|_B$

$$\epsilon_{AB} = 0$$

general state (of **fixed** N): $\epsilon_{AB} \neq 0$

→ measure of spatial coherence → good measure for entanglement for N=2

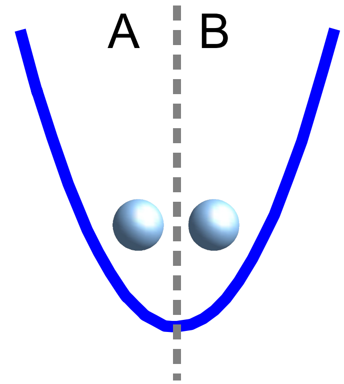
I. Bloch, T.W. Hänsch and T. Esslinger, Nature **403**, 166 (2000)

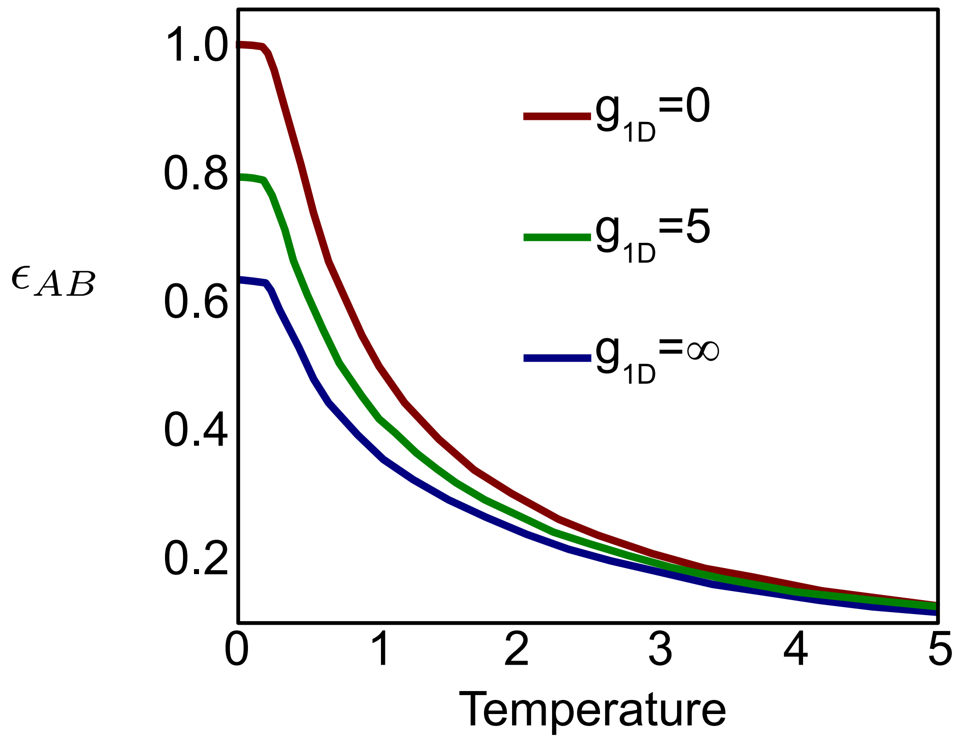
L. Heany, PhD Thesis, University of Leeds (2008)

Cold Boson Pair

Boson pair Hamiltonian (1D)

$$H = \sum_{i=1}^{2} \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx_i^2} + \frac{1}{2}m\omega^2 x_i^2 \right) + g_{1D}\delta(|x_i - x_j|)$$



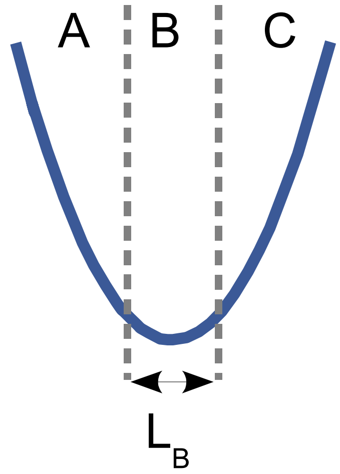


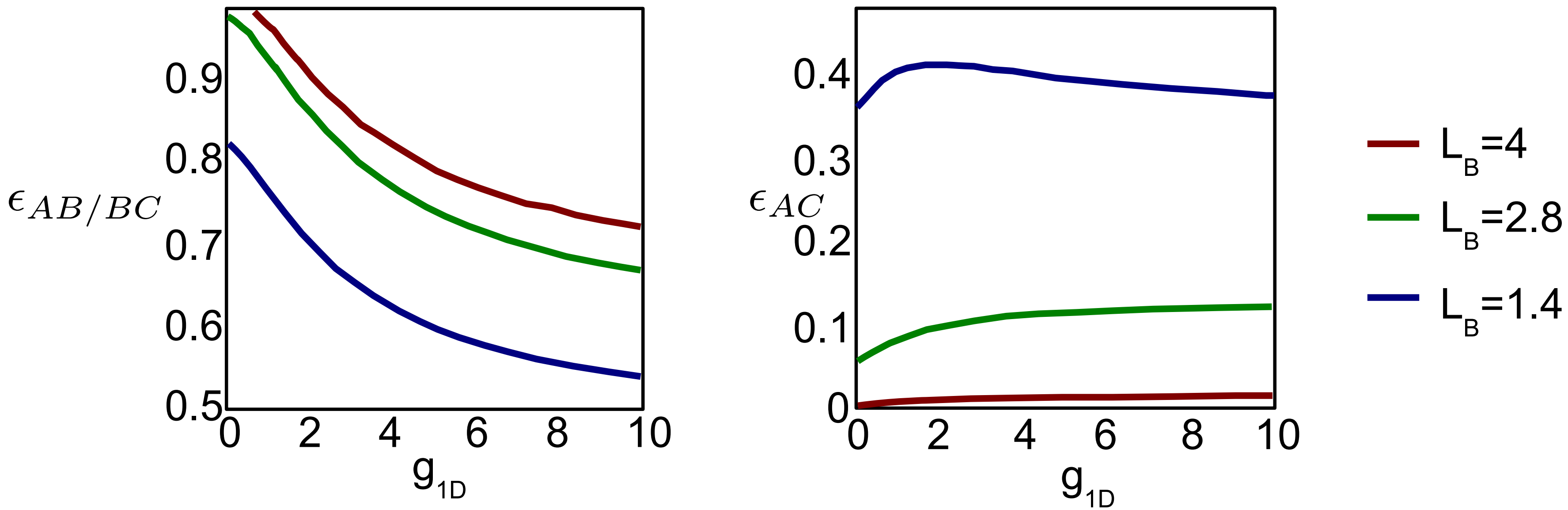
entanglement finite even at strong interactions

D.S. Murphy, J.F. McCann, J. Goold and TB, Phys. Rev. A **76**, 053616 (2007)

J. Goold, L. Heany, TB and V. Vedral, arxiv/0902.2096

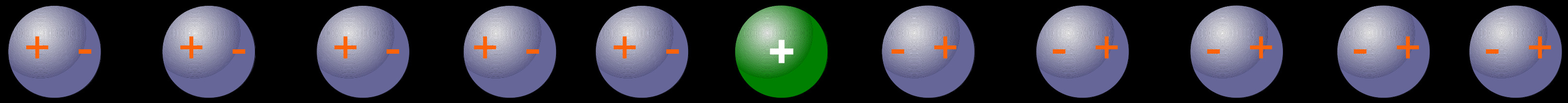
Cold Boson Pair



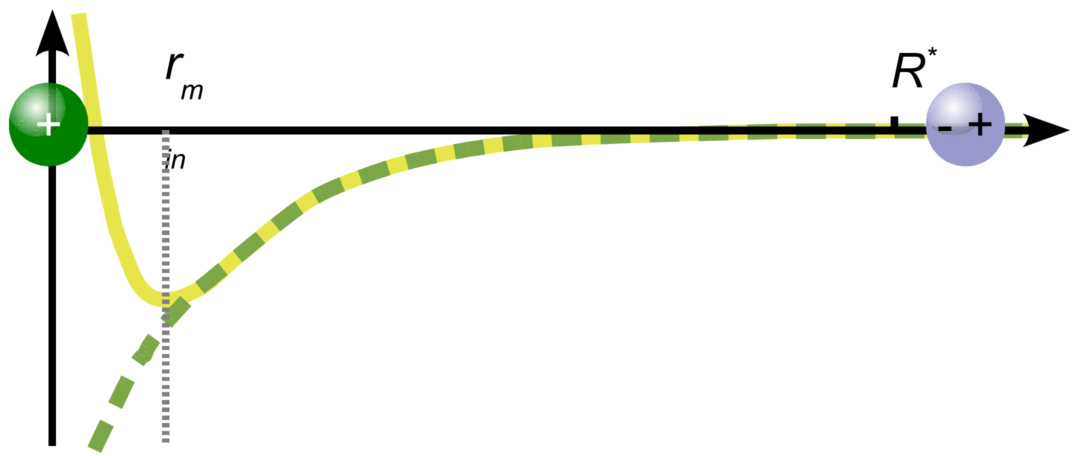


tuning the interaction parameter modifies the distribution of entanglement

Ultracold Ions in Tonks Gases



Born-Oppenheimer polarization potential



$$\lim_{r\to\infty} V(r) = \frac{-\alpha e^2}{2r^4}$$

Characteristic scales: $$\frac{\hbar^2}{2\mu(R^*)^2} = \frac{\alpha e^2}{2(R^*)^4}$$

Polarisation length $$R^* = \sqrt{\frac{2\mu\alpha e^2}{\hbar^2}}$$

Polarisation energy $$E^* = \frac{\hbar^2}{2\mu(R^*)^2}$$

Atom-Ion Hamiltonian

Consider the idealised situation where an atom and an ion sit in the same isotropic 3D harmonic trap

$$\mathcal{H}_{ia} = \sum_{\nu=i,a}\left(-\frac{\hbar^2}{2m_\nu}\frac{\partial^2}{\partial \mathbf{r}_\nu^2} + \frac{1}{2}m_\nu\omega_\nu^2\mathbf{r}_\nu^2\right) + V_{int}(|\mathbf{r}_i - \mathbf{r}_a|)$$

- ramp up transverse trapping frequencies $\omega_\perp \gg \omega_{||}$
- for low energies the problems becomes one-dimensional

$$\Psi(r_i, r_a) = \psi_\perp(\rho_i, \rho_a)\psi_{||}(x_i, x_a)$$

- go to relative and centre of mass co-ordinates:

$$\mathcal{H}_{rel} = -\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial x^2} + \frac{1}{2}\mu\omega^2 x^2 - \frac{\alpha e^2}{2x^4}$$

Z. Idziaszek, T. Calarco, and Peter Zoller Phys. Rev. A **76**, 033409 (2007)

Quantum Defect Theory

the interaction potential deviates from the $1/r^4$ law at short distance, which diverges towards $-\infty$

→ quantum defect theory (neglect harmonic potential)

$$\left(-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial x^2}-\frac{\alpha e^2}{2x^4}\right)\psi_n(x)=E_n\psi_n(x)$$

⇒

$$\psi_n^e \to |x|\sin\left(\frac{R^*}{|x|}+\underline{\phi_e}\right)$$

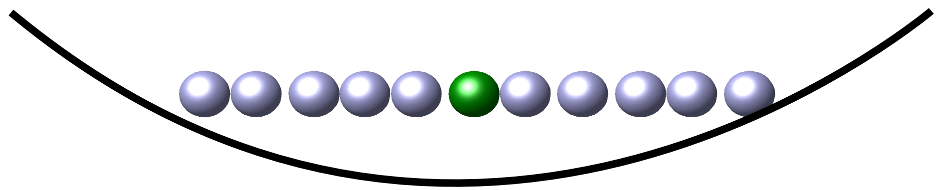
$$\psi_n^o \to x\sin\left(\frac{R^*}{|x|}+\underline{\phi_o}\right)$$

quantum defect parameters are energy independent short range phases

→ related to *s*- and *p*-wave scattering lengths via $a_{1D}^{e,o}=-\cot(\phi_{e,o})$

→ not known for current systems → numerical solution using the iterative Numerov method

Ion in Tonks Gas

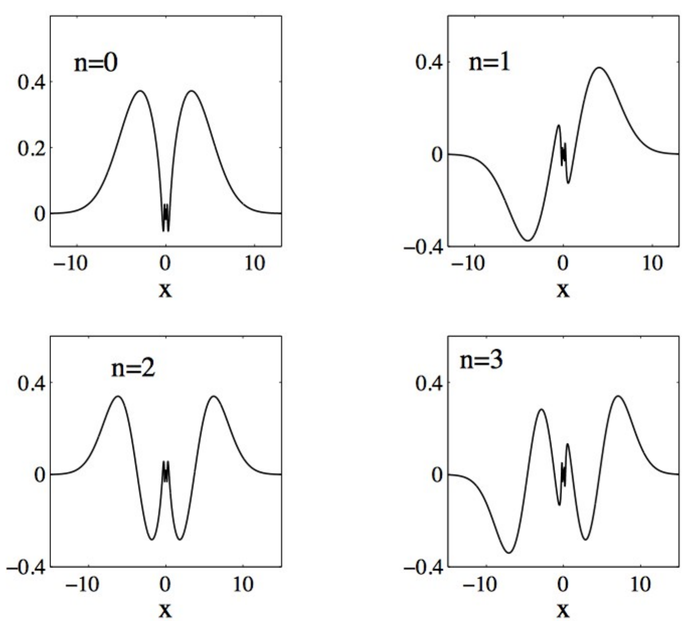


$$H = \sum_{n=1}^{N} \left(-\frac{d^2}{dx_n^2} + \xi x_n^2 - \frac{1}{x_n^4} \right) + g_{1D} \sum_{i<j} \delta(|x_i - x_j|)$$

$$\xi = \left(\frac{R^*}{a_0} \right)^4$$

single particle problem

atom-atom interaction



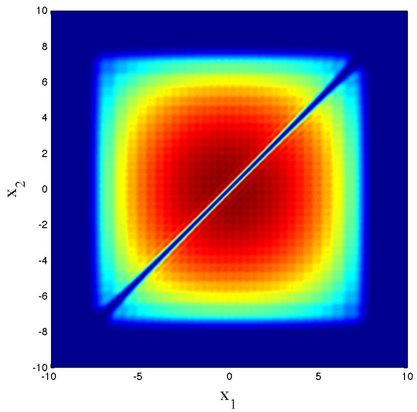
Fermi-Bose Mapping
(as before)

$$\phi_e = -\frac{\pi}{4} \quad \phi_o = \frac{\pi}{4}$$

Molecular Atom-Ion States?

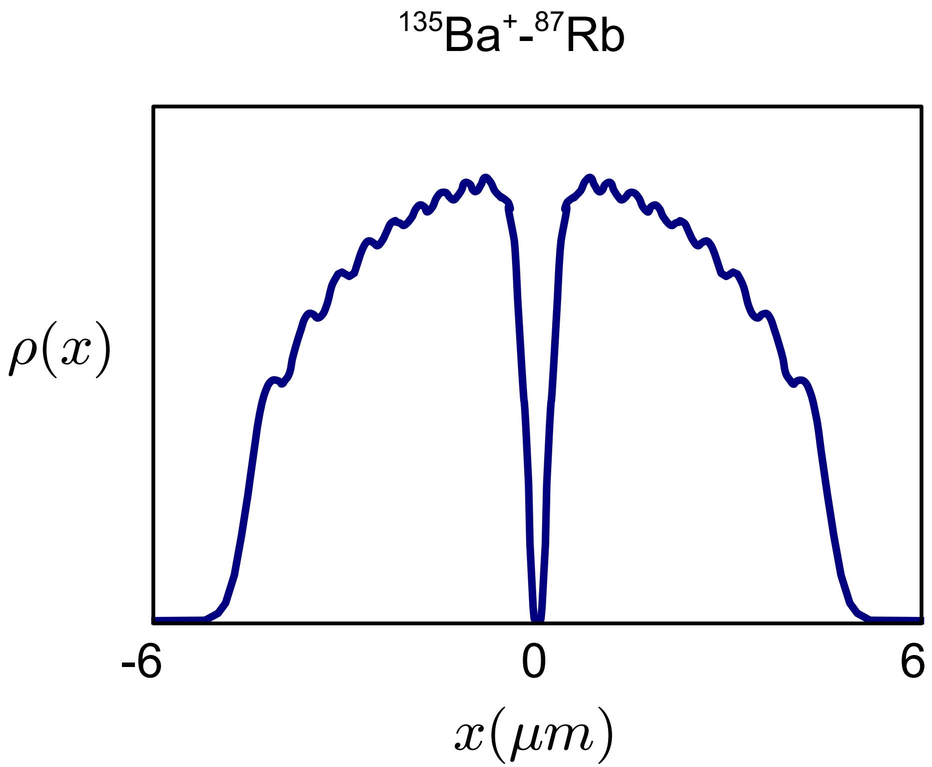
- access to bound states requires three body collisions

- but, in Tonks limit the *second order correlation function* shows that its diagonal elements are suppressed



- in one dimension the system has no access to the bound states!

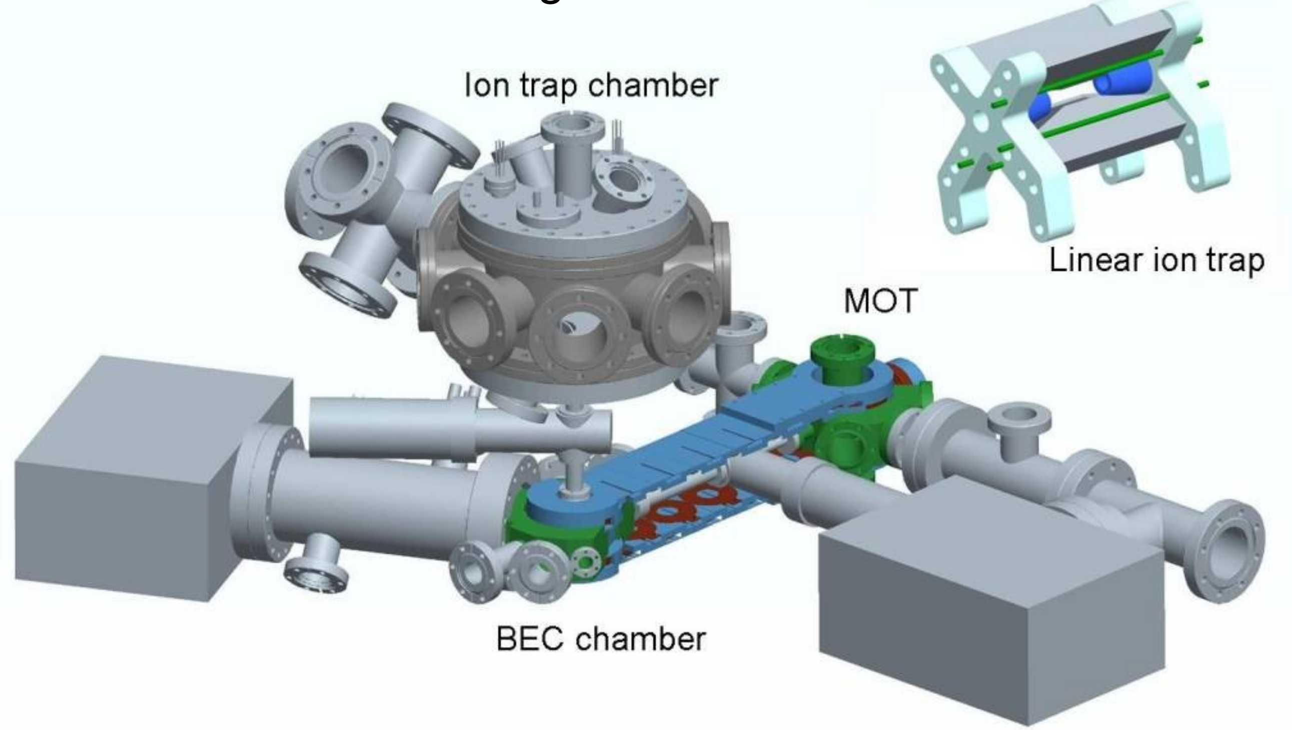
Tonks Gas Density

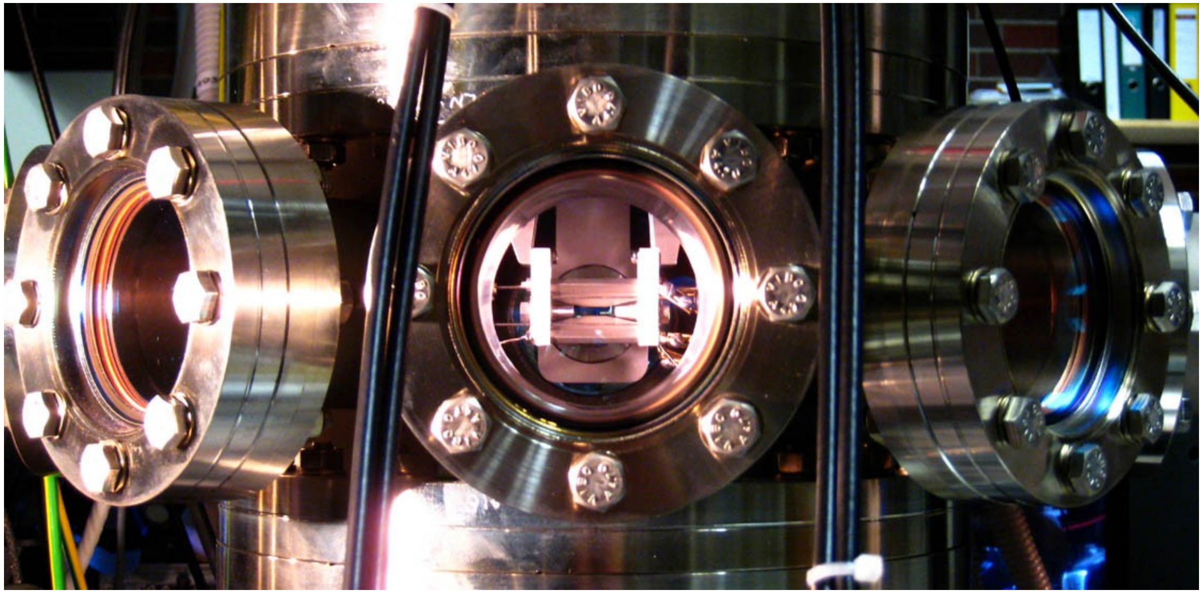


→ density dip in centre, despite attractive interaction!

Experiment Innsbruck

Prof. Johannes Denschlag





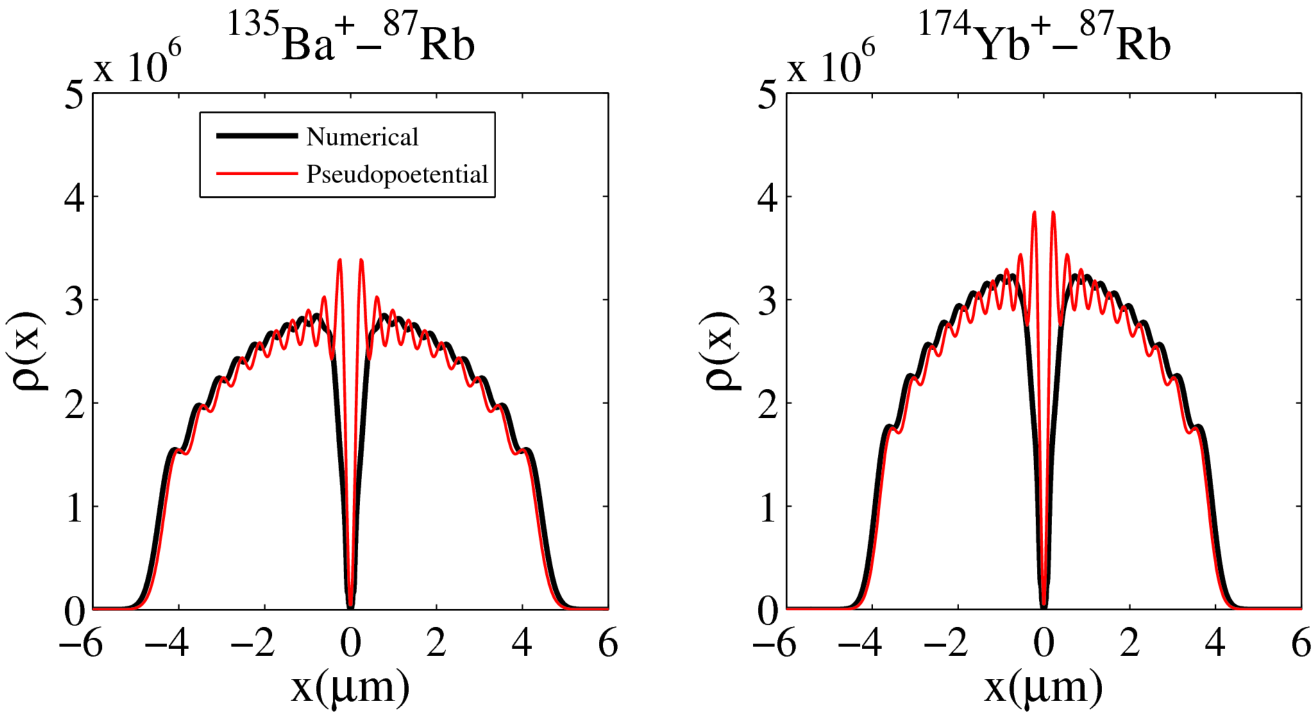
also: Dr. M. Köhl,
Cambridge

Pseudo-Potential Approximation

$$H = \sum_{n=1}^{N}\left(-\frac{d^2}{dx_n^2} + \xi x_n^2 - \frac{1}{x_n^4}\right) + g_{1D}\sum_{i<j}\delta(|x_i - x_j|)$$

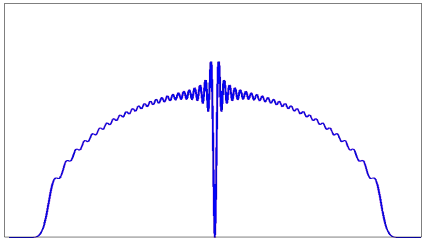
vs.

$$H = \sum_{n=1}^{N}\left(-\frac{d^2}{dx_n^2} + \frac{1}{2}x_n^2 + \kappa\delta(x)\right) + g_{1D}\sum_{i<j}\delta(|x_i - x_j|)$$

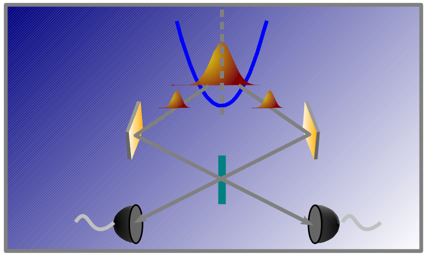


Conclusion

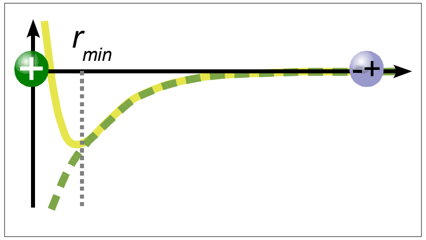
Tonks gas can be solved in a double well trap.



Mode- Entanglement properties can be calculated exactly



One-dimensional atom-ion systems can be treated in quantum defect and TG formalism



Co-workers

John Goold	PhD student
Domhnall Murphy Jim McCann	PhD student QUB (now Citi)
Libby Heaney Vlatko Vedral	PhD student (now Oxford)
Hauke Doerk-Bending Tommaso Calarco	PhD student (now Munich)